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On a Note on Apéry-like Series with an Application

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Abstract. The goal of this note is to use a hypergeometric series strategy to build many Apéry-like series. As an application, we obtain several results due to Sherman.

Keywords: Apéry-like series, factorials, hypergeometric function, summation formulas, Gauss summation theorem, contiguous results, binomial coefficients, combinatorial sums.

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1. Introduction and preliminaries

The following standard notations will be used throughout the paper:

$$\mathbb{N} := \{1, 2, 3, \dots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [12, p. 73, Eqn.(2)]

$$ {}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} . \frac{z^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the well-known Pochhammer's symbol (or the shifted or the raised factorial since $(1)_n = n!$) defined for any complex number $a(\neq 0)$ by

$$(a)_n = \begin{cases} a(a+1)\dots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases} . \tag{2}$$

Using the fundamental relation $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3}$$

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where Γ is the well known Gamma function.

For more details about this function, and its convergence conditions (including absolute convergence), we refer standard texts [12, 14].

It's worth noting that anytime a generalized hypergeometric function reduces to the gamma function, the results are crucial from the standpoint of applications. Thus, classical summation theorems like as those of Gauss, Gauss second, Kummer, and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple, and Saalschütz for the series ${}_3F_2$, and others, are relevant.

During 1992–2011, the classical summation theorems listed above have been extended and generalised to their most general form. For this we refer interesting research papers by Lavoie et al. [6–8], Kim et al. [5] and Rakha and Rathie [13].

The following summation formula for the series ${}_2F_1$ which can be obtained from a very general summation formula established earlier by Rakha and Rathie [13, Theorem 2 (for $i = 2$), p. 828] is required in our current inquiry.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b-1) \end{matrix} ; \frac{1}{2}\right] = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}-\frac{1}{2}\right)\left[\frac{\frac{1}{2}(a+b-1)}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})} + \frac{2}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\right]. \tag{4} $$

The result (4) is seen to be closely related to the following well-known and useful Gauss's second summation theorem [12, p. 69, Ex. 2; 14, p. 243, Eqn. (III.6)] viz.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1) \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2}+\frac{b}{2}+\frac{1}{2})}{\Gamma(\frac{a}{2}+\frac{1}{2})\Gamma(\frac{b}{2}+\frac{1}{2})}. \tag{5} $$

On the other hand, in 1979, Apéry [1] proved irrationality of $\zeta(3)$ and in the same manner, the irrationality of $\zeta(2)$ by making use of the following well-known identities viz,

$$ \zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n!)^2}{(2n)!\; n^3} $$

and

$$ \zeta(2) = 3\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!\; n^2}. $$

Also, following Apéry's proof, a large number of a similar series

$$ \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}f(n) = \sum_{n=0}^{\infty}\frac{f(n)}{\binom{2n}{n}} $$

which was commonly referred to as Apéry-like series have been studied by van der Poortan [11], Leschiner [10], Lehmer [9], Zucker [16] and Borwein et al. [3]. Berndt and Joshi [2], in a review of chapter 9 of Ramanujan's second notebook have also recorded many of such similar formulas. In addition to this, if we denote

$$ S_k = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}n^k\; 2^n, \tag{6} $$

then in the year 2000, Sherman [15] establishedthe results S_k for $k = 0, 1, 2, \ldots, 10$ given here in the Tab. 1.

On the other hand, it is not difficult to see that

$$ n^k = \sum_{r=0}^{k}(-1)^r\left\{\begin{matrix} k \\ r \end{matrix}\right\}(-n)_r $$

Table 1. For S_k

k	0	1	2	3	4	5	6
S_k	$\frac{\pi}{2}+2$	$\pi+3$	$\frac{7\pi}{2}+11$	$\frac{35\pi}{2}+55$	$113\pi+355$	$\frac{1787\pi}{2}+2807$	$\frac{16717\pi}{2}+26259$

k	7	8	9	10
S_k	$90280\pi+283623$	$\frac{2211181\pi}{2}+34733315$	$\frac{30273047\pi}{2}+47552791$	$229093376\pi+719718067$

where $\left\{ {k \atop r} \right\}$ denotes the well-known Sterling numbers of the second kind [4] written here in slightly modified form as:

$$\left\{ {k \atop r} \right\} = \frac{1}{r!}\sum_{i=0}^{r}(-1)^r\binom{r}{i}(r-i)^k.$$

Thus this note aims to offer closed expressions for Apéry-like series of the form

$$\sum_{n=k}^{\infty}\frac{(n!)^3 2^n}{(2n)!(n-k)!}$$

for $k = 1, 2, \ldots, 10$ via a hypergeometric series approach. As an application, we recover the above results of Apéry-like series obtained earlier by Sherman [15].

2. Main results

In this section, we shall establish the results asserted in the following theorem.

Theorem 2.1. *For $k \in \mathbb{N}_0$, the following general result holds true.*

$$\sum_{n=k}^{\infty}\frac{(n!)^3 2^n}{(2n)!(n-k)!} = 2^{-k}\pi\Gamma^2(k+1)\left[\frac{k+\frac{1}{2}}{\Gamma^2(\frac{1}{2}k+1)} + \frac{2}{\Gamma^2(\frac{1}{2}k+\frac{1}{2})}\right]. \tag{7}$$

Proof. In order to establish the result (7) asserted in the Theorem 2.1, we proceed as follows. Denoting the left hand side of (7) by Δ_k, we have

$$\Delta_k = \sum_{n=k}^{\infty}\frac{(n!)^3\ 2^n}{(2n)!(n-k)!}.$$

Replacing n by n+k, we have

$$\Delta_k = \sum_{n=0}^{\infty}\frac{((n+k)!)^3\ 2^{n+k}}{(2n+2k)!\ n!}.$$

But it is easy to see that $(n+k)! = \Gamma(n+k+1) = \Gamma(k+1)\frac{\Gamma(n+k+1)}{\Gamma(k+1)} = \Gamma(k+1)(k+1)_n$ (using (3)) and using Duplication formula for the gamma function

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\ \Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we see that

$$\begin{aligned}(2n+2k)! &= \Gamma(2n+2k+1) = \\ &= 2^{2n+2k}\pi^{-\frac{1}{2}}\Gamma\Big(n+k+\frac{1}{2}\Big)\Gamma(n+k+1) = \\ &= 2^{2n+2k}\pi^{-\frac{1}{2}}\Gamma\Big(k+\frac{1}{2}\Big)\Gamma(k+1)\Big(k+\frac{1}{2}\Big)_n(k+1)_n \quad (using(3)).\end{aligned}$$

Thus we have after some algebra

$$\Delta_k = \frac{2^{-k}\pi^{\frac{1}{2}}\Gamma^2(k+1)}{\Gamma(k+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{(k+1)_n(k+1)_n}{2^n(k+\frac{1}{2})_n\ n!}.$$

Summing up the series using (1), we have

$$\Delta_k = \frac{2^{-k}\pi^{\frac{1}{2}}\Gamma^2(k+1)}{\Gamma(k+\frac{1}{2})}\ {}_2F_1\left[\begin{matrix} k+1, & k+1 \\ & k+\frac{1}{2}\end{matrix};\frac{1}{2}\right].$$

We now observe that the ${}_2F_1$ appearing can be evaluated with the help of the result (4) by letting $a = b = k+1$, and we easily arrive at the right hand side of (7). This completes the proof of the result (7) asserted in the Theorem 2.1. □

3. Corollaries

In this section, we shall provide several interesting special cases of our main result asserted in the Theorem 2.1 since

$$\Delta_k = \sum_{n=k}^{\infty}\frac{(n!)^3 2^n}{(2n)!(n-k)!} = \sum_{n=k}^{\infty}\frac{(n!)^2 2^n}{(2n)!}(n-k+1)_k. \tag{8}$$

Fortunately, the results Δ_k for k= 0 and 1, we get the same results $\Delta_0 = S_0$ and $\Delta_1 = S_1$ due to Sherma [15] recorded in Section 1. The results Δ_k for k = 2 to 10 are recorded in the Tab. 2.

Table 2. For Δ_k

k	2	3	4	5	6
Δ_k	$\frac{5\pi}{2}+8$	$9\pi+28$	$\frac{81\pi}{2}+128$	$225\pi+704$	$\frac{2925\pi}{2}+4608$

k	7	8	9	10
Δ_k	$11025\pi+34560$	$\frac{187425\pi}{2}+294912$	$893025\pi+2801664$	$\frac{18753525\pi}{2}+29491200$

Application of these results will be given in the next section.

4. Application

As an application of our newly obtained results given in section 3, in this section, we shall obtain the results given in the table S_k.

(a) Derivation of the result S_k for $k = 0$.

Denoting the left-hand side of the series given in (6) for $k = 0$ by S_0 and converting the factorials into the Pochhammer symbols, we have

$$S_0 = \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(\frac{1}{2})_n 2^n n!}.$$

Summing up the series we have

$$S_0 = {}_2F_1\left[\begin{matrix} 1, & 1 \\ & \frac{1}{2} \end{matrix} ; \frac{1}{2}\right].$$

This may be evaluated using the result (4) by letting $a = b = 1$, and we get the right-hand side of (6) for $k = 0$ right away.

(b) Derivation of the result S_k for $k = 1$.

Denoting the left-hand side of the series given in (6) for $k = 1$ by S_1, we have

$$S_1 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n 2^n.$$

Setting $n = m + 1$ and proceeding as above, we have

$$S_1 = {}_2F_1\left[\begin{matrix} 2, & 2 \\ & \frac{3}{2} \end{matrix} ; \frac{1}{2}\right].$$

The result follows by using the result (4) by letting $a = b = 2$

(c) Derivation of the result S_k for $k = 2$.

Denoting the left-hand side of the series given in (6) for $k = 2$ by S_2, we have

$$S_2 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n^2\ 2^n.$$

Expressing $n^2 = n(n-1) + n$ and separating into two series, we get

$$S_2 = \sum_{n=2}^{\infty} \frac{(n!)^2}{(2n)!} (n-1)_2\ 2^n + \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} n\ 2^n.$$

Finally, using the result given in the table Δ_k for $k = 2$ and S_k for $k = 1$, we get at once the right-hand side of S_k for $k = 2$.

In exactly the same manner, the results S_k for $k = 3, 4, \ldots, 10$ can be proven on similar lines by using the result (7) for $k = 3, 4, \ldots, 10$ and taking appropriate results of Δ_k given in the tabular form in Section 3 together with the result S_2 given in the tabular form in Section 2. So we left as an exercise to the interested reader.

5. Hypergeometric series representations of the result given in the equations (7) and (8)

It is interesting to mention here that the results given in the equations (7) and (8) can also be written in terms of generalized hypergeometric series. These are

$${}_2F_1\left[\begin{matrix} 1, & 1 \\ & \frac{1}{2} \end{matrix} ; \frac{1}{2}\right] = \frac{\pi}{2} + 2 \qquad (9)$$

$$ {}_2F_1\left[\begin{matrix} 2, & 2 \\ & \frac{3}{2} \end{matrix}; \frac{1}{2}\right] = \pi + 3 \tag{10}$$

$$ {}_3F_2\left[\begin{matrix} 2,2,2 \\ \frac{3}{2},1 \end{matrix}; \frac{1}{2}\right] = \frac{7\pi}{2} + 11 \tag{11}$$

$$ {}_4F_3\left[\begin{matrix} 2,2,2,2 \\ \frac{3}{2},1,1 \end{matrix}; \frac{1}{2}\right] = \frac{35\pi}{2} + 55 \tag{12}$$

$$ {}_5F_4\left[\begin{matrix} 2,2,2,2,2 \\ \frac{3}{2},1,1,1 \end{matrix}; \frac{1}{2}\right] = 113\pi + 355 \tag{13}$$

$$ {}_6F_5\left[\begin{matrix} 2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1 \end{matrix}; \frac{1}{2}\right] = \frac{1787\pi}{2} + 2807 \tag{14}$$

$$ {}_7F_6\left[\begin{matrix} 2,2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1,1 \end{matrix}; \frac{1}{2}\right] = \frac{16717\pi}{2} + 26259 \tag{15}$$

$$ {}_8F_7\left[\begin{matrix} 2,2,2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1,1,1 \end{matrix}; \frac{1}{2}\right] = 90280\pi + 283623 \tag{16}$$

$$ {}_9F_8\left[\begin{matrix} 2,2,2,2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1,1,1,1 \end{matrix}; \frac{1}{2}\right] = \frac{2211181\pi}{2} + 34733315 \tag{17}$$

$$ {}_{10}F_9\left[\begin{matrix} 2,2,2,2,2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1,1,1,1,1 \end{matrix}; \frac{1}{2}\right] = \frac{30273047\pi}{2} + 47552791 \tag{18}$$

$$ {}_{11}F_{10}\left[\begin{matrix} 2,2,2,2,2,2,2,2,2,2,2 \\ \frac{3}{2},1,1,1,1,1,1,1,1,1 \end{matrix}; \frac{1}{2}\right] = 229093376\pi + 719718067 \tag{19}$$

$$ {}_2F_1\left[\begin{matrix} 3, & 3 \\ & \frac{5}{2} \end{matrix}; \frac{1}{2}\right] = \frac{3}{4}\left(\frac{5\pi}{2} + 8\right) \tag{20}$$

$$ {}_2F_1\left[\begin{matrix} 4, & 4 \\ & \frac{7}{2} \end{matrix}; \frac{1}{2}\right] = \frac{5}{12}(9\pi + 28) \tag{21}$$

$$ {}_2F_1\left[\begin{matrix} 5, & 5 \\ & \frac{9}{2} \end{matrix}; \frac{1}{2}\right] = \frac{32}{192}\left(\frac{81\pi}{2} + 128\right) \tag{22}$$

$$ {}_2F_1\left[\begin{matrix} 6, & 6 \\ & \frac{11}{2} \end{matrix}; \frac{1}{2}\right] = \frac{21}{320}(225\pi + 704) \tag{23}$$

$$ {}_2F_1\left[\begin{matrix} 7, & 7 \\ & \frac{13}{2} \end{matrix}; \frac{1}{2}\right] = \frac{77}{3840}\left(\frac{2925\pi}{2} + 4608\right) \tag{24}$$

$$ {}_2F_1\left[\begin{matrix} 8, & 8 \\ & \frac{15}{2} \end{matrix}; \frac{1}{2}\right] = \frac{143}{26880}(11025\pi + 34560) \tag{25}$$

$$ {}_2F_1\left[\begin{matrix} 9, & 9 \\ & \frac{17}{2} \end{matrix}; \frac{1}{2}\right] = \frac{45045}{896}\left(\frac{187425\pi}{2} + 294912\right) \tag{26}$$

$$ {}_2F_1\left[\begin{matrix} 10, & 10 \\ & \frac{19}{2} \end{matrix}; \frac{1}{2}\right] = \frac{2431}{9289728}(893025\pi + 2801664) \tag{27}$$

$$ {}_2F_1\left[\begin{matrix} 11, & 11 \\ & \frac{21}{2} \end{matrix}; \frac{1}{2}\right] = \frac{46189}{928972800}\left(\frac{18753525\pi}{2} + 29491200\right) \tag{28}$$

Concluding remark

In this note we have established the closed expressions for the Apéry-like series of the form

$$\sum_{n=k}^{\infty} \frac{(n!)^3\ 2^n}{(2n)!(n-k)!} \tag{*}$$

for $k = 1, 2, \dots, 10$ via a hypergeometric series approach. As an application, we obtained the Apéry-like series of the form

$$\sum_{n=0}^{\infty} \frac{(n!)^2 n^k\ 2^n}{(2n)!} \tag{**}$$

for $k = 1, 2, \dots, 10$ established earlier by Sherman [15].

We conclude this note by remarking that the results (*) and (**) in the most general forms for $k \in \mathbb{N}_0$ are under investigations and will form a part of the subsequent paper in this direction.

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Заметка об Апери-подобном ряде с приложением

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Аннотация. Цель этой заметки — использовать стратегию гипергеометрических рядов для построения многих рядов, подобных Апери. В качестве приложения мы получаем несколько результатов, принадлежащих Шерману.

Ключевые слова: Апери-подобные ряды, факториалы, гипергеометрическая функция, формулы суммирования, теорема суммирования Гаусса, смежные результаты, биномиальные коэффициенты, комбинаторные суммы.