A Logarithmic Barrier Approach Via Majorant Function for Nonlinear Programming

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Abstract. In this paper, we are interested in solving an optimization nonlinear programming problem using a logarithmic barrier interior point method, in which the penalty term is taken as a vector $r \in \mathbb{R}_+^n$. The descent direction has been calculated using a classical Newton method, however the step size has been calculated with a new technique of majorant functions and a secant technique. The numerical simulations show us the efficiency of our approach compared to the classical line search method.

Keywords: nonlinear convex programming, logarithmic penalty method, line search, majorant function, secant technique.

1. Introduction and preliminaries

In this paper, we are interested in the barrier logarithmic penalty method when using a new majorant function technique instead of the classical line search method to determine the step size ([1, 2, 4]).

1.1. The problem formulation

The problem to be studied in this paper is as follows:

$$\begin{cases} \min\ g(x) \\ x \in K \subseteq \mathbb{R}^n \end{cases} . \tag{P1}$$

In which: $K = \{x \in \mathbb{R}^n : \ Bx = c,\ x \geqslant 0\}$ is the set of feasible solution of $(P1)$.

1.1.1. Assumptions

A1 g is nonlinear, convex, twice continuously differentiable function on K.

A2 $B \in \mathbb{R}^{m \times n}$ is a full rank matrix, $c \in \mathbb{R}^m\ (m < n)$.

A3 There exists $x^0 > 0$ such that $Bx^0 = c$.

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A4 The set of optimal solutions of $(P1)$ is nonempty and bounded.

For x^* be an optimal solution in the problem $(P1)$, there exists two Lagrange multipliers $u^* \in \mathbb{R}^m, v^* \in \mathbb{R}^n_+$ such as:

$$\begin{cases} \nabla g(x^*) + B^t u^* - v^* = 0 \\ Bx^* = c \\ <v^*, x^*> = 0 \end{cases} . \tag{1}$$

We can write $g^* = g^*(x^*) = \min_{x \in B^*} g(x)$.

In the following, we replace the nonlinear constrained problem $(P1)$ with a perturbed problem. What is new in our work is that the term of penalty is taken as a vector $r \in \mathbb{R}^n_+$.

1.2. The perturbed problem

In this section, we firstly define the function $\psi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicountinuous and proper function.

ψ defined as follows:

$$\psi(r,x) = \begin{cases} \displaystyle\sum_{i=1}^{n} r_i \ln(r_i) - \sum_{i=1}^{n} r_i \ln(x_i) & \text{if } x, r > 0 \\ 0 & \text{if } r = 0, \ x \geqslant 0 \\ +\infty & \text{if not} \end{cases} . \tag{2}$$

Now, the convex, lower semicountinuous and proper function $\phi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by:

$$\phi_r(x) = \Phi(r,x) = \begin{cases} \displaystyle g(x) + \sum_{i=1}^{n} r_i \ln(r_i) - \sum_{i=1}^{n} r_i \ln(x_i) & \text{if } Bx = c; \ x, r \geqslant 0 \\ +\infty & \text{if not} \end{cases} . \tag{3}$$

Finally, the convex function m is defined by:

$$m(r) = \inf_x \{\phi_r(x); \quad x \in \mathbb{R}^n\} \tag{P2}$$

m is clearly convex since of the convexity of ϕ_r.

We notice that the two problems $(P1)$ and $(P2)$ are coincided when $\| r \| \to 0$, then $g^* = m(0)$.

Our idea is to develop a new approach, which consist to determine the step size using a majorant function technique. We begin by studying the existence and the uniqueness of the optimal solution of the perturbed problem $(P2)$ followed by the convergence study. The resolution of the perturbed problem is based on the Newton descent direction and the majorant function technique to determine the step size.

1.2.1. Existence and uniqueness of optimal solution of perturbed problem

In order to prove that $(P2)$ admits one unique optimal solution, suffice it to prove that the cone of recession of ϕ_r is reduced to zero.

Proof. According to the fourth assumption, $(P1)$ admits one unique optimal solution then the cone of recession C_g of g is reduced to zero, we have:

$$C_g = \{d \in \mathbb{R}^n : [g]_\infty(d) \leqslant 0, \ Bd = 0, \ d \geqslant 0\} = \{0\}$$

$[g]_\infty(d)$ is the asymptotic function of b, which define by:

$$[g]_\infty(d) = \lim_{t \to +\infty} \frac{b(x_0 + td) - b(x_0)}{t}.$$

We have:

$$[\phi_r]_\infty = \begin{cases} g_\infty(d) & \text{if} \quad Bd = 0, \ d \geqslant 0 \\ +\infty & \text{if not} \end{cases}.$$

Then we deduce that: $\{d \in \mathbb{R}^n; \quad [\phi_r]_\infty \leqslant 0\} = \{0\}$, wich means that $C_\phi = \{0\}$.

By taking into account that ϕ_r is strictly convex, we come to conclusion that the perturbed problem $(P2)$ admits one unique optimal solution which is denoted by $x(r) \in \overline{K}$, the set of strictly feasible solution of $(P2)$, in which

$$\overline{K} = \{x \in \mathbb{R}^n : Bx = c, \ x > 0\}.$$

$\square$

1.2.2. Convergence of perturbed problem

According to the necessary and sufficient optimality conditions, there exists $\lambda(r) \in \mathbb{R}^m$ (assumption 2) verify:

$$\begin{cases} \nabla g(x(r)) - rX_r^{-1} + B^t\lambda(r) = 0 \\ Bx(r) - c = 0 \end{cases}. \tag{4}$$

In which X is the diagonal matrix with diagonal entries $X_{ii} = x_i \ \forall i = \overline{1, n}$.
We impose that

$$F(x(r), \lambda(r)) = \begin{pmatrix} \nabla g(x(r)) - rX_r^{-1} + B^t\lambda(r) \\ Bx(r) - c \end{pmatrix} = 0.$$

The two functions $r \to x(r)$ and $r \to \lambda(r)$ are differentiable on $\mathbb{R}_+^n$, by using the implicit function theorem, we get

$$\begin{pmatrix} \nabla^2 g(x(r)) + RX_r^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \nabla x(r) \\ \nabla \lambda(r) \end{pmatrix} = \begin{pmatrix} X_r^{-1} \\ 0 \end{pmatrix}, \tag{5}$$

where R is the diagonal matrix with diagonal entries $R_{ii} = r_i \ \forall i = \overline{1, n}$. And

$$\nabla x(r) = \begin{pmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_1}{\partial r_2} & \cdots & \dfrac{\partial x_1}{\partial r_n} \\[2mm] \dfrac{\partial x_2}{\partial r_1} & \dfrac{\partial x_2}{\partial r_2} & \cdots & \dfrac{\partial x_2}{\partial r_n} \\[2mm] \vdots & \vdots & \ddots & \vdots \\[2mm] \dfrac{\partial x_n}{\partial r_1} & \dfrac{\partial x_n}{\partial r_2} & \cdots & \dfrac{\partial x_n}{\partial r_n} \end{pmatrix}, \quad \nabla \lambda(r) = \begin{pmatrix} \dfrac{\partial \lambda_1}{\partial r_1} & \dfrac{\partial \lambda_1}{\partial r_2} & \cdots & \dfrac{\partial \lambda_1}{\partial r_n} \\[2mm] \dfrac{\partial \lambda_2}{\partial r_1} & \dfrac{\partial \lambda_2}{\partial r_2} & \cdots & \dfrac{\partial \lambda_2}{\partial r_n} \\[2mm] \vdots & \vdots & \ddots & \vdots \\[2mm] \dfrac{\partial \lambda_m}{\partial r_1} & \dfrac{\partial \lambda_m}{\partial r_2} & \cdots & \dfrac{\partial \lambda_m}{\partial r_n} \end{pmatrix}.$$

Remember that the function m wich is differentiable on $\mathbb{R}_+^n$ is define by:

$$m(r) = g(x(r)) + \sum_{i=1}^{n} r_i \ln(r_i) - \sum_{i=1}^{n} r_i \ln(x_i(r)).$$

We have
$$\nabla m(r) = (\nabla x(r))^t \left(\nabla g(x(r)) - X_r^{-1} r \right) + (e + z_1 - z_2).$$

In which $e = (1, 1, \ldots, 1)^t$, $z_1 = (\ln r_1, \ln r_2, \ldots, \ln r_n)^t$ and $z_2 = (\ln x_1, \ln x_2, \ldots, \ln x_n)^t$. According to (4) and (5), we get

$$\begin{aligned}
\nabla m(r) &= -(\nabla x(r))^t B^t \lambda(r) + (e + z_1 - z_2) = \\
&= -(B \nabla x(r))^t \lambda(r) + (e + z_1 - z_2) = \\
&= e + z_1 - z_2.
\end{aligned}$$

For $x(r) \in K$ and since of the convexity of m, we get:

$$\begin{aligned}
m(0) \ &\geqslant m(r) - r^t \nabla m(r) \geqslant \\
&\geqslant g(x(r)) + \sum_{i=1}^{n} r_i \ln r_i - \sum_{i=1}^{n} r_i \ln x_i(r) - r^t(e + z_1 - z_2) \geqslant \\
&\geqslant g(x(r)) + \sum_{i=1}^{n} r_i \ln r_i - \sum_{i=1}^{n} r_i \ln x_i(r) - \sum_{i=1}^{n} r_i - \sum_{i=1}^{n} r_i \ln r_i + \sum_{i=1}^{n} r_i \ln x_i(r) \geqslant \\
&\geqslant g(x(r)) - \sum_{i=1}^{n} r_i.
\end{aligned}$$

Taking into account that: $g^* = m(0) = \min_x g(x(r))$.

Then, we come conclusion $g^* \leqslant g(x(r)) \leqslant g^* + \sum_{i=1}^{n} r_i$.

For the rest, we are interesting on the trajectory of $x(r)$ when $\| r \|$ tends to zero.

a) The case in which g is only convex.

This case is a little complicated, we impose that $\|r\|_\infty \leqslant 1$, and for that we note

$$x(r) \in \{x; \ Bx = c, \ x > 0, \ g(x) \leqslant n + g^*\}.$$

This set is convex, bounded and non empty, its cone of recession is reduced to zero.
It follows that each accumulation point of x_r is an optimal solution of $(P1)$ only if $\|r\| \to 0$.

b) The case in which g is strongly convex with coefficient γ strictly positif.

We have

$$\sum_{i=1}^{n} r_i \ \geqslant \ g(x(r)) - g(x^*) \ \geqslant \ < \nabla g(x^*), x(r) - x^* > + \frac{\gamma}{2} \| x(r) - x^* \|^2 .$$

Using (1), we obtain

$$\sum_{i=1}^{n} r_i \ \geqslant \ < v^*, x(r) > \ \geqslant \ 0.$$

Then

$$\| x(r) - x^* \| \leqslant \left(\frac{2}{\gamma} \sum_{i=1}^{n} r_i \right)^{\frac{1}{2}} .$$

We come to conclusion that the convergence of $x(r)$ to x^* is of order $\dfrac{1}{2}$.

Remark 1.1. *If the problem $(P1)$ or the perturbed problem $(P2)$ will have an optimal solution and the values of their objective functions are equal and finite, the other problem has an optimal solution.*

The general prototype of our method is as follows:

0 Starting by $(r_0, x_0) \in \mathbb{R}_+^n \times \overline{K}$.

1 Find an approximate solution of $(P2)$ has been noted by x_{k+1} such that:

$$\phi(r_k, x_{k+1}) \leqslant \phi(r_k, x_k).$$

2 Take: $\|r_{k+1}\|_\infty \leqslant \|r_k\|_\infty.$

The iterations continue until we obtained the approximate solution.

2.   Some useful inequalities

Taking into consideration the statistical serie of n real numbers $\{z_1, \ldots, z_n\}$, we define their arithmetic mean $\overline{z}$ and their standard deviation σ_z. These quantities are defined as follows:

$$\overline{z} = \frac{1}{n} \sum_{i=1}^n z_i, \quad \sigma_z^2 = \frac{1}{n} \sum_{i=1}^n z_i^2 - \overline{z}^2 = \frac{1}{n} \sum_{i=1}^n (z_i - \overline{z})^2.$$

For the following result see [3, 6]

Proposition 2.1.

$$\overline{z} - \sigma_z \sqrt{n-1} \leqslant \min_i z_i \leqslant \overline{z} - \frac{\sigma_z}{\sqrt{n-1}},$$

$$\overline{z} + \frac{\sigma_z}{\sqrt{n-1}} \leqslant \max_i z_i \leqslant \overline{z} + \sigma_z \sqrt{n-1}.$$

In the case where z_i are all positifs, we deduce that:

$$\ln \left(\overline{z} - \sigma_z \sqrt{n-1} \right) \leqslant \sum_{i=1}^n \ln(x_i) \leqslant \ln \left(\overline{z} + \sigma_z \sqrt{n-1} \right).$$

Theorem 2.1 ([2]). *Assume that $z_i > 0$ for all $i = \overline{1, n}$, then:*

$$A_1 \leqslant \sum_{i=1}^n \ln(z_i) \leqslant A_2,$$

with:

$$A_1 = (n-1) \ln \left(\overline{z} + \frac{\sigma_z}{\sqrt{n-1}} \right) + \ln \left(\overline{z} - \sigma_z \sqrt{n-1} \right),$$

$$A_2 = \ln \left(\overline{z} + \sigma_z \sqrt{n-1} \right) + (n-1) \ln \left(\overline{z} - \frac{\sigma_z}{\sqrt{n-1}} \right).$$

3.   Solving the perturbed problem

Consider the following perturbed problem defined as follows:

$$m(r) = \min_x \left\{ \phi_r(x) = g(x) + \sum_{i=1}^n \psi(r_i, x_i) \ : \ Bx = c, \ x \geqslant 0 \right\}.$$

In this section we are interested in the numerical solution of the problem $(P1)$, we begin our work by calculate the descent direction and the step size in which we use a new technique of majorant functions.

3.1. The descent direction and line search function

A descent direction d can be computed by various methods, in this task we choose the Newton's method and therefore d is given by solving the following quadratic convex minimization problem:

$$\begin{cases} \min_d \left(\dfrac{1}{2} < \nabla^2\phi_r(x)d, d > + < \nabla\phi_r(x), d > \right) \\ Bd = 0 \end{cases}.$$

According to the necessary and sufficient optimality conditions, there exists $\mu \in \mathbb{R}^m$ such that:

$$\begin{cases} \nabla^2\phi_r(x)d + \nabla\phi_r(x) + B^t\mu = 0 \\ Bd = 0 \end{cases}.$$

Which is equivalent to

$$\begin{pmatrix} \nabla^2 g(x) + RX^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} d \\ \mu \end{pmatrix} = \begin{pmatrix} X^{-1}r - \nabla g(x) \\ 0 \end{pmatrix}.$$

From which we get

$$\begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} \nabla^2 g(x) + RX^{-2} & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} d \\ \mu \end{pmatrix} = \begin{pmatrix} d^t & 0 \end{pmatrix} \begin{pmatrix} X^{-1}r - \nabla g(x) \\ 0 \end{pmatrix}.$$

Then

$$< \nabla^2 g(x)d, d > + < \nabla g(x), d > = < r, X^{-1}d > - < RX^{-1}d, X^{-1}d > . \tag{6}$$

This system is equivalent to

$$\begin{pmatrix} X\nabla^2 g(x)X + R & XB^t \\ BX & 0 \end{pmatrix} \begin{pmatrix} X^{-1}d \\ \mu \end{pmatrix} = \begin{pmatrix} r - X\nabla g(x) \\ 0 \end{pmatrix}. \tag{7}$$

The Newton descent direction being calculated.

3.2. Computation of the step size

Generally, the most used methods in the search line are the classical itterative methods as Armijo–Goldstein, Wolfe, Fibonnaci, . . . , but the computational cost in there becomes high when n is very large.

In this part, we are interested to avoid this difficulty. The method that we use bellow is simple and more effective than the first, it consists on the use of majorant function of the function θ. The choice of the step size $t^* > 0$ must give us a significant decrease of the convex function θ, we have:

$$\begin{aligned} \theta_0(t) &= \phi_r(x + td) - \phi_r(x) = \\ &= g(x + td) - g(x) - \sum_{i=1}^{n} r_i ln(1 + ty_i), \qquad y = X^{-1}d. \end{aligned}$$

According to Proposition 2.1, we have: $\rho \leqslant \min_i r_i \leqslant r_i \quad \forall i = \overline{1, n}$.
In which $\rho = \overline{r} - \sigma_r\sqrt{n-1}$.
Then, we obtain

$$\theta(t) = \frac{\theta_0(t)}{\rho} \leqslant \theta_1(t) = \frac{1}{\rho} \left(g(x + td) - g(x) \right) - \sum_{i=1}^{n} ln(1 + ty_i).$$

We have

$$\theta^{'}(t) = \frac{1}{\rho}\left(<\nabla g(x+td), d> - \sum_{i=1}^{n} r_i \frac{y_i}{1+ty_i}\right),$$

$$\theta^{''}(t) = \frac{1}{\rho}\left(<\nabla^2 g(x+td)d, d> + \sum_{i=1}^{n} r_i \frac{y_i^2}{(1+ty_i)^2}\right).$$

And

$$\theta_1^{'}(t) = \frac{1}{\rho} <\nabla g(x+td), d> - \sum_{i=1}^{n} \frac{y_i}{1+ty_i},$$

$$\theta_1^{''}(t) = \frac{1}{\rho} <\nabla^2 g(x+td)d, d> + \sum_{i=1}^{n} \frac{y_i^2}{(1+ty_i)^2}.$$

We deduce from (6), that $\theta^{'}(0) + \theta^{''}(0) = 0$, and we have $\theta^{''}(0) \geqslant 0$ wich give us that $\theta^{'}(0) \leqslant 0$. Now it must to prove that $\theta_1^{'}(0) \leqslant 0$, we have:

a If $y_i \geqslant 0$, it is clearly that $\theta_1^{'}(0) \leqslant 0$.

b If $y_i < 0$, we deduce from (6) that $\theta_1^{'}(0) + \theta_1^{''}(0) \leqslant 0$ and as $\theta_1^{''}(0) \geqslant 0$ we come conclusion that $\theta_1^{'}(0) \leqslant 0$.

What is prove the significant decrease of θ_1.

3.3. The first majorant function

The choice of t^* in which $\theta^{'}(t^*) = \theta^{'}(t_{opt}) = 0$ consists of some numerical complications, so generally we can't obtain t^* directly. To solve this problem, we propose to find an approximation function of θ.

This method is based on the use of a majorant function θ_2 of the function θ.

In the following, we take: $x_i = 1 + ty_i$, $\overline{x} = 1 + t\overline{y}$, and $\sigma_x = t\sigma_y$.

Applying the inequality $\sum_{i=1}^{n} ln(x_i) \geqslant A_1$ (Theorem 2.2), we get that $\theta_1(t) \leqslant \theta_2(t)$ such that

$$\theta_2(t) = \frac{1}{\rho}\left(g(x+td) - g(x)\right) - (n-1)ln(1+t\alpha) - ln(1+t\beta).$$

In which

$$\alpha = \overline{y} + \frac{\sigma_y}{\sqrt{n-1}}, \quad \beta = \overline{y} - \sigma_y\sqrt{n-1}.$$

We have

$$\theta_2^{'}(t) = \frac{1}{\rho} <\nabla g(x+td), d> - (n-1)\frac{\alpha}{1+t\alpha} - \frac{\beta}{1+t\beta},$$

$$\theta_2^{''}(t) = \frac{1}{\rho} <\nabla^2 g(x+td)d, d> + (n-1)\frac{\alpha^2}{(1+t\alpha)^2} + \frac{\beta^2}{(1+t\beta)^2}.$$

The domains of θ_2 is $H_2 =]0, T[$ in which $T = \max\{t: 1+t\beta > 0\}$, this domain is content in the domain of the line search function θ.

We notice that : $\theta(0) = \theta_1(0) = \theta_2(0) = 0$, $\theta_1^{'}(0) = \theta_2^{'}(0) < 0$ and $\theta_1^{''}(0) = \theta_2^{''}(0) > 0$.

We prove that the strictly convex function θ_2 is a good approximation of θ_1 in a neighbourhood of 0, hence the unique minimum t^* of θ_2 guarantee a significant decrease of the function θ_1, and we have the follows inequalities:

$$\theta(t^*) \leqslant \theta_1(t^*) \leqslant \theta_2(t^*) < 0.$$

3.4. Case when g is linear

We impose that $g(x) = c^t x$, $x, c \in \mathbb{R}^n$, the auxiliary function ω is given by the following form:

$$\omega(t) = n\eta t - (n-1)ln(1 + t\alpha) - ln(1 + t\beta)$$

in which: $\eta = \dfrac{c^t d}{n}$.

ω have the same properties as θ_2, the unique root of $\omega'(t) = 0$ is the minimum of θ_2. The unique $\bar{t}$ that we have guarantee a significant decrease of the function ϕ_r along the newton descent direction d.

3.5. Case when g is only convex

In this case, the equation $\theta_2'(t) = 0$ is no longer reduces to an equation of second degree, we thought to look at another function greater than θ_2, for this we use the secant technique. Given $\bar{t} \in]0, T[$ for all $t \in]0, \bar{t}]$, we have

$$\frac{g(x + td) - g(x)}{\rho} \leqslant \frac{g(x + \bar{t}d) - g(x)}{\rho \bar{t}} t.$$

Then the auxiliary function ω is define as follows:

$$\omega(t) = n\eta t - (n-1)ln(1 + t\alpha) - ln(1 + t\beta).$$

Such as, we take

$$\eta = \frac{g(x + \bar{t}d) - g(x)}{n\rho \bar{t}},$$

and we calculate t^* the root of the equation $\omega'(t) = 0$.

1. If $\bar{t} = 1$ and $T > 1$ then $\bar{t}$ is the optimal solution.

2. If $\bar{t} \neq 1$, then

   a If $t^* \leqslant \bar{t}$, in this case we have $\theta(t^*) \leqslant \theta_1(t^*) \leqslant \theta_2(t^*) \leqslant \omega(t^*)$, which means that we assure a significant decrease of the function ϕ_r along the direction d.

   b If $t^* > \bar{t}$, we must to choose another $\bar{t} \in]t^*, T[$ and calculate t^* for the new auxiliary function and repeat this until we have that $t^* \leqslant \bar{t}$, for example we choose

   $$\bar{t} = t^* + \zeta(T - t^*); \quad \zeta \in [0, 1].$$

3.6. Minimization of the auxiliary function ω

We have

$$\omega(t) = n\eta t - (n-1)ln(1 + t\alpha) - ln(1 + t\beta).$$

It is easy to calculate

$$\omega'(t) = n\eta - (n-1)\frac{\alpha}{1 + t\alpha} - \frac{\beta}{1 + t\beta},$$

$$\omega''(t) = (n-1)\frac{\alpha^2}{(1 + t\alpha)^2} + \frac{\beta^2}{(1 + t\beta)^2}.$$

Then: $\omega(0) = 0$, $\omega'(0) = n(\eta - \bar{y})$, $\omega''(0) = n(\bar{y}^2 + \sigma_x^2) = \| y \|^2$.

We impose that $\omega'(0) \leqslant 0$ and $\omega''(O) \geqslant 0$.

For getting t^*, we need to calculate the root of the equation $\omega'(t) = 0$:

Equivalent to

$$\eta\alpha\beta t^2 + (\eta(\alpha + \beta) - \alpha\beta)t + \eta - \overline{y} = 0$$

1. if $\eta = 0$, $t^* = \dfrac{-\overline{y}}{\alpha\beta}$,

2. if $\alpha = 0$, $t^* = \dfrac{\overline{y} - \eta}{\eta\beta}$,

3. if $\beta = 0$, $t^* = \dfrac{\overline{y} - \eta}{\eta\alpha}$,

4. if $\eta\alpha\beta \neq 0$, in this case we have two roots of the equation of the second degree but there is just only root t^* which belongs to the domain of definition of ω, both roots are:

$$t_1^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\alpha} - \frac{1}{\beta} - \sqrt{\Delta}\right), \quad t_2^* = \frac{1}{2}\left(\frac{1}{\eta} - \frac{1}{\alpha} - \frac{1}{\beta} + \sqrt{\Delta}\right).$$

In which

$$\Delta = \frac{1}{\eta^2} + \frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{2}{\alpha\beta} - \left(\frac{2n-4}{n\eta}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right).$$

3.7. The second majorant function

Here, we thought to find another approximation of θ_1 simpler than θ_2 and has one logarithm. Remember that:

$$\theta_1(t) = \frac{1}{\rho}\left(g(x + td) - g(x)\right) - \sum_{i=1}^{n} \ln(1 + ty_i); \ \rho = \overline{r} - \sigma_r\sqrt{n-1}.$$

Using the inequality:

$$\sum_{i=1}^{n} \ln(1 + ty_i) \geqslant (\| y \| + n\overline{y})t + \ln(1 - t \| y \|).$$

Then, we get a second majorant function of θ noting by θ_3 such that:

$$\theta_3(t) = \frac{1}{\rho}\left(g(x + td) - g(x)\right) - (\| y \| + n\overline{y})t - \ln(1 - t \| y \|)$$

and

$$\theta_3'(t) = \frac{1}{\rho} < \nabla g(x + td), d > - \| y \| - n\overline{y} + \frac{\| y \|}{1 - t \| y \|},$$

$$\theta_3''(t) = \frac{1}{\rho} < \nabla g(x + td)d, d > + \frac{\| y \|^2}{(1 - t \| y \|)^2}.$$

The domains of θ_3 is $H_3 = [0, T_3[$, with $T_3 = \max\{t; 1 - t \| y \| > 0\}$.

We remark that:

- $\theta_3(0) = \theta_1(0) = 0$,

- $\theta_3'(0) = \dfrac{1}{\rho} < \nabla g(x), d > - n\overline{y} = \theta_1'(0) < 0$,

- $\theta_3''(0) = \dfrac{1}{\rho} < \nabla^2 g(x)d, d > + \parallel y \parallel^2 = \theta_1''(0) > 0.$

The strictly convex function θ_3 is a good approximation of θ_1 in a neighbourhood of 0, the unique minimum t^* of θ_3 guarantee a significant decrease of the function θ_1, and we have:

$$\theta_1(t^*) \leqslant \theta_2(t^*) \leqslant \theta_3(t^*).$$

3.7.1. Minimization of an auxiliary function

Let us define the convex function ω_2, where it's minimum is reached at t^*.

$$\omega_2(t) = n\eta t - (\parallel y \parallel + n\overline{y})t - ln(1 - t \parallel y \parallel).$$

It is easy to calculate

$$\omega_2'(t) = n\eta - \parallel y \parallel - n\overline{y} + \frac{\parallel y \parallel}{1 - t \parallel y \parallel},$$
$$\omega_2''(t) = \frac{\parallel y \parallel^2}{(1 - t \parallel y \parallel)^2}.$$

Then: $\omega_2(0) = 0$, $\omega_2'(0) = n(\eta - \overline{y})$, $\omega_2''(0) = \parallel y \parallel^2$.
We impose that $\omega_2'(0) \leqslant 0$ and $\omega_2''(O) \geqslant 0$.
For getting t^*, we need to calculate the root of the equation $\omega_2'(t) = 0$.

4. Description of the algorithm

In this part, we present the algorithm which resume our study to obtain the optimal solution x^* of the problem $(P1)$.

4.1. Algorithm

1. **Input** $\epsilon > 0$, $r_s > 0$, $x_0 \in K$, X with $X(i, i) = x_0(i)$, $r \in \mathbb{R}_+^n$, $\delta \in [0, 1]^n$.

2. **Iteration**

   * Calculate d and $y = X^{-1}d$

   (a) If $\parallel y \parallel > \epsilon$ do
       a1 Calculate η, α, β and solve the equation $\omega'(t) = 0$ for obtain t^*.
       a2 Calculate $x = x + t^*d$, and return to $(*)$.
   (b) If $\parallel y \parallel \leqslant \epsilon$, we obtained a good approximation of $m(r)$.
       i If $\|r\| \geqslant r_s$, $r = \delta \times r$ and return to $(*)$.
           With $\delta \times r = (\delta_1 \times r_1, \ldots, \delta_n \times r_n)$.
       ii If $\|r\| \leqslant r_s$, Stop. We have a good approximation of the optimal solution.

5. Numerical tests

In the tables bellow, Iter represents the number of iterations to obtain x^*, Min represents the minimum and T(s) represents the time in seconds. Method 1 corresponds to the method of majorant function introduced in this work, method 2 corresponds to the method of majorant function introduced in [1] and method 3 corresponds to the classical line search method.

5.1. Examples with variable size

Example 1. Quadratic case [4]

We consider the following quadratic problem with $n = m + 2$

$$g^* = \min\{b(x) : Bx = c, x \geqslant 0\}.$$

In which:

$$g(x) = \frac{1}{2} < x, Qx > .$$

With

$$Q[i,j] = \begin{cases} 2 & \text{if } i = j = 1 \quad \text{or} \quad i = j = m \\ 4 & \text{if } i = j \text{ and } i \neq \{1, m\} \\ 2 & \text{if } i = j - 1 \quad \text{or} \quad i = j + 1 \\ 0 & \text{otherwise} \end{cases} \quad , \quad B[i,j] = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i = j - 1 \\ 3 & \text{if } i = j - 2 \\ 0 & \text{otherwise} \end{cases} .$$

$g_i = 1 \, \forall i = \overline{1, n}, \, \forall j = \overline{1, m}.$

We test this example for different value of n.

Example 2. Erikson's problem [5]

Consider the following convex problem:

$$g^* = \min[g(x) : Bx = c, x \geqslant 0].$$

Where $g(x) = \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{a_i}\right), \; a_i, b_i \in \mathbb{R}$ are fixed, and

$$B[i,j] = \begin{cases} 1 & \text{if } i = j \text{ or } j = i + m \\ 0 & \text{if not} \end{cases} .$$

We test this example for different values of n, a_i and b_i.

6.   Tables

Table 1. Numerical simulations for Example 1

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
4	0.285	8	0.0061	0.285	9	0.007	0.285	14	0.019
50	5.37	6	0.019	5.372	7	0.021	5.374	14	0.053
100	10.924	6	0.065	10.927	7	0.08	10.93	14	0.188
500	55.3722	8	14.5	55.372	7	13.8	55.374	14	29.815

Table 2. The case where $a_i = 1$ and $b_i = 6, \forall i = \overline{1.n}$ (Example 2)

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
10	32.94	3	0.0038	32.95	3	0.004	32.95	4	0.018
50	164.79	4	0.026	164.79	4	0.025	164.79	6	0.082
500	329.57	5	0.1	329.58	5	0.076	329.58	6	0.23

Table 3. The case where $a_i = 2$ and $b_i = 5, \forall i = \overline{1.n}$ (Example 2)

	Method 1			Method 2			Method 3		
n	Min	Iter	T(s)	Min	Iter	T(s)	Min	Iter	T(s)
10	0.62×10^{-8}	2	0.0021	0.66×10^{-7}	2	0.002	9.09×10^{-6}	3	0.01
50	0.1×10^{-7}	3	0.0028	0.11×10^{-8}	3	0.003	1.86×10^{-4}	4	0.07
500	0.75×10^{-7}	3	0.0045	0.76×10^{-7}	3	0.04	1.03×10^{-4}	5	0.22

Conclusion

The effective numerical simulations show that our approach is a very important alternative and gives an encouraging results compared to the classical line search method. However, it competes with the method introduced in [1] where $r \in \mathbb{R}$.

References

[1] L.B.Cherif, B.Merikhi, A penalty method for nonlinear programming, *RAIRO Oper. Res.*, **53**(2019), 29–38. DOI doi.org/10.1051/ro/2018061

[2] J.P.Crouzeix, B.Merikhi, A logarithm barrier method for semidefinite programming, *RAIRO Oper. Res.*, **42**(2008), 123–139.

[3] J.P.Crouzeix, A.Seegerm, New bounds for the extreme values of a finite sample of real numbers, *J. Math. Anal. Appl.*, **197**(1996), 411–426.

[4] M.Ouriemchi, Résolution de problémes non linéaires par les méthodes de points intérieurs, Théorie et algorithmes, *Thése de doctorat, Univerité du Havre, France*, 2006.

[5] E.Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.*, **27**(1948), 379–423 and 623–656 .

[6] H.Wolkowicz, G.-P.-H.Styan, Bounds for eigenvalues using traces, *Linear Algebra Appl.*, **29**(1980), 471–506 .

Логарифмический барьерный подход с использованием мажорантной функции для нелинейного программирования

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Аннотация. В данной статье нас интересует решение оптимизационной задачи нелинейного программирования с использованием метода внутренних точек с логарифмическим барьером, в котором штрафной член берется в виде вектора $r \in \mathbb{R}_+^n$. Направление спуска было рассчитано с использованием классического метода Ньютона, однако размер шага был рассчитан с использованием новой техники мажорантных функций и техники секущих. Численное моделирование показывает нам эффективность нашего подхода по сравнению с классическим методом линейного поиска.

Ключевые слова: нелинейное выпуклое программирование, метод логарифмических штрафов, линейный поиск, мажорантная функция, метод секущих.