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Analog of the Weierstrass Theorem and the Blaschke Product for $A(z)$-analytic Functions

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Abstract. We consider $A(z)$-analytic functions in the case when $A(z)$ is an antiholomorpic function. For $A(z)$-analytic functions analogs of the Weierstrass theorem and of the Blaschke theorem are proved.

Keywords: $A(z)$-analytic function, Cauchy's integral theorem, Weierstrass theorem, Jensen's theorem, Blaschke theorem.

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1. Introduction and preliminaries

The paper is devoted to the solutions of the Beltrami equation

$$\bar{D}_A f(z) := \frac{\partial f(z)}{\partial \bar{z}} - A(z)\frac{\partial f(z)}{\partial z} = 0 \tag{1}$$

which is directly related to the theory of quasi-conformal mappings (see [1,13]). The function $A(z)$ in general is assumed to be measurable with the condition $|A(z)| \leqslant C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. Solutions of equation (1) are often called $A(z)$-analytic functions. The most interesting case is $\partial A = 0$, i.e. $A(z)$ is an anti-analytic function in D and such that $|A(z)| \leqslant C < 1$ $\forall z \in D$. Then according to (1) the class $f \in O_A(D)$ of $A(z)$-analytic functions in D is characterized by the fact that $\bar{D}_A f = 0$. Since any anti-analytic function is smooth, it follows that $O_A(D) \subset C^\infty(D)$ (see [13]).

Here we study the analogs of the well-known Weierstrass and Blaschke theorems for $A(z)$-analytic functions in convex domains, when $A(z)$ is an anti-analytic function. The requirement for the convexity of the domain is due to the fact that for non-convex domains the required kernel of the integral formula, which is involved in the proof of the main results, may not exist. For analytic functions, the Weierstrass and Blaschke factorizations are well studied (see [7, 8]).

Let us present some facts from the theory of $A(z)$- analytic functions that we will need below. Consider the integral

$$\psi(z,\xi) = z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\,d\tau} \in O_A(D),$$

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where $\gamma(\xi, z)$ is a smooth curve connecting points ξ, $z \in D$. If the domain D simply connected, then the integral

$$I(z) = \int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau$$

does not depend on the integration path; it coincides with the primitive, $I'(z) = \bar{A}(z)$. The function $\psi(z,\xi)$ for convex domains has a single zero at the point $z = \xi$. In particular, the set $L(\xi, r) = \left\{ z \in D : \left| \psi(z,\xi) \right| = \left| z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau} \right| < r \right\}$ is an open connected set in D. For sufficiently small $r > 0$ it belongs compactly to D and contains the point ξ. This set is called the $A(z)$-lemniscate centered at ξ and denoted as $L(\xi, r)$. Put

$$K(z,\xi) = \frac{1}{2\pi i} \cdot \frac{1}{z - \xi + \overline{\overline{\int\limits_{\gamma(\xi,z)} \bar{A}(\tau)\, d\tau}}}. \tag{2}$$

Theorem 1.1 (analog of Cauchy's formula, see [4, 9])**.** *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ be its subdomain with a piecewise smooth boundary ∂G. Then for any function $f(z) \in O_A(G) \bigcap C(\bar{G})$ we have*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\xi)}{z - \xi + \overline{\overline{\int\limits_{\gamma(\xi,z)} \overline{A(\tau)} d\tau}}} \left(d\xi + A(\xi)\, d\bar{\xi} \right), \quad z \in G. \tag{3}$$

2. Generalized Weierstrass theorem for $A(z)$-analytic functions.

The main result of the section is following theorem.

Theorem 2.1. *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ its compact subdomain. Then, whatever sequence of points $a_n \in G$ that has no limit points in G, there exists an $A(z)$-analytic in G function f that has zeros at all points of a_n and only at these points.*

Proof. Note that if the set $\{a_n\} = \{a_1, a_2, \ldots, a_n\}$ is finite, then the product $\prod\limits_{n=1}^{m} \psi(z, a_n)$ can be taken as the function $f(z)$. However, when the set $\{a_n\}$ is countable this product may diverge. In this case, the function $f(z)$ is constructed in the form of an infinite product, also with the help of $\psi(z,\xi)$, which for convex domains has a single zero $z = \xi$. But the $\psi(z, a_n)$ is multiplied by some additional function, that do not vanish, so that the considered infinite product converges uniformly.

For each point a_n, we find a point $b_n \in \partial G$, which is closest to the point a_n. Then the value of $r_n = \psi(b_n, a_n) \to 0$ at $n \to \infty$. Since

$$\psi(z, b_n) - \psi(a_n, b_n) = z - b_n + \overline{\int\limits_{\gamma(b_n,z)} \bar{A}(\tau)\, d\tau} - (a_n - b_n) - \overline{\int\limits_{\gamma(b_n,a_n)} \bar{A}(\tau)\, d\tau} =$$

$$= z - a_n + \overline{\int\limits_{\gamma(a_n,z)} \bar{A}(\tau)\, d\tau} = \psi(z, a_n),$$

we get

$$\frac{\psi(z, a_n)}{\psi(z, b_n)} = \frac{\psi(z, b_n) - \psi(a_n, b_n)}{\psi(z, b_n)} = 1 - \frac{\psi(a_n, b_n)}{\psi(z, b_n)}.$$

We fix $n \in \mathbb{N}$ and consider the decomposition

$$\ln \frac{\psi(z, a_n)}{\psi(z, b_n)} = \ln \left(1 - \frac{\psi(a_n, b_n)}{\psi(z, b_n)}\right) = -\sum_{k=1}^{\infty} \frac{\psi^k(a_n, b_n)}{k\psi^k(z, b_n)}. \tag{4}$$

The series converges uniformly on the compact set $\{z \in G : \; |\psi(z, b_n)| \geqslant 2r_n\}$. Therefore, we can choose a natural number p_n so that

$$\left|\ln \frac{\psi(z, a_n)}{\psi(z, b_n)} + \sum_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}\right| < \frac{1}{2^n}, \;\; |\psi(z, b_n)| \geqslant 2r_n, \;\; (n = 1, 2, \dots). \tag{5}$$

With this choice of p_n, the infinite product

$$f(z) = \prod_{n=1}^{\infty} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}} \tag{6}$$

converges uniformly inside the domain $G \setminus \{a_n\}$.

Indeed, for any compact set $K \subset\subset G$, there is N such that $a_n \notin K, \; |\psi(z, b_n)| \geqslant 2r_n$ for all $n \geqslant N$ and all $z \in K$. Then the series of $A(z)$-analytic functions

$$\sum_{n=N}^{\infty} \left(\ln \frac{\psi(z, a_n)}{\psi(z, b_n)} + \sum_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}\right)$$

and, therefore, the infinite product $\prod\limits_{n=N}^{\infty} \frac{\psi(z, a_n)}{\psi(z, b_n)} e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}}$ due to (5) converges on K uniformly. Therefore, the product

$$f(z) = \prod_{n=1}^{\infty} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}} = \prod_{n=1}^{N-1} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}} \times \prod_{n=N}^{\infty} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}}$$

is an $A(z)$-analytic function in G that vanishes only at points $a_n \in G$. □

Corollary 1. *Let $D \subset \mathbb{C}$ be a convex domain and $G \subset\subset D$ an arbitrary simply connected compact subdomain. Then, any function $f(z) \in O_A(G)$ admits a factorization*

$$f(z) = e^{g(z)} \prod_{n} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}}, \tag{7}$$

where $\{a_n\}$ is a set (finite or countable) of zeros of the function $f(z) \in O_A(G)$, p_n, b_n the values defined in the proof of Theorem 2, and $g(z)$ is some $A(z)$-analytic function in G. Note that if $\{a_n\}$ is finite, then representation (7) is very simple,

$$f(z) = e^{g(z)} \prod_{n} \psi(z, a_n) \, .$$

Proof. The corollary is easily obtained if we take into account that the ratio

$$f(z) / \prod_{n} \frac{\psi(z, a_n)}{\psi(z, b_n)} \, e^{\sum\limits_{k=1}^{p_n} \frac{\psi^k(b_n, a_n)}{k\psi^k(z, b_n)}}$$

is an $A(z)$-analytic and non-vanishing function in G. Since $G \subset\subset D$ is simply connected, the logarithm

$$g(z) = \ln\left\{ f(z)/\prod_n \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}} \right\} \in O_A(G)$$

and

$$f(z) = e^{g(z)} \prod_n \frac{\psi(z,a_n)}{\psi(z,b_n)}\, e^{\sum_{k=1}^{p_n} \frac{\psi^k(b_n,a_n)}{k\psi^k(z,b_n)}}.$$

□

3. The Blaschke product for $A(z)$-analytic functions.

In this section, we study the zero densities of an $A(z)$-analytic function $f(z) \in O_A(L)$, bounded in lemniscate $L = L(a,R) = \{|\psi(a,z)| < R\}$ in a convex domain $D \subset \mathbb{C}$. Let us start with the formulation of the following Jensen formula

Theorem 3.1 (Jensen's formula)**.** *Let $f \in O_A(L(a,R))$. Denote by $n(t)$ the number of zeros, taking into account the multiplicities of the function $f(z)$ in $\bar{L}(a,t)$, $t < R$. Assume that $f(a) \neq 0$, i.e. $n(0) = 0$. Then, the following formula holds*

$$\int_0^r \frac{n(t)\,dt}{t} = \frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + A(z)\,d\bar{z}| - \ln|f(a)|. \tag{8}$$

Proof. Suppose that $a_1, a_2, a_3, \ldots$ are the zeros of the function f in $L(a,R)$, in the non-decreasing order of $r_n = |\psi(a,a_n)|$, and each $a_1, a_2, a_3, \ldots$ zero in the sequence occurs as many times as its multiplicity. First we show that under the condition $r_n < r_{n+1}$ for $r \in (r_n, r_{n+1})$ we have

$$\frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + A(z)\,d\bar{z}| = \ln\frac{r^n\,|f(a)|}{r_1 r_2 r_3 \ldots r_n} = \ln|f(a)| + n\ln r - \ln r_1 r_2 \ldots r_n. \tag{9}$$

To do this, consider the finite product

$$B(z) = \prod_{k=1}^{n} r \cdot \frac{|\psi(a_k,a)|}{\psi(a_k,a)} \frac{\psi(a_k,a) - \psi(z,a)}{r^2 - \overline{\psi}(a_k,a)\psi(z,a)}.$$

It represents an $A(z)$-analytic function in the lemniscate $L(a, r_{n+1})$ that vanishes only at the points $a_1, a_2, \ldots, a_n$. Therefore, the following representation is true

$$f(z) = e^{g(z)} B(z) = e^{g(z)} \prod_{k=1}^{n} r \cdot \frac{|\psi(a_k,a)|}{\psi(a_k,a)} \frac{\psi(a_k,a) - \psi(z,a)}{r^2 - \overline{\psi}(a_k,a)\psi(z,a)}, \quad g(z) \in O(L(a, r_{n+1})).$$

From here

$$\ln|f(z)| = \operatorname{Re} g(z) + \sum_{k=1}^{n} \ln\left| r \frac{\psi(a_k,a) - \psi(z,a)}{r^2 - \overline{\psi}(a_k,a)\psi(z,a)} \right|, \quad \ln|f(a)| = \operatorname{Re} g(a) + \sum_{k=1}^{n} \ln\frac{r_k}{r}.$$

Since $\operatorname{Re} g(z)$ is $A(z)$-analytic function, we have (see [6])

$$\frac{1}{2\pi r} \int_{|\psi(z,a)|=r} \operatorname{Re} g(z)\,|dz + A(z)\,d\bar{z}| = \operatorname{Re} g(a).$$

Since $\left|r\dfrac{\psi(a_k,a)-\psi(z,a)}{r^2-\overline{\psi}(a_k,a)\psi(z,a)}\right|=1$ for $|\psi\,(z,a)|=r$, we get

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln\left|r\frac{\psi(a_k,a)-\psi(z,a)}{r^2-\overline{\psi}(a_k,a)\psi(z,a)}\right||dz+A\,(z)\,d\bar{z}|=0.$$

Therefore,

$$\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln|f\,(z)|\,|dz+A\,(z)\,d\bar{z}|=\operatorname{Re}g\,(a)=\ln|f\,(a)|+n\ln r-\ln r_1r_2\ldots r_n,$$

which proves the validity of formula (9).

It is clear that

$$\ln|f\,(a)|+n\ln r-\ln r_1r_2\ldots r_n=\ln|f\,(a)|+n\ln r-\sum_{k=1}^{n}\ln r_k=\ln|f\,(a)|+$$
$$+\sum_{k=1}^{n-1}k\,(\ln r_{k+1}-\ln r_k)+n\,(\ln r-\ln r_n)=\ln|f\,(a)|+\sum_{k=1}^{n-1}k\int\limits_{r_k}^{r_{k+1}}\frac{dt}{t}+n\int\limits_{r_n}^{r}\frac{dt}{t}=\ln|f\,(a)|+$$
$$+\sum_{k=1}^{n-1}\int\limits_{r_k}^{r_{k+1}}\frac{n\,(t)\,dt}{t}+\int\limits_{r_n}^{r}\frac{n\,(t)}{t}dt=\int\limits_{0}^{r_n}\frac{n\,(t)}{t}dt+\int\limits_{r_n}^{r}\frac{n\,(t)}{t}dt+\ln|f\,(a)|=\int\limits_{0}^{r}\frac{n\,(t)\,dt}{t}+\ln|f\,(a)|\,.$$

It follows that formula (9) can be written as

$$\int\limits_{0}^{r}\frac{n\,(t)\,dt}{t}=\frac{1}{2\pi r}\int\limits_{|\psi(z,a)|=r}\ln|f\,(z)|\,|dz+A\,(z)\,d\,\bar{z}|-\ln|f\,(a)|\,. \tag{10}$$

Note that we proved formula (10) under the condition $r_n<r<r_{n+1}$. If we show the continuous increase of both parts of this formula with the continuous increase of r from $r_{n+1}-0$ to $r_{n+1}+0$, then this will prove the validity of formula (10) for an arbitrary $r<R$. For the left side of (10) this is obvious. For the right side, let $r_n<r_{n+1}=r_{n+2}=\ldots.=r_{n+m}<r_{n+m+1},\ \ m\geqslant 1$. Then in some ring $L\,(a,r'')\setminus\bar{L}\,(a,r')\,,\ r_n<r'<r_{n+1}<r''<r_{n+m+1}$, (see [7])

$$f\,(z)=g\,(z)\prod_{k=1}^{m}\left[\psi\,(a_{n+k},a)-\psi\,(z,a)\right]=g\,(z)\prod_{k=1}^{m}\psi\,(a_{n+k},a)\left[1-\frac{\psi\,(z,a)}{\psi\,(a_{n+k},a)}\right]$$

for all $z\in L\,(a,r'')\setminus\bar{L}\,(a,r')$. Therefore,

$$\ln|f\,(z)|=\ln|g\,(z)|+\sum_{k=1}^{m}\ln\left[|\psi\,(a_{n+k},a)|+\left|1-\frac{\psi\,(z,a)}{\psi\,(a_{n+k},a)}\right|\right]=\ln|g\,(z)|+$$
$$+\sum_{k=1}^{m}\ln r_{n+k}+\sum_{k=1}^{m}\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|=\ln|g\,(z)|+m\ln r_{n+1}+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|,\ \ 0\leqslant t\leqslant 2\pi.$$

From here,

$$\ln|f\,(z)|=\ln|g\,(z)|+m\ln r_{n+1}+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|=\eta\,(z)+m\ln\left|1-\frac{r}{r_{n+1}}e^{it}\right|,$$

where

$$\eta(z) = \ln|g(z)| + m \ln r_{n+1}$$

is continuous in a neighborhood of $r' < r < r''$. Now it is sufficient to prove that the integral

$$I(r) = \int_0^{2\pi} \ln\left|1 - \frac{r}{r_n} e^{it}\right| dt, \quad I(r_n) = 0,$$

is continuous at the point $r = r_{n+1}$. For

$$\frac{r}{r_{n+1}} \geqslant \left|1 - \frac{r}{r_{n+1}} e^{it}\right|^2 = 1 - 2\frac{r}{r_{n+1}} \cos t + \frac{r^2}{r_{n+1}^2} = \sin^2 t + \left(\cos t - \frac{r}{r_{n+1}}\right)^2 \geqslant \sin^2 t.$$

Hence, for fixed $\varepsilon > 0$, $\delta \in (0, \pi)$ we have

$$I(r) - I(r_{n+1}) = I(r) = \int_0^{2\pi} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt =$$
$$= \int_{-\delta}^{\delta} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt + \int_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt.$$

$$\left|\int_{-\delta}^{\delta} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt\right| < \int_{-\delta}^{\delta} (\ln 3 + |\ln|\sin t||)\, dt < \int_{-\delta}^{\delta} (\ln 3 + |\ln|t||)\, dt <$$
$$< (2 + \ln 9)\,\delta + 2\delta \ln\frac{1}{\delta} < (4 + \ln 9)\,\delta \ln\frac{1}{\delta}.$$

We fix δ so small that the right side is smaller than $\frac{\varepsilon}{2}$. The integral

$$\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt$$

is continuous at the point $r = r_n$. Therefore, for $r \to r_{n+1}$ we have

$$\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt \to \int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r_n}{r_{n+1}} e^{it}\right| dt = 0$$

and we get that for sufficiently close r to r_{n+1} the integral

$$\left|\int\limits_{[0,2\pi]\setminus[-\delta,+\delta]} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt\right| < \frac{\varepsilon}{2}.$$

Hence, $|I(r) - I(r_{n+1})| < \varepsilon$ i.e. $I(r) \to I(r_{n+1})$ for $r \to r_{n+1}$ and the integral

$$\int_0^{2\pi} \ln\left|1 - \frac{r}{r_{n+1}} e^{it}\right| dt$$

is continuous at the point $r = r_{n+1}$. □

4. Properties of the Blaschke product for $A(z)$-analytic functions

If $0 < |\psi(a_n, a)| < R$, $n = 1, 2, 3, \ldots$, and an infinite product

$$\prod_{n=1}^{\infty} R \cdot \frac{|\psi(a_n, a)|}{\psi(a_n, a)} \frac{\psi(a_n, a) - \psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)} \tag{11}$$

converges uniformly inside $\{|\psi(z, a)| < R\} \setminus \{a_n\}$, then it represents some $A(z)$-analytic in the lemniscate $L(a, R)$ function $B(z)$. It is called the Blaschke product. One can admit a finite number of zeros in the lemniscate $L(a, R)$. In this case, the number of factors in (11) will be finite.

Now we study the convergence of the Blaschke product (11).We have

$$R\frac{|\psi(a_n, a)|}{\psi(a_n, a)} \frac{\psi(a_n, a) - \psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)} = R\left[|\psi(a_n, a)| \frac{1 - \frac{\psi(z,a)}{\psi(a_n,a)}}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)}\right] =$$

$$= R\,\frac{1}{R^2}\left[|\psi(a_n, a)| + \frac{\left(\overline{\psi}(a_n, a) - \frac{R^2}{\psi(a_n,a)}\right)|\psi(a_n, a)|\,\psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)}\right] =$$

$$= \frac{1}{R}\left[|\psi(a_n, a)| + \frac{|\psi(a_n, a)|^2 - R^2}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)} \frac{|\psi(a_n, a)|\,\psi(z, a)}{\psi(a_n, a)}\right].$$

Here

$$R\frac{|\psi(a_n, a)|}{\psi(a_n, a)} \frac{\psi(a_n, a) - \psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)} =$$

$$= \frac{1}{R}\left\{R + (|\psi(a_n, a)| - R)\left\{1 + \frac{(|\psi(a_n, a)| + R)\,|\psi(a_n, a)|}{\psi(a_n, a)\left[R^2 - \overline{\psi}(a_n, a)\psi(z, a)\right]}\psi(z, a)\right\}\right\}.$$

Therefore, the considered infinite product converges uniformly inside $\{|\psi(z, a)| < R\} \setminus \{a_n\}$ if and only if

$$\sum_{n=1}^{\infty} (R - |\psi(a_n, a)|) < \infty\,.$$

Note that

$$\left|R\frac{\psi(a_n, a) - \psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)}\right|^2 = \frac{|\psi(a_n, z)|^2}{|\psi(a_n, z)|^2 + |R - \psi(a_n, a)|^2 + |R - \psi(a_n, a)|^2} \leqslant 1 \ \forall z \in L(a, R)\,.$$

Under the condition

$$\sum_{n=1}^{\infty} (R - |\psi(a_n, a)|) < \infty\,,$$

the $A(z)$-analytic Blaschke product $B(z)$ in $L(a, R)$ does not exceed 1 in absolute value, i.e., $|B(z)| \leqslant 1$.

Let $\sum\limits_{n=1}^{\infty} (R - |\psi(a_n, a)|) < \infty$, so that

$$\prod_{n=1}^{\infty} R\frac{|\psi(a_n, a)|}{\psi(a_n, a)} \frac{\psi(a_n, a) - \psi(z, a)}{R^2 - \overline{\psi}(a_n, a)\psi(z, a)}$$

converges in $L(a,R)$ and represents the Blaschke product $B(z)$, which is $A(z)$-analytic in $L(a,R)$, $|B(z)| < 1$.

The following assertion implies that at almost all points of the boundary $\partial L(a,R)$ the Blaschke product has radial limits

Lemma 4.1. *If a function $f \in O_A(L(a,R))$ and is bounded in $L(a,R)$, $|f| \leqslant M$, then it has the radial limit $\lim\limits_{z\to\xi\in\partial L(a,R)} f(z)$ almost everywhere on $\partial L(a,R)$.*

Proof. We expand the function $f(z)$ into a series: $f(z) = \sum\limits_{n=0}^{\infty} c_n \psi^n(z,a), z \in L(a,R)$ (see [9]). First we show that $\sum\limits_{n=1}^{\infty} |c_n|^2 R^{2n} < \infty$. Setting $\psi(z,a) = re^{it}$, we have

$$|f(z)|^2 = f(z)\overline{f(z)} = \sum_{n=0}^{\infty} c_n r^n e^{int} \sum_{n=0}^{\infty} \overline{c_n} r^n e^{-int} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_j \overline{c_{n-j}} e^{it(2j-n)} \right) r^n, \quad r < R.$$

The series

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} c_j \overline{c_{n-j}} e^{it(2j-n)t} \right) r^n$$

converges uniformly in $[0, 2\pi]$ and integrating it, we get

$$\int\limits_{|\psi(z,a)|=r} |f(z)|^2 \, |dz + A(z)\, d\bar{z}| = \sum_{n=0}^{\infty} |c_n|^2 r^{2n},$$

That is why

$$\sum_{j=0}^{\infty} |c_n|^2 r^{2n} \leqslant M^2.$$

Since this inequality is true for all $r < R$, we have

$$\sum_{n=0}^{\infty} |c_n|^2 R^{2n} \leqslant M^2.$$

According to the Riesz–Fischer theorem, it follows from the condition $\sum\limits_{n=1}^{\infty} |R^n c_n|^2 < \infty$ that $\sum\limits_{n=-\infty}^{\infty} c_n R^n e^{itn} = \varphi(t) \in L_2[0; 2\pi]$ is a Fourier series. So that $\int\limits_{[0;2\pi]} \left| \sum\limits_{n=1}^{\infty} c_n R^n e^{itn} - \varphi(t) \right|^2 dt = 0$. This means that the series is Cesaro summable and converges to $\varphi(t)$ for almost all $t \in [0; 2\pi]$. But then it is Abel summable (see [8, 12]), i.e.

$$\lim_{z\to\xi\in\partial L(a,R)} f(z) = \lim_{r\to R-0} \sum_{n=0}^{\infty} c_n r^n e^{int}$$

for almost all $t \in [0, 2\pi]$. □

The Lemma just proven states that for almost all $\xi \in \partial L(a,R)$ the limit function

$$\lim_{z\to\xi\in\partial L(a,R)} B(z) = B^*(\xi)$$

exists.

Theorem 4.2. $|B^*(z)| \underset{\partial L(a,R)}{\overset{a.e}{=}} 1$ *holds almost everywhere on* $L(a,R)$.

Proof. Without loss of generality, we can assume that all points $a_n \neq a$ (otherwise we would consider the function $B^*(z) = \dfrac{B(z)}{\psi^N(z,a)}$, where N is the order of zero of the function $B(z)$ at the point a). Then $\ln|B(a)| = \sum\limits_{n=1}^{\infty} \ln \dfrac{|\psi(a_n,a)|}{R}$ and the fact that $\sum\limits_{n=1}^{\infty} (R - |\psi(a_n,a)|) < \infty$ implies

$$\sum_{n=1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R} > -\infty.$$

Take $r \in (0;R)$ not equal to any of the values $|\psi(a_n,a)|$. Then, according to the analogue of the Jensen formula

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)=r|} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = \ln|B(a)| - \sum_{|\psi(a_n,a)|<r} \ln \frac{|\psi(a_n,a)|}{r}.$$

Substituting

$$\ln|B(a)| = \sum_{n=1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R},$$

we get

$$\sum_{n=1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R} = \sum_{|\psi(a_n,a)|<r} \ln \frac{|\psi(a_n,a)|}{r} + \frac{1}{2\pi r} \int\limits_{|\psi(z,a)=r|} \ln|B(z)|\,|dz + A(z)\,d\bar{z}|,$$

or

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = \sum_{n=1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R} - \sum_{|\psi(a_n,a)|<r} \ln \frac{|\psi(a_n,a)|}{r}.$$

We fix some number n_0 such that

$$\sum_{n=n_0+1}^{\infty} \ln \frac{|\psi(a_n,a)|}{R} < \varepsilon$$

and take $r < R$ so large that for $n \in \{1, 2, \ldots, n_0\}$ all points of z_n lie in $L(a,r)$. Then from the previous relation we get

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant \sum_{n=1}^{n_0} \ln \frac{|\psi(a_n,a)|}{R} - \sum_{n=1}^{n_0} \ln \frac{|\psi(a_n,a)|}{r} - \varepsilon.$$

From here it follows that

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant -2\varepsilon,$$

if we take $r < R$ close enough to R. Due to the arbitrariness of the number $\varepsilon > 0$, we obtain

$$\varliminf_{r\to R-0} \frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| \geqslant 0. \tag{12}$$

But from the conditions $\lim\limits_{z\to\xi\in\partial L(a,R)} B(z) = B^*(\xi)$ almost everywhere and $\ln|B(z)| \leqslant 0$, $z \in L(a,r)$ according to (12) we get $\frac{1}{2\pi R}\int\limits_{|\psi(z,a)|=R} \ln|B(z)|\,|dz + A(z)\,d\bar{z}| = 0$. This means that $|B^*(z)| \underset{\partial L(a,R)}{\overset{a.e}{=}} 1$. □

Theorem 4.3 (An analogue of Blaschke's theorem)**.** *Let the function* $f(z) \in \mathrm{O}_A(L(a,R))$ *and* $a_1, a_2, a_3, \ldots$ *be the zeros of the function* f *in* $L(a,R)$, $r_n = |\psi(a,a_n)|$. *If*

$$M = \sup_{0<r<R} \frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + Ad\bar{z}| < \infty$$

then

$$\sum_n (R - |\psi(a_n,a)|) < \infty$$

and the Blaschke product

$$B(z) = \prod_n R \cdot \frac{|\psi(a,a_n)|}{\psi(a,a_n)} \frac{\psi(a,a_n) - \psi(z,a)}{R^2 - \overline{\psi}(a,a_n)\psi(z,a)}$$

is $A(z)$*-analytic in* $\{|\psi(z,a)| < R\}$, $f(z) = B(z)\cdot G(z)$, *where the function* $G(z)$ *is* $A(z)$*-analytic and has no zeros at* $\{|\psi(z,a)| < R\}$.

Proof. Without loss of generality, we can assume that $f(a) \neq 0$. Then by the Jensen formula

$$\frac{1}{2\pi r} \int\limits_{|\psi(z,a)|=r} \ln|f(z)|\,|dz + Ad\bar{z}| = \ln \frac{r^n f(a)}{r_1 r_2 \ldots r_n},\ r < R,$$

it follows, that

$$\sum_{|\psi(a_n,a)|<r} \ln\left|\frac{r}{\psi(a_n,a)}\right| \leqslant -\ln|f(a)|\,.$$

Letting r tend to R, we get that

$$\sum_n \ln \frac{R}{|\psi(a_n,a)|} < \infty.$$

Note that the convergence of this series is equivalent to the convergence of the series

$$\sum_n (R - |\psi(a_n,a)|) < \infty.$$

The existence of the Blaschke product $B(z)$ now follows according to Theorem 3.

Finally, if we define a function $G(z)$ in $\{|\psi(z,a)| < R\}$ by the formula $G(z) = \frac{f(z)}{B(z)} \in O_A(L(a,R))$, then $G(z) \neq 0$ and $f(z) = B(z)\cdot G(z)$. □

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Обобщенная теорема Вейерштрасса и произведение Бляшке для $A(z)$-аналитических функций

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Аннотация. Мы рассматриваем $A(z)$-аналитические функции в случае, когда $A(z)$ является антиголоморфной функцией. В статье для $A(z)$-аналитических функций доказаны аналог теоремы Вейерштрасса и аналог теоремы Бляшке.

Ключевые слова: $A(z)$-аналитическая функция, интегральная теорема Коши, теорема Вейерштрасса, теорема Йенсена, теорема Бляшки.