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Strongly Algebraically Closed MV-algebras

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Abstract. The aim of this paper is to fully characterize strongly algebraic closed MV-algebras, extending a result of Lacava. Moreover we provide some computation relating orbit algebras, Wajsberg algebras and Łukasiewicz semirings.

Keywords: MV-algebra, strongly algebraically closed, orbit algebra.

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Introduction

The present paper continues the research started in [15] and [23]. In [23], J. Schmid proved that a distributive lattice is an algebraically closed lattice if and only if it is a Boolean lattice. Also, he shows that any strongly algebraically closed lattice is a complete Boolean lattice. Later, it is proved by the author in [20] that if a complete Boolean lattice is q'-compact, then it is an strongly algebraically closed lattice. We recall from [17–19] that an algebra A is an strongly algebraically closed in a class of algebras, if every set of equations (finite or infinite) with coefficients from A, which is solvable in some algebras of the class of algebras containing A, already has a solution in A. Similarly, the purpose of this paper is to study strongly algebraically closed MV-algebras.

MV-algebras were introduced by C. C. Chang in 1958 to give an algebraic proof of the completeness of Łukasiewicz logic reducing the problem to require the semisimplicity of the Lindenbaum–Tarski algebra. Boolean algebras stand to Boolean logic as MV-algebras stand to Łukasiewicz infinite-valued logic (see [6]).

This paper continues the examination of the structure of MV-algebras. Algebraically closed MV-algebras are studied by Lacava in [15] and [16], where an MV-algebra A is called algebraically closed if every polynomial with coefficients in A having a root in some extension of A has already a root in A. Similarly, we provide a new axiomatization of strongly algebraically closed MV-algebras and prove that an MV-algebra A is an strongly algebraically closed MV-algebra if and only if it is regular, divisible, and equationally compact. We also describe orbit algebras with other algebraic structures as Wajsberg algebras and Łukasiewicz semirings. Recall that Wajsberg algebras are special algebraic structures that naturally arise from Łukasiewicz logic and Łukasiewicz near semirings were introduced by S. Bonzio, I. Chajda, and A. Ledda in [1].

1. Algebraically closed MV-algebras

A structure $(A, \oplus, \ominus, \neg, 0, 1)$ is an MV-algebra iff A satisfies the following equations for all $x, y, z \in A$:

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1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $\neg 0 = 1$;
6. $\neg 1 = 0$;
7. $x \ominus y = \neg(\neg x \oplus \neg y)$;
8. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

By definition following two new operations $\vee$ and $\wedge$ on A, the structure $(A, \vee, \wedge, 0, 1)$ will be a bounded distributive lattice:
$x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \ominus y) \ominus y$.

We recall from [11] that MV-algebras form a variety and the notion of MV-homomorphism is just the particular cases of the corresponding universal algebraic notion. In [21], an algebra A in class of MV-algebras is called an absolute retract in the class of MV-algebras if and only if every embedding $A \hookrightarrow B$ has a left inverse (i.e., there is a homomorphism h of B onto A such that $h(a) = a$ for each $a \in A$), whenever B is in the class of MV-algebras and A is a subalgebra of B, then A is a retract of B. Also, A is equationally compact if and only if every finite subset of set of equations is satisfiable in A, then the set of equations is satisfiable in A.

Recall from [7] that a maximal ideal M of an MV-algebra A is said to have a finite rank n, for some integer $n = 2$ if $A/M \cong L_n$, otherwise one says that M has infinite rank, where $L_n = [0, 1] \cap \mathbb{Z}\dfrac{1}{n-1}$. One should observe that every maximal ideal of a Boolean algebra has finite rank. An MV-algebra A is called regular if for every prime ideal N of its Boolean center, the ideal of A generated by N is a prime ideal of A [4].
By [22], an MV-algebra A is divisible if and only if for any $a \in A - \{0\}$ and integer $n \geqslant 0$ there exist a unique least element $b \in A$ such that $\underbrace{b \oplus b \ldots \oplus b}_{n\text{-}times} = a$ and $a \cdot (\underbrace{\neg b \ominus \neg b \ominus \ldots \ominus \neg b}_{(n-1)\text{-}times}) = b$.

By an equation in an algebra A we mean a formal expression

$$p(a_1, \ldots, a_m, x_1, \ldots, x_n) \approx q(a_1, \ldots, a_m, x_1, \ldots, x_n)$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$, p and q are $(m+n)$-ary terms (in the language of A), the elements $a_1, \ldots, a_m$ belong to A and they are called parameters (or coefficients), and $x_1, \ldots, x_n$ are the unknowns of this equation.

Definition 1.1. *An MV-algebra A is called algebraically closed if every polynomial with coefficients in A having a root in some extension of A has already a root in A.*

Definition 1.2. *An MV-algebra A of the class of MV-algebras is strongly algebraically closed in the class if for every extension B of A in the class and for any system of equations with parameters taken from A, if the system has a solution in B, then it also has a solution in A.*

In [15], Lacava proved an MV-algebra is an algebraically closed if and only if it is regular and divisible. Also, following Schmid [23], if we replace "any system" by "any finite system", then we obtain the concept of an algebraically closed algebra A in the class, which this two definitions are the same. Consequently, we will have that two definition above are same. Now, we provide a new axiomatization of strongly algebraically closed MV-algebras.

Lemma 1.3. *An MV-algebra A is strongly algebraically closed algebra in variety of MV-algebras in V if it is an absolute retract in V.*

Proof. By [5], let A be strongly algebraically closed in V and $B \in V$ be an extension of A. We need to show the existence of a retraction $f : B \to A$. We can assume that A is a proper

subalgebra of B, because the identity map of B would obviously be a retraction $B \to A$ if $A = B$. For each element b of $B\backslash A$, we take an unknown x_b and we define $x_{\neg b} \approx \neg x_b$. For each pair $(a, b) \in B \times B$ of elements such that at least one of a and b is element of $B\backslash A$, we define an equation $E_{\oplus}(a, b)$ according to the following six rules:
(1) – If a is element of A, b is element of $B\backslash A$, and $a \oplus b$ is element of A, then $E_{\oplus}(a, b)$ is $a \oplus x_b \approx a \oplus b$.
(2) – If a is element of $B\backslash A$, b is element of A, and $a \vee b$ is element of A, then $E_{\oplus}(a, b)$ is $x_a \oplus b \approx a \oplus b$.
(3) – If a and b are elements of $B\backslash A$ and $a \oplus b$ is element of A,
then $E_{\oplus}(a, b)$ is $x_a \oplus x_b \approx a \oplus b$.
(4) – If a is element of A, b and $a \oplus b$ are elements of $B\backslash A$, then $E_{\oplus}(a, b)$ is $a \oplus x_b \approx x_{a\oplus b}$.
(5) – If a and $a \oplus b$ are elements of $B\backslash A$ and b is element of A, then $E_{\oplus}(a, b)$ is $x_a \oplus b \approx x_{a\oplus b}$.
(6) – If a, b, and $a \oplus b$ are all elements of $B\backslash A$, then $E_{\oplus}(a, b)$ is $x_a \oplus x_b \approx x_{a\oplus b}$.

Let $\widehat{E}$ be the system of all equations we have defined so far. Clearly, $\widehat{E}$ has a solution in B. Indeed, we can let $x_a := a$ for all elements b of $B\backslash A$ to obtain a solution of $\widehat{E}$. Since we have assumed that A is strongly algebraically closed in V and $\widehat{E}$ also has a solution in A. This allows us to fix a solution of $\widehat{E}$ in A. That is, we can choose an element $u_b \in A$ for each element b of A such that the equations (1)–(6) turn into true equalities when the unknowns x_b, for $b \in B \setminus A$, are replaced by the elements u_b.

Next, consider the map

$$f : B \to A, \text{ defined by } c \mapsto \begin{cases} c & \text{if } c \text{ is an element of } A, \\ u_c & \text{if } c \text{ is a element of } B\backslash A. \end{cases}$$

We claim that f is a retraction. Clearly, f acts identically on A. So we need only to show that f is a MV-homomorphism or $f(0) = 0$, $f(x \oplus y) = f(x) \oplus f(y)$ and $f(\neg x) = \neg f(x)$, for every x, $y \in A$. It suffices to verify that f commutes with $\oplus$. If $a, b \in A$, then $a \oplus b$ is also in A, and we have that $f(a) \oplus f(b) = a \oplus b = f(a \oplus b)$, as required. If, say, $a, a \oplus b \in A$ and $b \in B \setminus A$, then (1) applies and we obtain that $f(a) \oplus f(b) = a \oplus u_b = a \oplus b = f(a \oplus b)$, as required. If $a, b, a \oplus b$ are all elements of A, then we can use (6) to obtain that $f(a) \oplus f(b) = u_a \oplus u_b = u_{a\oplus b} = f(a \oplus b)$, as required. The rest of the cases follow similarly from (3)–(5). Thus, we conclude that f commutes with $\oplus$. Therefor, f is a retraction and A is an absolute retract in V. $\square$

Now, we are in the position to state the main theorem of the paper.

Theorem 1.4. *An MV-algebra A is strongly algebraically closed algebra if and only if it is algebraically closed and equationally compact.*

Proof. Suppose that A is strongly algebraically closed MV-algebra. By Lemma 2.4, A is an absolute retract. Now, we prove that A is regular, divisible, and equationally compact. Notice that Banaschewski–Nelson in [2] and Weglorz in [26] proved that the MV-algebra A is equationally compact if and only if every pure embedding $A \hookrightarrow B$ has a left inverse, see [21]. Since any absolute retract is a pure absolute retract, and here A is equationally compact. Now, to prove that A is regular and divisible it suffices to show that A is algebraically closed MV-algebra. Suppose $i : A \hookrightarrow B$ is an arbitrary extension of A and we prove that i is pure. To do this, for any system $\Sigma(\overline{x})$ of equations with parameters taken from A, if $\Sigma(\overline{b})$ is a solution in extension B of A, then we give $p : B \longrightarrow A$, where p a retraction (left inverse) of i and $\Sigma(p(\overline{x}))$ a solution in A.

Conversely, suppose A is algebraically closed and equationally compact. We know that A is regular and divisible. If $i : A \hookrightarrow B$ is an arbitrary extension of A, then i is pure. On the other hand, A is a pure absolute retract and thus i has a left inverse. Consequently, A is an absolute retract in V.

Now, we have that $A \in V$ is an absolute retract and $B \in V$ is an extension of A, and a system $\widehat{G}$ of equations with constants taken from A has a solution in B.

Let $x, y, z, \ldots$ denote the unknowns occurring in $\widehat{G}$ (possibly, infinitely many), and let $b_x, b_y, b_z, \dots \in B$ form a solution of $\widehat{G}$. Since we have assumed that A is an absolute retract for V, we can take a retraction $f : B \to A$. We define

$$d_x := f(b_x),\ \ d_y := f(b_y),\ \ d_z := f(b_z),\ \ \ldots;$$

they are elements of A. Let

$$p(a_1, \ldots, a_k, x, y, z, \ldots) = q(a_1, \ldots, a_k, x, y, z, \ldots)$$

be one of the equations of $\widehat{G}$; here p and q are MV-algebra terms, the constants $a_1, \ldots, a_k$ are in A, and only finitely many unknowns occur in this equation. Using that f commutes with MV-algebra terms and, at $=^*$, using also that b_x, b_y, b_z, $\ldots$ form a solution of the equation in question, we obtain that

$$\begin{aligned} p(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots) &= p(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) = \\ &= f(p(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) =^* f(q(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) = \\ q(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) &= q(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots). \end{aligned}$$

This shows that $d_x, d_y, d_z, \dots \in A$ form a solution of $\widehat{G}$ in A. Therefore, A is strongly algebraically closed in V. □

We recall from [9] that the ordinary polynomials in the language of MV-algebras are called MV-polynomials and built from variables and function symbols of the language. And as usual, the value of a polynomial is calculated inductively from the value of its variables. In [13], MV-polynomials generalized to DMV-polynomials, which are built from MV-algebra symbols plus a unary function symbol δ_n for every positive integer n and DMV-polynomials have a value in every divisible MV-algebra (not in every MV-algebra, however) and $\mathbf{x}$ and $\mathbf{y}$ are finite vectors of variables. By [9], for every MV-polynomial $f(\mathbf{x},\ \mathbf{y})$ there is a single DMV-polynomial $g_f(\mathbf{y})$ such that [0, 1] verifies the following formula:

$$\varphi_f :\ \forall \mathbf{y}, (\exists \mathbf{x}, f(\mathbf{x}, \mathbf{y}) = 0 \Longleftrightarrow g_f(\mathbf{y}) = 0).$$

Now, we can state the following theorem:

Theorem 1.5. *An MV-algebra A is an algebraically closed MV-algebra if and only if*

1. *A is divisible;*
2. *for every MV-polynomial f, A verifies the formula φ_f;*
3. *A equationally compact.*

Proof. By [22] and Theorem 7 from [8], since A is strongly algebraically closed, is divisible and for every MV-polynomial f, A verifies the formula φ_f. Using Theorem 2.5, A is equationally compact.
Conversely, suppose that A is divisible and A models the formula φ_f. By [8], Theorem 7], A is an algebraically closed algebra. On the other hand, if A is equationally compact and algebraically closed, then A is an absolute retract in V. By Theorem 2.5, we conclude that A is an strongly algebraically closed MV-algebra. □

2. Representation of orbit algebras

We recall that a Boolean algebra $(\mathfrak{B}, \wedge, \vee, \neg, 0, 1)$ with a countable dense subset is called separable, where a subset A of a Boolean algebra B is dense in B if and only if every element of B is a join of a subset of A. Suppose $\mathfrak{B}$ is a separable Boolean algebra and $a, b \in \mathfrak{B}$. We define $a \sim b$ if $b = g(a)$ for some $g \in G$ of group of automorphisms of $\mathfrak{B}$. Furthermore, for $g \in Aut(\mathfrak{B})$ and $a \in \mathfrak{B}$, we denote by $g\lceil a$ a the restriction of g to the interval $[0, a]$. We define $a \to b = \neg a \vee b$ for $a, b \in \mathfrak{B}$. We furthermore write $a \perp b$ if there are no non-zero $a_0 \leqslant a$ and $b_0 \leqslant b$ such that $a_0 \sim b_0$. In a Boolean algebra $\mathfrak{B}$ with a countable dense subset if G is a group of automorphisms of $\mathfrak{B}$, then we call the pair $(\mathfrak{B}, G)$ a Boolean ambiguity algebra.

We recall from [25] that $(\mathfrak{B}, G)$ is a Boolean ambiguity algebra. Assume that the following infima and suprema exists for all $a, b \in \mathfrak{B}$:

$$[a] \odot [b] = \inf\{[a' \wedge b'] : a' \sim a,\ b' \sim b\},$$
$$[a] \to [b] = \inf\{[a' \to b'] : a' \sim a,\ b' \sim b\},$$

where $[a]$ and $[b]$ are equivalence classes with respect to $\sim$. Following [25] we call the structure $(O(\mathfrak{B}, G), \leqslant, \odot, \to, 0, 1)$ the orbit algebra of $(\mathfrak{B}, G)$, where $O(\mathfrak{B}, G)$ is the quotient by equivalence relation $\sim$. Also, we have $\neg[a] = [a] \to 0$ for $a \in \mathfrak{B}$.
A Wajsberg algebra is a structure $(W, \rightharpoondown, ^*, 1)$, where $\rightharpoondown$ is a binary operation, $*$ is a unary operation and 1 is a constant such that the following identities hold:
1. $1 \rightharpoondown a = a$;
2. $(a \rightharpoondown b) \rightharpoondown ((b \rightharpoondown c) \rightharpoondown (a \rightharpoondown c)) = 1$;
3. $(a \rightharpoondown b) \rightharpoondown b = (b \rightharpoondown a) \rightharpoondown a$;
4. $(a^* \rightharpoondown b^*) \rightharpoondown (b \rightharpoondown a) = 1$,
for all $a, b, c \in W$.
This leads us to state the following theorem.

Theorem 2.1. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \oplus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, ^*, 1)$ is a strongly algebraically closed algebra if it is algebraically closed and equationally compact, where * is an unary operation and the implication $\rightharpoondown$ is defined by $x \rightharpoondown y = x \oplus y$ and $1 = 0^*$, for all $x, y \in O_{(\mathfrak{B},G)}$.*

Proof. First we prove that $(O_{(\mathfrak{B},G)}, \leqslant, \oplus, 1)$ is an ordered monoid. If we suppose that $a' \sim a$ and $b' \sim b$ such that $[a] \oplus [b] = [a' \wedge b']$, for all $a, b, c \in \mathfrak{B}$ then we will have

$$([a] \oplus [b]) \oplus [c] = \min\{[d \wedge c'] : d \sim a' \wedge b',\, c' \sim c\} = \min\{[a'' \wedge b'' \wedge c'] : a'' \sim a,\, b'' \sim b,\, c' \sim c\},$$

whence associativity of $\oplus$ follows. Obviously, $\oplus$ is in both arguments isotone. On the other hand, $[a] \oplus [b] \leqslant [c]$ if and only if $[a] \leqslant [b] \to [c]$. Now, for any $a, b \in \mathfrak{B}$, we can obtain $ab' \sim b$ such that $[a \wedge b'] = [a] \wedge [b]$ and $[a \vee b'] = [a] \vee [b]$. On the other hand, $\neg(a' \to b') \leqslant a'$ and then $[a] \to [b] = [a' \to b']$. Finally, $[a] \oplus ([a] \to [b]) = [a'] \oplus [a' \to b'] = [a \wedge (a' \to b')] = [a' \wedge b'] = [a] \wedge [b]$. Therefore, it is divisible and $\neg[a] = [\neg a]$ for any $a \in \mathfrak{B}$; so $\neg$ is involutive. Therefor, $(O_{(\mathfrak{B},G)}, \oplus, \to, 0)$ is an MV-algebra. Using Theorem 1.4, completes the poof. □

By [1], we have that if $(O_{(\mathfrak{B},G)}, \oplus, \to, 0)$ is an MV-algebra, then

$$(O_{(\mathfrak{B},G)}, \rightharpoondown, *, 1)$$

is a Wajsberg algebra, where $a \rightharpoondown b = \neg a \oplus b$, for any $a, b \in O_{(\mathfrak{B},G)}$ and $1 = 0^*$. Thus we will have the following corollary:

Corollary 2.2. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \oplus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, *, 1)$ is a Wajsberg algebra.*

We recall from [14], a BE-algebra we shall mean an algebra $(X, ^*, 1)$ of type $(2, 0)$ satisfying the following axioms:
1. $x * x = 1$;
2. $1 * x = x$;
3. $x * (y * z) = y * (x * z)$;
4. $x * 1 = 1$,
for all $x, y, z \in X$.

A BE-algebra $(X, ^*, 1)$ is called bounded if there exists the smallest element 0 of X (i.e. $0 * x = 1$, for all $x \in X$). Recall that Imai and Iski (1966) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. A BCI-algebra is a non-empty set X endowed with a binary operation $*$ and a constant 0 satisfies the following axioms, for all $x, y, z \in X$:
1. $((x * y) * (x * z)) * (z * y) = 0$;
2. $x * 0 = x$;
3. $x * y = 0$ and $y * x = 0$ imply that $x = y$.
Every BCI-algebra satisfying $0 * x = 0$ for all $x \in X$ is a BCK-algebra. We recall from [27] that an algebra $(X, ^*, 1)$ of type $(2, 0)$ is called a dual BCK-algebra (or briefly, DBCK-algebra) if
1. $x * x = 1$;
2. $x * 1 = 1$;
3. $x * y = y * x \Longrightarrow x = y$;
4. $(x * y) * ((y * z) * (x * z)) = 1$;
5. $x * ((x * y) * y) = 1$ for all $x, y, z \in X$.

We now study the relations between BE-algebras and Łukasiewicz semirings.

Theorem 2.3. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, *, 1, 0)$ is a bounded commutative BE-algebra, where $x*y = \neg x \odot y$, for all $x, y \in \mathfrak{B}$ and $1 = \neg 0$.*

Proof. We claim the structure $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ is equivalent to a bounded commutative BE-algebra. By [27], an MV-algebra $(O_{(\mathfrak{B},G)}, \odot, \neg, o)$ is a bounded commutative dual BCK-algebra $(O_{(\mathfrak{B},G)}, ^*, 1, 0)$ with the operation $*$ and the top element 1 defined as follows:
$x * y = \neg x \odot y,\ 1 = \neg 0$, for $x, y \in O_{(\mathfrak{B},G)}$. [14], any DBCK-algebra is a bounded commutative BE-algebra. □

BL-algebras were introduced by Hajek [12] as algebraic structures of basic logic, where a *BL-algebra* is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that:
(i) $(A, \vee, \wedge, 0, 1)$is a bounded lattice;
(ii) $(A, \odot, 1)$ is a commutative monoid;
(iii) the following statements hold for every $x, y, z \in A$:
(a) $z \leqslant x \rightarrow y$ iff $x \odot z \leqslant y$;
(b) $x \wedge y = x \odot (x \rightarrow y)$;
(c) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Recall that from [24] and [9] that a BL-algebra A to be an RS-BL-algebra if, for all elements $a \in A$ holds

$$\{x \in A, a \longrightarrow x = x\} = \{x \in A \,|\, x \longrightarrow a = a\}.$$

Theorem 2.4. *If $O_{(\mathfrak{B},G)}$ is a RS-BL-algebra, then $(O_{(\mathfrak{B},G)}, \neg, ^*, 0)$ is a Wajsberg algebra.*

Proof. Since MV-algebras are such BL-algebras that $(a \rightarrow x) \rightarrow x = (x \rightarrow a) \rightarrow a$ holds for all $x, a \in O_{(\mathfrak{B},G)}$, it is an easy task to show that MV-algebras are RS-BLalgebras. To show that MV-algebras re the only BL-algebras that are RS-BL-algebras, observe first that the following holds in all RS-BL-algebras if $x^* = 0$, then $x = 1$, where $x^* = x \rightarrow 0$. Consequently, by [24] and [19] RS-BL-algebras are equivalent to MV-algebras. By Theorem 2.1, this completes the proof. □

We recall from [8] that a semiring $(R, +, 0, \cdot, 1)$ is an algebraic structure where 0 and 1 are distinct elements of R, $+$ and $\cdot$ are binary operations on R satisfying:
(i) $(R, +)$ is a commutative monoid with identity 0;
(ii) $(R, \cdot)$ is a monoid with identity 1;
(iii) Multiplication distributes over addition;
(iv) $0 \cdot r = r \cdot 0 = 0$, for every $r \in R$.

Also, by [8], a semiring $(R, +, 0, \cdot, 1)$ is called lattice-ordered semiring iff it has the structure of a lattice such that for all $a, b \in R$:
(i) $a + b = a \vee b$;
(ii) $a \cdot b \leqslant a \wedge b$.
Groupoids were introduced by Brandt in his 1926 paper [3] and semilattices can be equivalently presented as ordered sets as well as groupoids. An algebra is a structure (A, F) where A is an arbitrary non-empty set and F is a system of operations. A type of algebra is a mapping from F to $\mathbb{N}$ (natural numbers including zero) which maps any $f \in F$ to its arity. An algebra $(S, \cdot)$ of type $\langle 2 \rangle$ is called a groupoid.
In closing this section, we mention that the Łukasiewicz semirings are also closely related with the lattice ordered semirings. We recall from [1] that a near semiring is an algebra $(R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that:
(i) $(R, +, 0)$ is a commutative monoid;
(ii) $(R, \cdot, 1)$ is a groupoid satisfying $x \cdot 1 = x = 1 \cdot x$ (a unital groupoid);
(iii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$;
(iv) $x \cdot 0 = 0 \cdot x = 0$;
for all $x, y, z \in R$.

In [1] a near semiring is called a semiring if $(R, \cdot, 1)$ is a monoid and satisfies left distributivity: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, for all $x, y, z \in R$.

A near semiring R is called idempotent if it satisfies $x + x = x$, for all $x \in R$. It is clear that in this case $(R, +)$ is a semilattice. In particular, $(R, +)$ can be considered as a join-semilattice, where the induced order is defined as $x \leqslant y$ iff $x + y = y$ and the constant 0 is the least element [1, Remark 1].

Following [1], a map α of an idempotent near semiring, with $\leqslant$ the induced order, $(R, +, \cdot, 0, 1)$ to $(R, +, \cdot, 0, 1)$ is called an *involution* on R if it satisfies the following conditions, for each $x, y \in R$:
(1) $\alpha(\alpha(x)) = x$;
(2) if $x \leqslant y$ then $\alpha(y) \leqslant \alpha(x)$.

As is defined in [1], an involutive near semiring R is said a Łukasiewicz near semiring if it satisfies the following additional identity:

$$\alpha(x \cdot \alpha(y)) \cdot \alpha(y) = \alpha(y \cdot \alpha(x)) \cdot \alpha(x).$$

A Łukasiewicz semiring A is a Łukasiewicz near semiring such that the reduct $(A, \cdot, 1)$ is a monoid.

Recall that if $(\mathfrak{B}, G)$ is a complete Boolean ambiguity algebra, then $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, 0)$ is an MV-algebra. On the other hand, the reducts $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ is an lc-semiring, where $x \vee y = x \odot (\neg x \ominus y)$, for every $x, y \in O_{(\mathfrak{B},G)}$. In following corollaries we shall use the name lc-semiring for lattice ordered commutative semiring:

Corollary 2.5. *Let $(\mathfrak{B}, G)$ be a complete Boolean ambiguity algebra and $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, o)$ be the orbit algebra. Then $(O_{(\mathfrak{B},G)}, \rightharpoondown, {}^*, 1)$ and $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ are MV-algebra and lc-semiring, where * is an unary operation and the implication $\rightharpoondown$ is defined by $x \rightharpoondown y = x \oplus y$, $x \vee y = x \odot (\neg x \ominus y)$, and $1 = 0^*$, for all $x, y \in O_{(\mathfrak{B},G)}$.*

Corollary 2.6. *Let* $(\mathfrak{B}, G)$ *be a complete Boolean ambiguity algebra and* $(O_{(\mathfrak{B},G)}, \odot, \ominus, \neg, o)$ *be the orbit algebra. Then* $(O_{(\mathfrak{B},G)}, \rightharpoondown, ^*, 1)$ *and* $(O_{(\mathfrak{B},G)}, \vee, 0, \ominus, 1)$ *are strongly algebraically closed algebra if they are algebraically closed and equationally compact, where* * *is an unary operation and the implication* $\rightharpoondown$ *is defined by* $x \rightharpoondown y = x \oplus y$, $x \vee y = x \odot (\neg x \ominus y)$, *and* $1 = 0^*$, *for all* $x, y \in O_{(\mathfrak{B},G)}$.

Example 2.7. *Let* R *denote the set of real numbers and let* Q *denote the set of rational numbers. For any* $n \in \omega$, $n \geqslant 1$ *we define* $L_{n+1} = \{0, 1/n, \ldots, (n-1)/n, 1\}$. *If* a *and* b *are real numbers we define* $a \odot b = \min(a+b, 1)$, *and* $\neg a = 1-a$. *Suppose* A *is* $(Q \cap [0, 1], \odot, \neg, 0)$ *or* $(L_{n+1}, \odot, \neg, 0)$, *where they are MV-algebras. If* $\mathfrak{B}(A)$ *denotes its* R*-generated Boolean algebra and* $G(A)$ *is a subgroup of the automorphism group of* $\mathfrak{B}(A)$, *it turns out that* $(B(A), G(A))$ *forms an MV-pair. Independently, a similar study of certain type of* $(\mathfrak{B}, G)$*-pairs which yield an MV-algebra, so called ambiguity algebras.*

Conclusions

Lacava in [16] proved that an MV-algebra is algebraically closed if and only if it is regular and divisible. So, gathering up the theorems in Section 1, we obtain a representation of strongly algebraically closed MV-algebras as regular, divisible, and equationally compact. Therefore our results give further tools which can be suitable for Łukasiewicz logic and this can be the starting point to develop a sort of Algebraic Geometry based on MV-algebras.

References

[1] S.Bonzio, I.Chajda, A.Ledda, Representing quantum structures as near semirings, *Logic Journal of IGPL*, **24**(2016), 1–24. DOI :10.1093/jigpal/jzw031

[2] B.Banaschewski, E.Nelson, Equational compactness in equational classes of algebras, *Algebra Universalis*, **2**(1972), 152–165.

[3] H.Brandt, Uber eine Verallgemeinerung der Gruppen-begriffes, *Math. Ann.*, **96**(1926), 360–366.

[4] L.P.Belluce, A.Di Nola, S.Sessa, The prime spectrum of an MV-algebra, *Math. Log. Qart.*, **40**(1994), 331–346.

[5] G.Czedli, A.Molkhasi, Absolute retracts for finite distributive lattices and slim semimodular lattices, Order, 2022.

[6] C.C.Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88**(1958), 467-490.

[7] R.Cignoli, D.Mundici, Stone duality for Dedekind s-complete ℓ-groups with orderunit, *J. of Algebra*, **302**(2006), 848–861.

[8] A.Di Nola, B.Gerla, Algebras of Łukasiewicz's logic and their semiring reducts, Idempotent mathematics and mathematical physics, *Contemp. Math.*, **377**(2005), Amer. Math. Soc., Providence, RI, 131–144. DOI: 10.1090/conm/377/06988

[9] A.Di Nola, G.Georgescu, A.Iorgulescu, Pseudo-BL algebras: Part II, *Mult. Val. Logic*, **8**(2002), 717–750.

[10] A.Di Nola, A.R.Ferraioli, G.Lenzi, Algebraically closed MV-algebras and their sheaf representation, *Annals of Pure and Applied Logic*, **164**(2013), 349–355. DOI: 10.1016/j.apal.2012.10.017

[11] B.Davey, Sheaf spaces and sheaves of universal algebras, *Math. Z.*, **134**(1973), no. 4, 275–290.

[12] P.Hajek, Metamathematics of fuzzy logic (Trends in Logic, Studia Logica Library), Kluwer, Dordrecht, 1998.

[13] B.Gerla, Rational Łukasiewicz logic and DMV-algebras, *Neural Netw. World*, **6**(2001), 579–594.

[14] K.H.Kim Y.H.Yon, Dual BCK-algebra and MV-algebra, On algebras with a generalized implication, *Sci. Math. Jpn.*, (2007), 393–399.

[15] F.Lacava, Sulle classi delle ℓ-algebre e degli ℓ-gruppi abeliani algebricamente chiusi, *Boll. Un. Mat. Ital. B*, **1**(1987), no. 7, 703–712.

[16] F.Lacava, Algebra di Łukasiewicz quasi-locali, *Boll. Un. Mat. Ital. 1-B*, (1997), no. 7, 961–972.

[17] A.Molkhasi, On some strongly algebraically closed semirings, *Journal of Intelligent and Fuzzy Systems*, **36**(2019), no. 6, 6393–6400. DOI: 10.3233/JIFS-182661

[18] A.Molkhasi, K.P.Shum, Strongly algebraically closed orthomodular near semirings, *Rend. Circ. Mat. Palermo, II*, **69**(2020), 803–812. DOI: 10.1007/s12215-019-00434-z

[19] S.Motamed, L.Torkzadeh, A new class of BL-algebras, *Soft Comput*, **21**(2017), 686–693. DOI: 10.1007/s00500-016-2043-z

[20] A.Molkhasi, On strongly algebraically closed lattices, *J. Sib. Fed. Univ. Math. Phys.*, **9**(2016), no. 2, 202–208.

[21] P.Ouwehand, H.Rose, Small congruence distributive varieties: retracts, injectives, equational compactness and amalgamation, *Periodica Math. Hung.*, **33**(1996), 207–228.

[22] P.Ptak, S.Pulmannova, Orthomodular structures as quantum logics, Kluwer Academic Publishers, 1991.

[23] J.Schmid, Algebraically and existentially closed distributive lattices, *Zeitschr Math. Logik U. G. M.*, **25**(1979), 525–530.

[24] E.Turunen, RS-BL-algebras are MV-algebras, *Iranian Journal of Fuzzy System*, **6**(2016), 153–154. DOI: 10.22111/IJFS.2016.2826

[25] T.Vetterlein, Boolean algebras with an automorphism group: a framework for Łukasiewicz logic, *Soft Comput.*, **10**(2006), 995–1000.

[26] B.Weglorz, Equationally compact algebras, *I. Fund. Math.*, **59**(1966), 289–298.

[27] A.Walendziak, On commutative BE-algebras, *Scientiae Mathematicae Japonicae Online*, (e-2008), 585–588.

Сильно алгебраически замкнутые MV-алгебры

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Аннотация. Цель этой статьи — полностью охарактеризовать сильно алгебраические замкнутые MV-алгебры, обобщая результат Лакавы. Кроме того, мы приводим некоторые вычисления, связанные с алгебрами орбит, алгебрами Вайсберга и полукольцами (Лукашевич). **Ключевые слова:** MV-алгебра, сильно алгебраически замкнутая, алгебра орбит.