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Creeping Three-dimensional Convective Motion in a Layer with Velocity Field of a Special Type

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Abstract. Problem of three-dimensional motion of a heat-conducting fluid in a channel with solid parallel walls is considered. Given temperature distribution is maintained on solid walls. The liquid temperature depends quadratically on the horizontal coordinates, and the velocity field has a special form. The resulting initial-boundary value problem for the Oberbeck–Boussinesq model is inverse and reduced to a system of five integro-differential equations. For small Reynolds numbers (creeping motion), the resulting system becomes linear. A stationary solution has been found for this system, and a priori estimates have been obtained. On the basis of these estimates, sufficient conditions for exponential convergence of a smooth non-stationary solution to a stationary solution have been established. The solution of the inverse problem has been found in the form of quadratures for the Laplace images under weaker conditions for the temperature regime on the walls of the layer. Behaviour of the velocity field for a specific liquid medium have been presented. The results were obtained with the use of numerical inversion of the Laplace transform.

Keywords: Oberbec–Boussinesq model, three-dimensional motion, inverse problem, a priori estimates, stability, Laplace transform.

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Problem statement and derivation of basic equations

Two-dimensional flows of the Himentz type [1] are known as flows near the critical point and they are characterized by the presence of zones with higher pressure and temperature than in the surrounding region. Such flows can be observed both in macro-scales (for example, the use of hydraulic fracturing technologies in the oil industry) and in micro-scales (for example, liquid biochips in medicine). The study of characteristics of such flows is necessary to assess the technological parameters, as well as to predict the dynamics and evolution of the liquid layer. Exact solutions of the defining equations are the most effective way to study processes in a liquid, as well as to obtain estimated characteristics. At present, solutions of problems describing Himentz-type flows in various geometries are presented: axisymmetric [2] and three-dimensional [3, 4] analogues of the Himentz solution, including flows in cylindrical geometry [5, 6]. A brief overview of the exact solutions that are close to the Himentz solution is given in [7].

Three-dimensional motion of a viscous incompressible heat-conducting fluid with special velocity field is studied in this paper. The velocity field is of the Himentz type: the horizontal

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components of the velocity field are linear in the corresponding coordinates, temperatures are set on solid walls.

The system of Oberbeck–Boussinesq equations of three-dimensional motion has the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\cdot\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\Delta\mathbf{u} + \mathbf{g}(1-\beta T), \quad \operatorname{div}\mathbf{u} = 0, \tag{1}$$

$$T_t + \mathbf{u}\cdot\nabla T = \chi\Delta T, \tag{2}$$

where $\mathbf{u}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$ is the velocity vector, u, v, w are components of the velocity vector in the Cartesian coordinate system; $\mathbf{g} = (0,0,-g)$; t is time; $T(x,y,z,t)$ is temperature; positive constants ρ, ν, χ, β, g are density, kinematic viscosity, thermal conductivity coefficient, coefficient of thermal expansion and acceleration of gravity, respectively. The solution of problem (1), (2) is taken in the following form

$$\begin{gathered} u(x,y,z,t) = (f(z,t)+h(z,t))x, \quad v(x,y,z,t) = (f(z,t)-h(z,t))y, \\ w(x,y,z,t) = -2\int_0^z f(\xi,t)d\xi, \quad p(x,y,z,t) = \bar{p}(x,y,z,t) - \rho g z, \\ T(x,y,z,t) = a(z,t)x^2 + b(z,t)xy + c(z,t)y^2 + \theta(z,t). \end{gathered} \tag{3}$$

Relations (3) are interpreted as fluid motion between two flat parallel fixed plates $z = 0$ and $z = l$ (see Fig. 1). Then adhesion conditions are set on fixed plates: $u(x,y,0,t) = v(x,y,0,t) = w(x,y,0,t) = 0$, $u(x,y,l,t) = v(x,y,l,t) = w(x,y,l,t) = 0$. Temperature is given in the form $T(x,y,0,t) = a_1(t)x^2 + b_1(t)xy + c_1(t)y^2$, $T(x,y,l,t) = a_2(t)x^2 + b_2(t)xy + c_2(t)y^2$. Considering (3), using conditions of adhesion and setting the temperature, boundary conditions for functions $a(z,t)$, $b(z,t)$, $c(z,t)$ $\theta(z,t)$ $f(z,t)$ $h(z,t)$ are derived

$$\begin{gathered} f(0,t) = f(l,t) = h(0,t) = h(l,t) = 0, \quad \int_0^l f(\xi,t)d\xi = 0, \\ a(0,t) = a_1(t), \quad b(0,t) = b_1(t), \quad c(0,t) = c_1(t), \quad \theta(0,t) = 0, \\ a(l,t) = a_2(t), \quad b(l,t) = b_2(t), \quad c(l,t) = c_2(t), \quad \theta(l,t) = 0, \end{gathered} \tag{4}$$

where functions $a_j(t)$, $c_j(t)$, $j = 1,2$ are set at some interval $[0,t_0]$. In addition, initial conditions are set

$$\begin{gathered} a(z,0) = a_0(z), \quad c(z,0) = c_0(z), \quad b(z,0) = b_0(z), \quad \theta(z,0) = 0, \\ f(z,0) = f_0(z), \quad h(z,0) = h_0(z). \end{gathered} \tag{5}$$

Remark 1. *Since* $\operatorname{rot}\mathbf{u} = ((h_z - f_z)y, (h_z + f_z)x, 0) \neq 0$, *then the motion is vortex.*

Remark 2. *Suppose, without the loss of generality, that* $a_j(t) \neq 0$, $j = 1,2$ *and* $b(z,t) = 0$. *Then when* $a_j(t) < 0$, $c_j(t) < 0$ *functions* $T_j(x,y,t)$ *have a maximum at the point* $x = 0$, $y = 0$, *and when* $a_j(t) > 0$, $c_j(t) > 0$ *functions* $T_j(x,y,t)$ *have a minimum. If* $a_j(t)$ *and* $c_j(t)$ *have the same signs then* $T_j(x,y,t)$ *is an elliptical paraboloid. If* $a_j(t)$ *and* $c_j(t)$ *have different signs then* $T_j(x,y,t)$ *is a hyperbolic paraboloid. In other words, the above solution describes the convection of a liquid near the points of temperature extremes on solid walls. There may be other cases, for example, the temperature has a maximum on the lower wall and a minimum on the upper wall or vice versa.*

The first step is to derive a system of equations for f, h, a, b, c, θ.

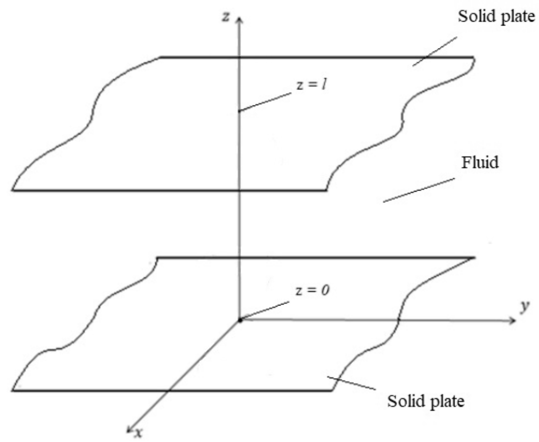

Fig. 1. Flow area diagram

Taking into account equations (2) and (3), the following relations are obtained

$$
\begin{aligned}
a_t + 2a(f+h) - 2a_z \int_0^z f(\xi,t)d\xi &= \chi a_{zz}, \ b(z,t) = 0, \\
c_t + 2c(f-h) - 2c_z \int_0^z f(\xi,t)d\xi &= \chi c_{zz}, \\
\theta_t - 2\theta_z \int_0^z f(\xi,t)d\xi &= 2\chi(a+c) + \chi\theta_{zz}.
\end{aligned}
\tag{6}
$$

The mass conservation equation is satisfied identically, and momentum equation (1) is equivalent to the following equation

$$
\begin{aligned}
f_t + f^2 + h^2 - 2f_z \int_0^z f(\xi,t)d\xi &= \nu f_{zz} - \beta g \int_0^z [a(\xi,t) + c(\xi,t)]d\xi + n_1(t), \\
h_t + 2fh - 2h_z \int_0^z f(\xi,t)d\xi &= \nu h_{zz} - \beta g \int_0^z [a(\xi,t) - c(\xi,t)]d\xi + n_2(t),
\end{aligned}
\tag{7}
$$

where $n_1(t), n_2(t)$ are arbitrary functions of time that represent incremental pressure gradients. The modified pressure $\bar{p}(x,y,z,t)$ is found in the form of quadratures

$$
\frac{1}{\rho}\bar{p}(x,y,z,t) = x^2(g\beta \int_0^z a(\xi,t)d\xi - \frac{1}{2}(n_1(t)+n_2(t)))+
$$
$$
+y^2(g\beta \int_0^z c(\xi,t)d\xi - \frac{1}{2}(n_1(t)-n_2(t))) - 2\nu f(z,t) - gz+
$$
$$
+g\beta \int_0^z \theta(\xi,t)d\xi + 2\int_0^z (z-\xi)f_t(\xi,t)d\xi - 2(\int_0^z f(\xi,t)d\xi)^2 + \alpha_0(t),
$$

where $\alpha_0(t)$ is an arbitrary function of time.

Thus, the Oberbeck–Boussinesq system is reduced to five non-linear integro-differential equations.

The following notations are introduced

$$\xi = \frac{z}{l}; \quad \tau = \frac{\chi}{l^2}t; \quad a^* = \max(|\, a_j(t)\, |, |\, c_j(t)\, |), \;\; j = 1, 2, \quad u^* = \beta a^* l \chi;$$
$$a(z,t) = a^* A(\xi,\tau); \quad c(z,t) = a^* C(\xi,\tau); \quad \theta(z,t) = a^*\Theta(\xi,\tau); \quad f(z,t) = \frac{\chi}{l^2} Re F(\xi,\tau); \tag{8}$$
$$h(z,t) = \frac{\chi}{l^2} Re H(\xi,\tau); \quad n_j(t) = \frac{\chi^2}{l^4} N_j(\tau), \; j = 1, 2.$$

Here u^* is the characteristic rate of thermal expansion of the fluid, since $a^* l^2$ is the characteristic temperature of the walls, $\epsilon = \beta a^* l^2$ is the Boussinesq parameter [11], $Re = u^* l/\nu$ is the Reynolds number, $Re = \epsilon P$, where $P = \nu/\chi$ is the Prandtl number.

After substituting (8) into system (6), (7), the initial boundary value problem in dimensionless form is obtained

$$A_\tau + 2ReA(F + H) - 2ReA_\xi \int_0^\xi F(\xi,\tau)d\xi = A_{\xi\xi},$$
$$C_\tau + 2ReC(F - H) - 2ReA_\xi \int_0^\xi F(\xi,\tau)d\xi = C_{\xi\xi},$$
$$\Theta_\tau - 2Re\Theta_\xi \int_0^\xi F(\xi,\tau)d\xi = 2(A + C) + \Theta_{\xi\xi}, \tag{9}$$
$$F_\tau + ReF^2 + ReH^2 - 2ReF_\xi \int_0^\xi F(\xi,\tau)d\xi = PF_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) + C(\xi,\tau)]d\xi + N_1(\tau),$$
$$H_\tau + 2ReFH - 2ReH_\xi \int_0^\xi F(\xi,\tau)d\xi = PH_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) - C(\xi,\tau)]d\xi + N_2(\tau).$$

Parameter $\eta = gl^3(\nu\chi)^{-1}$ plays an important role in the theory of micro convection [11].

In system (9) $\tau \in [0, \tau_0 = \chi t_0 l^{-2}]$, $\xi \in [0, 1]$. To fully define unknowns A, C, Θ, F, H, N_1, N_2 it is necessary to consider initial and boundary conditions

$$A(\xi,0) = A_0(\xi), \quad C(\xi,0) = C_0(\xi), \quad \Theta(\xi,0) = 0,$$
$$F(\xi,0) = F_0(\xi), \quad H(\xi,0) = H_0(\xi). \tag{10}$$

$$A(0,\tau) = A_1(\tau), \quad C(0,\tau) = C_1(\tau), \quad \Theta(0,\tau) = F(0,\tau) = H(0,\tau) = 0,$$
$$A(1,\tau) = A_2(\tau), \quad C(1,\tau) = C_2(\tau), \quad \Theta(1,\tau) = F(1,\tau) = H(1,\tau) = 0. \tag{11}$$

$$\int_0^1 F(\xi,\tau)d\xi = 0, \quad \int_0^1 H(\xi,\tau)d\xi = 0. \tag{12}$$

Let us note that problem (9)–(12) is the inverse problem, since functions $N_j(t)$ are unknown.

Remark 3. *Conditions (12) actually mean that motion is considered in some cell bounded by x and y.*

Conditions for matching the input data are satisfied for a smooth solution

$$A_0(0) = A_1(0), \quad C_0(0) = C_1(0), \quad A_0(1) = A_2(0), \quad C_0(1) = C_2(0), \tag{13}$$

$$\int_0^1 F_0(\xi)d\xi = 0, \quad \int_0^1 H_0(\xi)d\xi = 0. \tag{14}$$

Remark 4. *Taking into account (8), it is assumed that $a_j(t) = a^* A_j(\tau)$, $c_j(t) = a^* C_j(\tau)$.*

For most liquid media, the Boussinesq number is $\epsilon \ll 1$. Therefore, one can look for a solution of the inverse initial-boundary value problem in the form of a series with respect to the Reynolds number Re. The main terms of the decomposition satisfy the linear system of equations (the designations of the desired functions are left the same)

$$\begin{gathered}
A_\tau = A_{\xi\xi}, \quad C_\tau = C_{\xi\xi}, \quad \Theta_\tau = 2(A+C) + \Theta_{\xi\xi}, \\
F_\tau = PF_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) + C(\xi,\tau)]d\xi + N_1(\tau), \\
H_\tau = PH_{\xi\xi} - \eta P \int_0^\xi [A(\xi,\tau) - C(\xi,\tau)]d\xi + N_2(\tau).
\end{gathered} \tag{15}$$

The initial and boundary conditions remain unchanged (see (4), (5)). The problem describes the so-called "crawling" movements and it is the subject of study of this work.

Stationary creeping motion

In this case, all functions do not depend on the dimensionless time τ and initial data (5) is not taken into account. Let us assume that $A^s(\xi)$, $C^s(\xi)$, $\Theta^s(\xi)$, $F^s(\xi)$, $H^s(\xi)$, $N_1^s(\xi)$, N_2^s is the required solution, A_j^s, C_j^s are the given constants. Without the loss of of generality, it is assumed that $A_1^s \neq 0$. Simple mathematical treatment shows that there are relations

$$\begin{gathered}
A^s(\xi) = A_1^s(1+\alpha_1\xi), \quad C^s(\xi) = A_1^s(\alpha_2 + \alpha_3\xi), \\
\alpha_1 = \frac{A_2^s - A_1^s}{A_1^s}, \quad \alpha_2 = \frac{C_1^s}{A_1^s}, \quad \alpha_3 = \frac{C_2^s - C_1^s}{A_1^s}; \\
\Theta^s(\xi) = A_1^s\left[(1+\alpha_2)(\xi - \xi^2) + \frac{\alpha_1+\alpha_2}{3}(\xi - \xi^3)\right]; \\
F^s(\xi) = \frac{\eta A_1^s P}{12}\left[(1+\alpha_2)(2\xi^3 - 3\xi^2 + \xi) + \frac{\alpha_1+\alpha_3}{10}(5\xi^4 - 9\xi^2 + 4\xi)\right], \\
H^s(\xi) = \frac{\eta A_1^s P}{12}\left[(1-\alpha_2)(2\xi^3 - 3\xi^2 + \xi) + \frac{\alpha_1-\alpha_3}{10}(5\xi^4 - 9\xi^2 + 4\xi)\right]; \\
N_1^s = \frac{1}{2}\eta A_1^s P^2\left[1 + \alpha_2 + \frac{3}{10}(\alpha_1 + \alpha_3)\right], \\
N_2^s = \frac{1}{2}\eta A_1^s P^2\left[1 - \alpha_2 + \frac{3}{10}(\alpha_1 - \alpha_3)\right].
\end{gathered} \tag{16}$$

When $A_1^s = C_1^s$ there is radial heating of the fluid on the wall. If $A_1^s, C_1^s < 0$ then heating is maximal at the point $x = 0, y = 0$. If $A_1^s, C_1^s > 0$ then heating is minimal. If $A_j^s = -C_j^s$ then heating of the fluid on the wall has the form of a hyperbola.

The characteristic vertical velocity profile $W^s(\xi) = w^s(\xi)/W^0$ is shown in Fig. 2 ($W^0 = -\eta A_1^s \chi$)

Physical constants were taken for water at a temperature of 20 °C: $P \sim 7$, $Re \sim 25.5 \cdot 10^-4$, values A_j^s, C_j^s, $j = 1, 2$ are shown in Fig. 2.

The solid line shows the case of radial heating of the fluid on the walls with a minimum of its value at the point $x = 0$, $y = 0$ while the fluid in the layer moves upwards.

The dashed line shows the vertical velocity profile when distribution of the fluid temperature has the form of a hyperbola on the lower wall and weak elliptical heating on the upper wall.

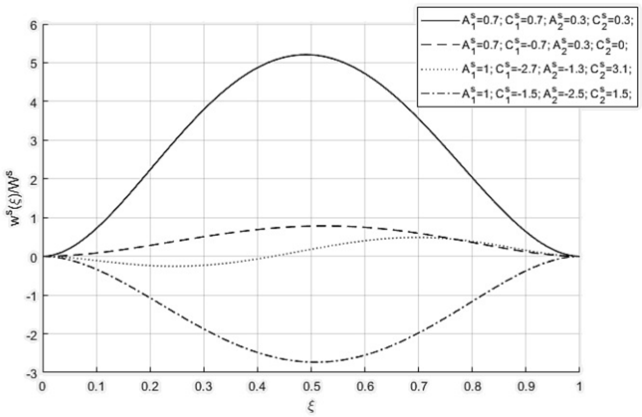

Fig. 2. Vertical velocity W^s as a function of dimensionless coordinate ξ.

In other cases, heating on both walls has the form of a hyperbola. The dotted line corresponds to such a temperature distribution that fluid in the lower part of the layer moves down, and in the upper part it moves up.

A priori estimates

The purpose of this paragraph is to establish sufficient conditions for the input data $A_j(t)$, $C_j(t)$, under which the solution of non-stationary problem converges to stationary solution (16) when dimensionless time increases. Functions $A(\xi,t)$, $C(\xi,t)$, $\Theta(\xi,t)$ are solutions of the first initial boundary value problem. They can be found in the form of trigonometric Fourier series. Using methods proposed in [12], it is possible to obtain a priori estimates of solutions. However, here it is easier to use results presented in [13] (pp. 201, 209). In fact, if $A_j(\tau)$, $C_j(\tau)$ are continuous for any $\tau \geqslant 0$ and

$$\lim_{\tau\to\infty} A_j(\tau) = A_j^s, \quad \lim_{\tau\to\infty} C_j(\tau) = C_j^s, \tag{17}$$

then

$$\lim_{\tau\to\infty} A_j(\xi,\tau) = A^s(\xi), \quad \lim_{\tau\to\infty} C_j(\xi,\tau) = C^s(\xi),$$

uniformly for any $\xi \in [0,1]$, where $A^s(\xi)$, $C^s(\xi)$ is stationary solution (16). If

$$|\, A_j(\tau) - A_j^s \,| \leqslant d(1+\tau)^{-\mu}, \quad |\, C_j(\tau) - C_j^s \,| \leqslant d(1+\tau)^{-\mu}, \tag{18}$$

with positive coefficients d, μ then

$$|\, A^s(\xi,\tau) - A^s(\xi) \,| \leqslant d_1(1+\tau)^{-\mu}, \quad |\, C^s(\xi,\tau) - C^s(\xi) \,| \leqslant d_1(1+\tau)^{-\mu}, \tag{19}$$

$d_1 > 0$ is a constant, $\xi \in [0,1]$. Considering inequalities

$$|\, A_j(\tau) - A_j^s \,| \leqslant d_2 e^{-\mu\tau}, \quad |\, C_j(\tau) - C_j^s \,| \leqslant d_2 e^{-\mu\tau}, \tag{20}$$

estimates

$$|\, A^s(\xi,\tau) - A^s(\xi) \,| \leqslant d_3 e^{-\mu_1\tau}, \quad |\, C^s(\xi,\tau) - C^s(\xi) \,| \leqslant d_3 e^{-\mu_1\tau}, \tag{21}$$

are obtained with constants $d_3 > 0$, $0 < \mu_1 \leqslant \mu$.

These estimates can be interpreted as the stability conditions of stationary solution $A^s(\xi)$, $C^s(\xi)$ under conditions (17), (18), (20).

Initial-boundary inverse problems for functions $F(\xi,\tau)$, $N_1(\tau)$ and $H(\xi,\tau)$, $N_2(\tau)$ are non classical ($A(\xi,\tau)$, $C(\xi,\tau)$ are known). Therefore, a priori estimates of their solutions have to be obtained.

Multiplying the last equation of system (15) by $H(\xi,\tau)$ and integrating with respect to ξ from zero to one, the following identity is obtained

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 H^2(\xi,\tau)d\xi + P\int_0^1 H_\xi^2(\xi,\tau)d\xi = -\eta P\int_0^1 H(\xi,\tau)\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\xi d\epsilon. \tag{22}$$

Here, boundary conditions (11) and redefinition condition (12) are taken into account. Since Steklov's inequality takes place

$$\int_0^1 H^2(\xi,\tau)d\xi \leqslant \frac{1}{\pi^2}\int_0^1 H_\xi^2(\xi,\tau)d\xi$$

then the left part of (22) is greater than or equal to

$$\frac{1}{2}\frac{d}{d\tau}\int_0^1 H^2(\xi,\tau)d\xi + \pi^2 P\int_0^1 H^2(\xi,\tau)d\xi.$$

The right part of (22) does not exceed

$$\eta P\Big(\int_0^1 H^2(\xi,\tau)\Big)^{\frac{1}{2}}\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}}.$$

Now for $E(\tau) = \Big(\int\limits_0^1 H^2(\xi,\tau)\Big)^{\frac{1}{2}}$ the following inequality is obtained

$$\frac{dE}{d\tau} + \pi^2 PE \leqslant \eta P\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}\int_0^1 H^2(\xi,\tau)d\xi \leqslant \Big\{\Big(\int_0^1 H_0^2(\xi)d\xi\Big)^{\frac{1}{2}} + \eta P\int_0^\tau e^{\pi^2P^2\tau}\Big[\int_0^1\int_0^\xi (A(\epsilon,\tau)-\\ -C(\epsilon,\tau))^2 d\xi d\epsilon\Big]^{\frac{1}{2}} d\tau\Big\}^2 e^{-2\pi^2P\tau} \equiv G_1(\tau)e^{-\pi^2P\tau}\end{aligned} \tag{23}$$

for any $\tau\in[0,\tau_0]$. Now recall that functions $A(\xi,\tau)$, $C(\xi,\tau)$ satisfy estimates (19) or (21), where $A^s(\xi)=0$, $C^s(\xi)=0$.

Function $H(\xi,\tau)$ also satisfies the following identity

$$\int_0^1 H^2(\xi,\tau)d\xi + \frac{P}{2}\frac{d}{d\tau}\int_0^1 H_\xi^2(\xi,\tau)d\xi = -\eta P\int_0^1 H_\tau(\xi,\tau)d\xi\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\xi d\epsilon.$$

Using the elementary inequality $ab\leqslant \epsilon_1 a^2/2 + b^2/(2\epsilon_1)$ when $\epsilon_1 = (2\eta P)^{-1}$, one can obtain from the previous identity that

$$\begin{aligned}\int_0^1 H_\xi^2(\xi,\tau)d\xi \leqslant 2\eta^2P^2\int_0^\tau\int_0^1\Big[\int_0^\xi (A(\epsilon,\tau)-C(\epsilon,\tau))d\epsilon\Big]^2 d\xi d\tau+\\ +\int_0^1 H_{0\xi}^2(\xi)d\xi \equiv G_2(\tau)\end{aligned} \tag{24}$$

Since $H(0,\tau)=0$ then

$$H^2(\xi,\tau)=2\int_0^{\xi} H(\xi,\tau)H_\xi(\xi,\tau)d\xi \leqslant$$
$$\leqslant 2\Big(\int_0^1 H^2(\xi,\tau)d\xi\Big)^{\frac{1}{2}}\Big(\int_0^1 H_\xi^2(\xi,\tau)d\xi\Big)^{\frac{1}{2}} \leqslant 2\sqrt{G_1(\tau)G_2(\tau)}e^{-\pi^2 P\tau}$$

due to inequalities (23), (24) and

$$\mid H(\xi,\tau)\mid \leqslant \sqrt{2}(G_1(\tau)G_2(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau} \tag{25}$$

for any $\xi\in[0,1]$, $\tau\in[0,\tau_0]$.

Similar estimate holds for $F(\xi,\tau)$ if $A(\xi,\tau)-C(\xi,\tau)$ is replaced with $A(\xi,\tau)+C(\xi,\tau)$ in expressions $G_1(\tau)$, $G_2(\tau)$, and they are denoted by $G_3(\tau)$ and $G_4(\tau)$. Therefore

$$\mid F(\xi,\tau)\mid \leqslant \sqrt{2}(G_3(\tau)G_4(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau}. \tag{26}$$

Let us start first with the evaluation of $N_2(\tau)$. Multiplying the equation for $H(\xi,\tau)$ by $\xi-\xi^2$, integrating over the interval $[0,1]$ and using the boundary conditions, one can obtain

$$N_2(\tau)=6\int_0^1(\xi-\xi^2)H_\tau(\xi,\tau)d\xi+6\int_0^1(\xi-\xi^2)\int_0^{\xi}(A(\varepsilon,\tau)-C(\varepsilon,\tau))d\varepsilon d\xi, \tag{27}$$

since $\int\limits_0^1(\xi-\xi^2)H_{\xi\xi}(\xi,\tau)d\xi=0$. To evaluate $N_2(\tau)$ it is necessary to obtain an estimate of $\mid H_\tau(\xi,\tau)\mid$ at $\xi\in[0,1]$, $\tau\in[0,\tau_0]$. If

$$\begin{aligned} &\mid A_j(\tau)\mid \leqslant d_2e^{-\mu\tau}, \quad \mid C_j(\tau)\mid \leqslant d_2e^{-\mu\tau},\\ &\mid A_{j\tau}(\tau)\mid \leqslant d_4e^{-\mu\tau}, \quad \mid C_{j\tau}(\tau)\mid \leqslant d_4e^{-\mu\tau}, \end{aligned} \tag{28}$$

$d_4>0$ then

$$\begin{aligned} &\mid A(\xi,\tau)\mid \leqslant d_3e^{-\mu_1\tau}, \quad \mid C(\xi,\tau)\mid \leqslant d_3e^{-\mu_1\tau},\\ &\mid A_\tau(\xi,\tau)\mid \leqslant d_5e^{-\mu_1\tau}, \quad \mid C_\tau(\xi,\tau)\mid \leqslant d_5e^{-\mu_1\tau} \end{aligned} \tag{29}$$

for any $\xi\in[0,1]$, $\tau\in[0,\tau_0]$. The first two equations of system (15) provide estimates of derivatives

$$\mid A_{\xi\xi}(\xi,\tau)\mid \leqslant d_5e^{-\mu_1\tau}, \quad \mid C_{\xi\xi}(\xi,\tau)\mid \leqslant d_5e^{-\mu_1\tau}. \tag{30}$$

To obtain estimates of derivatives (29), (30) it is enough to differentiate with respect to τ the corresponding initial boundary value problems, and use the results presented in [13]. Similarly, differentiating with respect to τ the last equation of system (15), a problem on $H_\tau(\xi,\tau)$ is obtained. It is similar to the problem on $H(\xi,\tau)$ when $A(\xi,\tau)-C(\xi,\tau)$ is replaced with $A_\tau(\xi,\tau)-C_\tau(\xi,\tau)$ and $N_2(\tau)$ is replaced with $N_{2\tau}(\tau)$. Therefore, there is an estimate (see (25))

$$\mid H_\tau(\xi,\tau)\mid \leqslant \sqrt{2}(G_3(\tau)G_4(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau}, \tag{31}$$

$\xi\in[0,1]$, $\tau\in[0,\tau_0]$, where $H_0(\xi)$ is replaced with $H_\tau(\xi,0)$ in relation for $G_3(\tau)$ (see (23)). Then

$$H_\tau(\xi,0)=\frac{1}{P}H_{0\xi\xi}(\xi)-\eta P\int_0^{\xi}(A_0(\xi)-C_0(\xi))d\xi+N_2(0). \tag{32}$$

The value of $N_2(0)$ can be found from another representation of $N_2(\tau)$:

$$N_2(\tau) = \frac{1}{P}(H_\xi(0,\tau) - H_\xi(1,\tau)) + \eta P \int_0^1 \int_0^\epsilon (A(\xi,\tau) - C(\xi,\tau)) d\xi d\epsilon.$$

Thus

$$N_2(0) = \frac{1}{P}(H_{0\xi}(0) - H_{0\xi}(1)) + \eta P \int_0^1 \int_0^\epsilon (A_0(\xi) - C_0(\xi)) d\xi d\epsilon.$$

Considering (27) and using inequalities (31), (29), the following estimate is obtained

$$| N_2(\tau) | \leqslant \frac{3}{\sqrt{2}}[(G_3(\tau)G_4(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau} + 4d_3 e^{-\mu_1\tau}], \quad \tau \in [0,\tau_0]. \tag{33}$$

A similar assessment takes place for $H_\tau(\xi,\tau)$, $N_1(\tau)$

$$\begin{gathered} | F_\tau(\xi,\tau) | \leqslant \sqrt{2}(G_5(\tau)G_6(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau}, \\ | N_1(\tau) | \leqslant \frac{3}{\sqrt{2}}[(G_5(\tau)G_6(\tau))^{\frac{1}{4}} e^{-\frac{\pi^2 P}{2}\tau} + 4d_3 e^{-\mu_1\tau}], \quad \tau \in [0,\tau_0]. \end{gathered} \tag{34}$$

where $G_5(\tau)$ and $G_6(\tau)$ follow from $G_3(\tau)$ and $G_4(\tau)$ when the term $A(\xi,\tau)+C(\xi,\tau)$ is replaced with $A_\tau(\xi,\tau) + C_\tau(\xi,\tau)$, and $F_0(\xi)$ is replaced with $F_\tau(\xi,0)$. Moreover (see (32))

$$F_\tau(\xi,0) = \frac{1}{P}F_{0\xi\xi}(\xi) - \eta P \int_0^\xi (A_0(\xi) + C_0(\xi)) d\xi + N_1(0),$$

$$N_1(0) = \frac{1}{P}(F_{0\xi}(0) - F_{0\xi}(1)) + \eta P \int_0^1 \int_0^\epsilon (A_0(\xi) + C_0(\xi)) d\xi d\epsilon.$$

Thus, if $A_j(\tau),\ C_j(\tau) \in C^1[0,\tau_0]$ and inequalities (28) are satisfied then solution of inverse initial boundary value problem (15), (10) and (14) satisfies a priori estimates (25), (26), (30)–(34). In addition, similarly to estimates (30), $F_{\xi\xi}(\xi,\tau)$, $H_{\xi\xi}(\xi,\tau)$ are bounded for any $\xi \in [0,1]$, $\tau \in [0,\tau_0]$.

Remark 5. *If $A_j(\tau),\ C_j(\tau) \in C^1[0,\tau_0]$, $A_0(\xi),\ C_0(\xi) \in C^2[0,1]$ then it follows from the maximum principle for parabolic equations that*

$$| A(\xi,\tau) | \leqslant \max\Big[\max_{\xi\in[0,1]} | A_0(\xi) |, \max_{\tau\in[0,\tau_0]} | A_j(\tau) | \Big],$$

$$| C(\xi,\tau) | \leqslant \max\Big[\max_{\xi\in[0,1]} | C_0(\xi) |, \max_{\tau\in[0,\tau_0]} | C_j(\tau) | \Big],$$

$$| A_\tau(\xi,\tau) | \leqslant \max\Big[\max_{\xi\in[0,1]} | A_{0\xi\xi}(\xi) |, \max_{\tau\in[0,\tau_0]} | A_{j\tau}(\tau) | \Big],$$

$$| C_\tau(\xi,\tau) | \leqslant \max\Big[\max_{\xi\in[0,1]} | C_{0\xi\xi}(\xi) |, \max_{\tau\in[0,\tau_0]} | C_{j\tau}(\tau) | \Big].$$

Therefore, the boundedness of $| F(\xi,\tau) |$, $| H(\xi,\tau) |$, $| F_\tau(\xi,\tau) |$, $| H_\tau(\xi,\tau) |$, $| F_{\xi\xi}(\xi,\tau) |$, $| H_{\xi\xi}(\xi,\tau) |$, $| N_1(\tau) |$, $| N_2(\tau) |$, with $\xi \in [0,1]$, $\tau \in [0,\tau_0]$ takes place for weaker conditions on functions $A_j(\tau)$, $C_j(\tau)$.

Relations for $G_1(\tau)$, $G_3(\tau)$, $G_5(\tau)$ contain integrals of exponent $e^{\pi^2 P\tau}$. Therefore, the use of a priori estimates for the behaviour of the solution at $\tau \gg 1$ requires the fulfilment of conditions (28), so that there are estimates (29) with some constant $\mu > 0$. Let us assume that $A_j(\tau)$, $C_j(\tau)$, $A_{j\tau}(\tau)$, $A_{j\tau}(\tau)$ are defined and continuously differentiable for all $\tau \geqslant 0$. If $\mu_1 = P\pi^2 + \gamma$,

$\gamma > 0$ then the specified integrals in relations for $G_1(\tau)$, $G_3(\tau)$, $G_5(\tau)$ and in the right-hand sides of inequalities (25), (26), (30)–(34) converge exponentially to zero.

Let us assume that inequalities (28) and estimates for derivatives (29) are satisfied. Considering the differences $F(\xi,\tau) - F^s(\xi)$, $H(\xi,\tau) - H^s(\xi)$, $N_j(\tau) - N_j^s$, $j = 1, 2$, let us ensure that they satisfy the same initial boundary value problems as $F(\xi,\tau)$, $H(\xi,\tau)$, $N_j(\tau)$. The difference is only in the initial conditions. They are replaced with $F_0(\xi) - F^s(\xi)$, $H_0(\xi) - H^s(\xi)$, $N_j(0) - N_j^s$, respectively. Therefore, the estimates follow from given above inequalities ($\mu_1 = \pi^2 + \gamma$)

$$(|\,F(\xi,\tau) - F^s(\xi)\,|, \quad |\,H(\xi,\tau) - H^s(\xi)\,|, \quad |\,F_\tau(\xi,\tau)\,|, \ |\,H_\tau(\xi,\tau)\,|,$$
$$|\,N_j(\tau) - N_j^s\,|) \leqslant De^{-\frac{\pi^2}{2}\tau}$$

with some constant $D > 0$.

Therefore, stationary solution (15) is exponentially stable under the given above conditions.

Solution of non-stationary problem by the Laplace method

Non-stationary solution of problem (10)–(12), (15) is found using the integral Laplace transform [14]. In our case, the method reduces the solution of non-stationary partial differential problem to the solution of a system of ordinary differential equations (ODEs).

Applying the Laplace transform to the initial boundary value problem

$$A_\tau = A_{\xi\xi},$$
$$A(\xi, 0) = A_0(\xi),$$
$$A(0,\tau) = A_1(\tau), \quad A(1,\tau) = A_2(\tau),$$

the following system of ODEs for the Laplace images is obtained

$$\hat{A}_{\xi\xi} - s\hat{A} = -A_0(\xi),$$
$$\hat{A}(0,s) = \hat{A}_1(s), \quad \hat{A}(1,s) = \hat{A}_2(s). \tag{35}$$

Taking into account (35), one can find $\hat{A}(\xi,s)$

$$\hat{A}(\xi,s) = \frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\hat{A}_2(s) + \frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})}\hat{A}_1(s) +$$
$$+\frac{1}{\sqrt{s}}\left[\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\int_0^1 A_0(\xi)\,\mathrm{sh}(\sqrt{s}(1-\xi))d\xi - \int_0^\xi A_0(\varepsilon)\,\mathrm{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right]. \tag{36}$$

Similarly, function $\hat{C}(z,s)$ is defined as

$$\hat{C}(\xi,s) = \frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\hat{C}_2(s) + \frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})}\hat{C}_1(s) +$$
$$+\frac{1}{\sqrt{s}}\left[\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\int_0^1 C_0(\xi)\,\mathrm{sh}(\sqrt{s}(1-\xi))d\xi - \int_0^\xi C_0(\varepsilon)\,\mathrm{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right]. \tag{37}$$

Therefore, $\hat{A}(\xi,s)$ and $\hat{C}(\xi,s)$ are known functions. Similarly, function $\hat{\Theta}(\xi,s)$ is

$$\hat{\Theta}(\xi,s) = \frac{2}{\sqrt{s}}\left(\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\int_0^1 (\hat{A}(\xi,s) + \hat{C}(\xi,s))\,\mathrm{sh}(\sqrt{s}(1-\xi))d\xi -\right.$$
$$\left. - \int_0^\xi (\hat{A}(\varepsilon,s) + \hat{C}(\varepsilon,s))\,\mathrm{sh}(\sqrt{s}(\xi-\varepsilon))d\varepsilon\right). \tag{38}$$

Equation for function $F(\xi,\tau)$ in Laplace images has the form

$$\hat{F}_{\xi\xi} - \frac{s}{P}\hat{F} = \eta\int_0^{\xi}(\hat{A}(\varepsilon,s)+\hat{C}(\varepsilon,s))d\varepsilon - \frac{1}{P}\hat{N}_1(s) - F_0(\xi),$$
$$\hat{F}(0,s) = \hat{F}(1,s) = 0. \tag{39}$$

Then solution of problem (39) is

$$\hat{F}(\xi,s) = \frac{\sqrt{P}\eta}{\sqrt{s}}\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P})}\int_0^1\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\times$$
$$\times\,\text{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon + \frac{P\eta}{s}\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\times$$
$$\times\,\text{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon - \frac{\text{ch}(\sqrt{s/P}\xi)-1}{s}(\hat{N}_1(s)-PF_0(\xi)) + P\frac{\text{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi +$$
$$+\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P}}\frac{\text{ch}(\sqrt{s/P})-1}{s}(\hat{N}_1(s)-PF_0(\xi)) - P\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P}}\frac{\text{sh}(\sqrt{s/P})}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi. \tag{40}$$

Let us find $\hat{N}_1(s)$ from (12). Introducing

$$r = (\text{sh}\,\sqrt{s/P}/\sqrt{s/P}-1)/P\sqrt{P} - \left((ch\sqrt{s/P}-1)/\sqrt{s/P}\right)^2/\sqrt{sP}\,\text{sh}\,\sqrt{s/P},$$

one can obtain

$$\hat{N}_1(s) = \frac{\eta\sqrt{P}}{r\sqrt{s}}\left[\int_0^1\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\text{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi -\right.$$
$$-\frac{\eta\sqrt{P}}{r\sqrt{s}}\frac{\text{ch}(\sqrt{s/P})-1}{\text{sh}(\sqrt{s/P})}\int_0^1\int_0^{\xi}(\hat{A}(\epsilon,s)+\hat{C}(\epsilon,s))\,\text{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi - \tag{41}$$
$$\left. -PF_0(\xi) + P\frac{\text{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}F_0(\xi)d\xi\right].$$

Similarly, find function $\hat{H}(\xi,s)$

$$\hat{H}(\xi,s) = \frac{\sqrt{P}\eta}{\sqrt{s}}\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P})}\int_0^1\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\times$$
$$\times\,\text{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon + \frac{P\eta}{s}\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\times$$
$$\times\,\text{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon - \frac{\text{ch}(\sqrt{s/P}\xi)-1}{s}(\hat{N}_1(s)-PH_0(\xi)) + P\frac{\text{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi +$$
$$+\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P})}\frac{\text{ch}(\sqrt{s/P})-1}{s}(\hat{N}_2(s)-PH_0(\xi)) - P\frac{\text{sh}(\sqrt{s/P}\xi)}{\text{sh}(\sqrt{s/P})}\frac{\text{sh}(\sqrt{s/P})}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi. \tag{42}$$

Function $\hat{N}_2(s)$ is defined from (12) as follows

$$\hat{N}_2(s) = \frac{\eta\sqrt{P}}{r\sqrt{s}}\left[\int_0^1\int_0^{\xi}\int_0^{\varepsilon}(\hat{A}(\zeta,s)-\hat{C}(\zeta,s))\,\text{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi -\right.$$
$$-\frac{\eta\sqrt{P}}{r\sqrt{s}}\frac{\text{ch}(\sqrt{s/P})-1}{\text{sh}(\sqrt{s/P})}\int_0^1\int_0^{\xi}(\hat{A}(\epsilon,s)-\hat{C}(\epsilon,s))\,\text{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi - \tag{43}$$
$$\left. -PH_0(\xi) + P\frac{\text{sh}(\sqrt{s/P}\xi)}{\sqrt{s}}\int_0^{\xi}H_0(\xi)d\xi\right].$$

Conditions for tendency of non-stationary solution to a given stationary solution

Suppose there are limits

$$\lim_{\tau\to\infty} A_j(\tau) = A_j^0, \quad \lim_{\tau\to\infty} C_j(\tau) = C_j^0, \; j = 1, 2, \tag{44}$$

and derivatives $A_j'(\tau)$, $C_j'(\tau)$ have Laplace images. Then [14]

$$\lim_{s\to 0} s\hat{A}_j(s) = \lim_{\tau\to\infty} A_j(\tau) = C_j^0, \quad \lim_{s\to 0} s\hat{C}_j(s) = \lim_{\tau\to\infty} C_j(\tau) = C_j^0. \tag{45}$$

Next, asymptotic expressions when $t \to 0$ for functions $\mathrm{sh}(t)$ and $\mathrm{ch}(t)$ are used: $\mathrm{sh}(t) \sim t + t^3/6, \; \mathrm{ch}(t) \sim 1 + t^2/2$.

The proof is given for function $\hat{\Theta}(\xi, s)$. The following relation is obtained for $s \to 0$

$$\begin{gathered} s\hat{\Theta}(\xi,s) \sim \frac{2}{\sqrt{s}}\Bigg(\xi\int_0^1 (s\hat{A}(\xi,s) + s\hat{C}(\xi,s))\left[(\sqrt{s}(1-\xi)) + \frac{(\sqrt{s}(1-\xi))^3}{6}\right]d\xi - \\ - \int_0^\xi (s\hat{A}(\varepsilon,s) + s\hat{C}(\varepsilon,s))\left[(\sqrt{s}(\xi-\varepsilon)) + \frac{(\sqrt{s}(\xi-\varepsilon))^3}{6}\right]d\varepsilon\Bigg) \sim \\ \sim \left(\frac{1}{3}[A_2^0 + C_2^0 - (A_1^0 + C_1^0)] + A_1^0 + C_1^0\right)\xi - \\ -\left(\frac{1}{3}[A_2^0 + C_2^0 - (A_1^0 + C_1^0)]\xi^3 + [A_1^0 + C_1^0]\xi^2\right) = \Theta^s(\xi). \end{gathered}$$

Lemma 1. *Under conditions (44), (45) the non-stationary solution of problem (10),(11), (12), (15) approaches stationary solution (16) when dimensionless time τ increases.*

Finding the originals of required functions

Functions (36), (37) are Laplace images. The inverse Laplace transform is used to determine the originals.

It is assumed that $A_j(\tau)$, $C_j(\tau)$ have the form

$$A_j(\tau) = A_j^0 + \epsilon_{j1}\exp[-\gamma_{j1}\tau]\sin(\omega_1\tau), \quad C_j(\tau) = C_j^0 + \epsilon_{j2}\exp[-\gamma_{j2}\tau]\sin(\omega_2\tau).$$

Then, their images are easily found from the Laplace transform table [15]

$$\hat{A}_j(s) = \frac{A_j^0}{s} + \frac{\epsilon_{j1}\omega_1}{(s+\gamma_{j1})^2 + \omega_1^2}, \quad \hat{C}_j(s) = \frac{C_j^0}{s} + \frac{\epsilon_{j2}\omega_2}{(s+\gamma_{j2})^2 + \omega_2^2}, \tag{46}$$

where $\gamma_{j1} > 0$, $\gamma_{j2} > 0$, i.e., the boundary mode is stabilized with time according to Lemma 1. If one of the values of γ_{j1}, γ_{j2} is negative then there is no stabilization effect of the solution.

At this point, for simplicity, it is assumed that motion arises from the state of rest and $A_0(\xi) = C_0(\xi) = 0$. In this case, compatibility conditions (13) are violated since $A_1(0) \neq A_0(0) = 0$, $C_1(0) \neq C_0(0) = 0$, that is, there are discontinuities of the 1st kind. This is acceptable since the integral Laplace transform is applicable for functions that have a finite number of discontinuities of the 1st kind [15].

Expressions for $\hat{A}(\xi,s)$, $\hat{C}(\xi,s)$, $\hat{F}(\xi,s)$, $\hat{N}_1(s)$ are simplified as

$$\hat{A}(\xi,s)=\left(\frac{A_2^0}{s}+\frac{\epsilon_{12}\omega_1}{(s+\gamma_{12})^2+\omega_1^2}\right)\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}+\left(\frac{A_1^0}{s}+\frac{\epsilon_{11}\omega_1}{(s+\gamma_{11})^2+\omega_1^2}\right)\frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})},$$

$$\hat{C}(\xi,s)=\left(\frac{C_2^0}{s}+\frac{\epsilon_{22}\omega_2}{(s+\gamma_{22})^2+\omega_2^2}\right)\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}+\left(\frac{C_1^0}{s}+\frac{\epsilon_{21}\omega_2}{(s+\gamma_{21})^2+\omega_2^2}\right)\frac{\mathrm{sh}(\sqrt{s}(1-\xi))}{\mathrm{sh}(\sqrt{s})},$$

$$\hat{F}(\xi,s)=\frac{\eta}{\sqrt{sP}}\left[\int_0^\xi\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon-\right.$$
$$\left.-\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(\xi-\varepsilon))d\zeta d\varepsilon\right]-$$
$$-\frac{\mathrm{ch}(\sqrt{s/P}\xi)-1}{s}\hat{N}_1(s)+\frac{\mathrm{sh}(\sqrt{s/P}\xi)}{\mathrm{sh}(\sqrt{s/P})}\frac{\mathrm{ch}(\sqrt{s/P})-1}{s}\hat{N}_1(s),$$

$$\hat{N}_1(s)=\frac{\eta}{r\sqrt{sP}}\left[\int_0^1\int_0^\xi\int_0^\varepsilon(\hat{A}(\zeta,s)+\hat{C}(\zeta,s))\,\mathrm{sh}(\sqrt{s/P}(1-\varepsilon))d\zeta d\varepsilon d\xi-\right.$$
$$\left.-\frac{\mathrm{ch}(\sqrt{s/P})-1}{\mathrm{sh}(\sqrt{s/P})}\int_0^1\int_0^\xi(\hat{A}(\epsilon,s)+\hat{C}(\epsilon,s))\,\mathrm{sh}(\sqrt{s/P}(1-\xi))d\epsilon d\xi\right].$$

Expressions for $\hat{H}(\xi,s)$, $\hat{N}_2(s)$ have the same form only terms $\hat{A}(\zeta,s)+\hat{C}(\zeta,s)$ are replaced by $\hat{A}(\zeta,s)-\hat{C}(\zeta,s)$.

Function $\hat{\Theta}(\xi,s)$ has the following form

$$\hat{\Theta}(\xi,s)=\frac{2}{\sqrt{s}}\left(\frac{\mathrm{sh}(\sqrt{s}\xi)}{\mathrm{sh}(\sqrt{s})}\int_0^1(\hat{A}(\eta,s)+\hat{C}(\eta,s))\,\mathrm{sh}(\sqrt{s}(1-\eta))d\eta-\right.$$
$$\left.-\int_0^\xi(\hat{A}(\eta,s)+\hat{C}(\eta,s))\,\mathrm{sh}(\sqrt{s}(\xi-\eta))d\eta\right).$$

After numerical inversion of the Laplace transform functions $N_j(\tau)$ are obtain (see Fig. 3), where $A_2^0=1.3$, $A_1^0=1$, $C_2^0=2.7$, $C_1^0=2$, $\epsilon_{12}=1.2$, $\epsilon_{11}=1$, $\epsilon_{22}=1.6$, $\epsilon_{21}=1.8$, $\omega_1=0.1$, $\omega_2=0.2$, $\gamma_{12}=0.04$, $\gamma_{11}=0.03$, $\gamma_{22}=0.07$, $\gamma_{21}=0.06$, $\chi=0.00143\ \mathrm{m}^2/\mathrm{sec}$, $\nu=0.01006\ \mathrm{m}^2/\mathrm{sec}$, $\beta=1.82\cdot10^{-4}$ 1/deg, $l=10^{-4}$ m, $g=9,81\ \mathrm{m/sec}^2$.

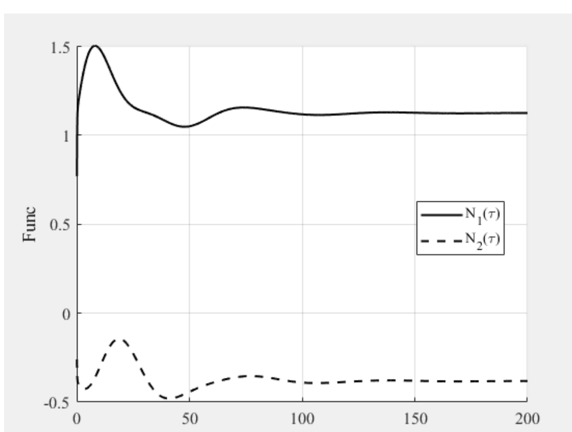

Fig. 3. Functions $N_j(\tau)$ versus dimensionless time

Dimensionless velocities

$$time\bar{u}(\xi,\tau) = \frac{l}{\chi Re}u = (F+H)\bar{x}, \quad \bar{v}(\xi,\tau) = \frac{l}{\chi Re}u = (F-H)\bar{y},$$

are shown in Fig. 4, 5 ($\bar{x} = \bar{y} = 1$).

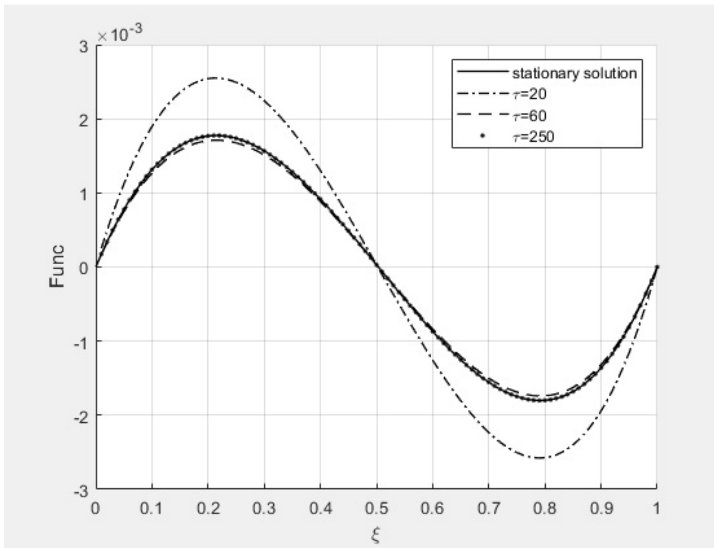

Fig. 4. Velocity $\bar{u}(\xi,\tau)$ as a function of dimensionless coordinate

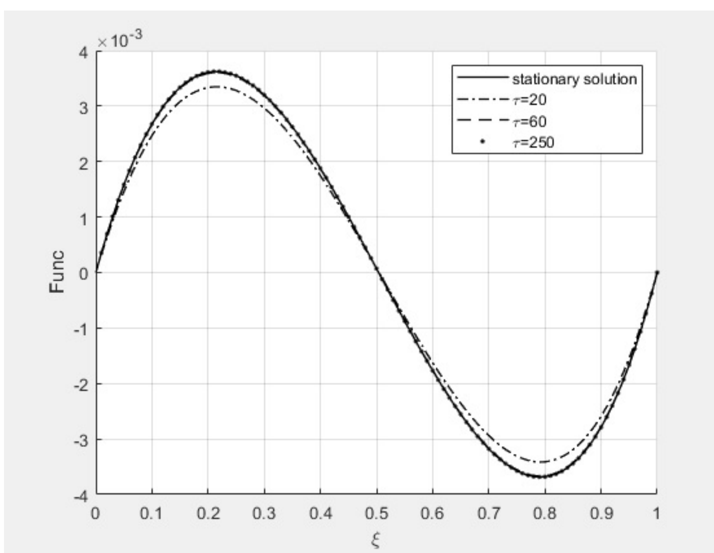

Fig. 5. Velocity $\bar{v}(\xi,\tau)$ as a function of dimensionless coordinate

Fig. 3 clearly shows that functions $N_j(\tau)$ approach constant values with increasing time. Figs. 4 and 5 show velocities along the x and y axes. One can see that distribution of velocities practically coincides with stationary distribution of velocities for large τ.

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Ползучее трехмерное конвективное движение в слое с полем скоростей специального вида

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Аннотация. Исследована задача о трехмерном движении теплопроводной жидкости в канале твердыми параллельными стенками, на которых поддерживается заданное распределение температуры. Температура в жидкостях квадратично зависит от горизонтальных координат, а поле скоростей имеет специальный вид. Возникающая начально-краевая задача для модели Обербека–Буссинеска является обратной и редуцирована к системе пяти интегродифференциальных уравнений. При малых числах Рейнольдса (ползущие движения) полученная система становится линейной. Для этой системы найдено стационарное решение, получены априорные оценки. На их основе установлены достаточные условия экспоненциальной сходимости гладкого нестационарного решения к стационарному режиму. В изображениях по Лапласу решение обратной задачи построено в виде квадратур, при более слабых условиях на температурный режим на стенках слоя. Приведены результаты расчетов, на основе численного обращения преобразования Лапласа, поведения поля скоростей для конкретной жидкой среды.

Ключевые слова: модель Обербека–Буссинеска, трехмерное движение, обратная задача, априорные оценки, устойчивость, преобразование Лапласа.