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On Maximal Operators Associated with a Family of Singular Surfaces

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Abstract. Maximal operator associated with singular surfaces is considered in this paper. The boundedness of this operator in the space of summable functions is proved when singular surfaces are given by parametric equations. Boundedness index of the maximal operator is also found for these spaces.

Keywords: maximal operator, averaging operator, fractional power series, singular surface, boundedness indicator.

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1. Introduction and preliminaries

The purpose of the paper is to study the boundedness of maximal operators defined by

$$\mathcal{M}f(y) := \sup_{t>0} \mid \mathcal{A}_t f(y) \mid, \tag{1}$$

where

$$\mathcal{A}_t f(y) := \int_S f(y - tx)\psi(x)dS(x) \tag{2}$$

is so called averaging operator, $S \subset \mathbb{R}^{n+1}$ is a hyper-surface, $\psi \geqslant 0$ is a fixed smooth function with compact support, i.e., $\psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $f \in C_0^\infty(\mathbb{R}^{n+1})$.

Maximal operator (1) is bounded in $L^p := L^p(\mathbb{R}^{n+1})$ if there exists a number $C > 0$ such that for any function $f \in C_0^\infty(\mathbb{R}^{n+1})$ the L^p inequality $\|\mathcal{M}f\|_{L^p} \leqslant C\,\|f\|_{L^p}$ holds.

For a hyper-surface S and for a fixed function $0 \leqslant \psi \in C_0^\infty(\mathbb{R}^{n+1})$ a critical exponent of maximal operator (1) is defined by

$$p(S) := \inf\{p : operator\,(1)\,is\,bounded\,in\,L^p\}.$$

Firstly, it was showed that when S is the unit $(n-1)$-dimensional sphere centred at the origin then maximal operator (1) is bounded in $L^p(\mathbb{R}^n)$ for $p > \dfrac{n}{n-1}, n \geqslant 3$ and it is not bounded in $L^p(\mathbb{R}^n)$ whenever $p \leqslant \dfrac{n}{n-1}$ [1]. The two dimensional case of this result was proved by J. Bourgain [2].

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It was proved that maximal operator (1) is bounded in $L^p(\mathbb{R}^{n+1})$ for $n \geqslant 2$ and $p > (n+1)/n$ when hyper-surface has everywhere non-vanishing Gaussian curvature [3]. Moreover, it was showed that if hyper-surface has at least $k (k \geqslant 2)$ non-vanishing principal curvatures then the maximal operator is bounded in $L^p(\mathbb{R}^{n+1})$ $(n \geqslant 2)$ for all $p > (k+1)/k$. A similar result for more difficult case $k = 1$ was obtained by C. D. Sogge [4].

Also, maximal operators (1) were considered in [5–11]. Maximal operators associated with smooth hyper-surfaces in $\mathbb{R}^{n+1}$ were studied and critical exponent of these operators in $L^p(\mathbb{R}^{n+1})$ was defined [12]. The boundedness of the maximal operators related to singular surfaces in 3-dimensional Euclidean space was investigated [13] and [14].

2. Problem statement

Let us consider a family of singular surfaces in $\mathbb{R}^3$ defined by the following parametric equations

$$x_1(u_1,u_2) = r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2), \quad x_2(u_1,u_2) = r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2),$$
$$x_3(u_1,u_2) = r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2), \tag{3}$$

where r_1, r_2, r_3 are any real numbers, $a_1, a_2, b_1, b_2, c_1, c_2$ are non-negative rational numbers, $u_1 \geqslant 0$, $u_2 \geqslant 0$ and $\{g_k(u_1,u_2)\}_{k=1}^3$ are fractional power series. For the definition of the fractional power series see [13] and [15].

Let us introduce the following designations

$$B_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad B_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Remark 1. *If at least one of the numbers B_1, B_2, B_3 is nonzero then the points of surface* (3) *that lie in a sufficiently small neighborhood of the singular point (r_1, r_2, r_3) outside the coordinate planes of a coordinate system which has its origin at the point (r_1, r_2, r_3) are non-singular. Points of surface* (3) *that lie on the coordinate planes of this coordinate system are singular points* (see definition 2 in [13]).

Let us define the averaging operator in (2) associated with surfaces (3) in the form

$$\mathcal{A}_t^{\varphi} f(y) = \int\limits_{\mathbb{R}_+^2} f\Big(y_1 - t\big(r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2)\big), y_2 - t\big(r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2)\big),$$
$$y_3 - t\big(r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2)\big)\Big)\psi_1(u_1,u_2)u_1^{d_1}u_2^{d_2}\varphi(u_1,u_2)du_1du_2, \tag{4}$$

where $\varphi(u_1,u_2)$ is fractional power series such that $\varphi(0,0) \neq 0$,

$$\psi_1(u_1,u_2) = \psi\big(r_1 + u_1^{a_1}u_2^{a_2}g_1(u_1,u_2),\ r_2 + u_1^{b_1}u_2^{b_2}g_2(u_1,u_2),\ r_3 + u_1^{c_1}u_2^{c_2}g_3(u_1,u_2)\big)$$

is non-negative fractional power series with a sufficiently small support, d_1, d_2 are real numbers and $f \in C_0^\infty(\mathbb{R}^3)$. The purpose is to prove L^p inequality for the maximal operator defined by

$$\mathcal{M}^{\varphi} f(y) := \sup_{t>0} |\mathcal{A}_t^{\varphi} f(y)|, \quad y \in \mathbb{R}^3.$$

Here this maximal operator is investigated in a small neighbourhood of singular point (r_1, r_2, r_3) of surfaces (3) in the case when $p > 2$.

3. The boundedness of the maximal operator $\mathcal{M}^{\varphi} f$

Let us denote the critical exponent of maximal operator $\mathcal{M}^{\varphi} f$ by $p'(S)$ and

$$p'(S) = \max\Big\{\frac{a_1}{d_1+1}, \frac{a_2}{d_2+1}, \frac{b_1}{d_1+1}, \frac{b_2}{d_2+1}, \frac{c_1}{d_1+1}, \frac{c_2}{d_2+1}\Big\}.$$

The extension of Theorem 1 in [13] and the main result of this paper is

Theorem 3.1. *Let $\varphi(u_1,u_2)$, $\{g_k(u_1,u_2)\}_{k=1}^{3}$ be fractional power series which defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ and it satisfy the following conditions: $\varphi(0,0) \neq 0$, $g_k(0,0) \neq 0$. Suppose $d_1 > -1$, $d_2 > -1$ and at least one of the following conditions is hold:*

1. $r_3 \neq 0, B_1 \neq 0$ and either $B_2B_3 \neq 0$, or $B_2(B_2+B_1) \neq 0$, or $B_3(B_3-B_1) \neq 0$;

2. $r_2 \neq 0, B_3 \neq 0$ and either $B_2B_1 \neq 0$, or $B_2(B_2-B_3) \neq 0$, or $B_1(B_1-B_3) \neq 0$;

3. $r_1 \neq 0, B_2 \neq 0$ and either $B_1B_3 \neq 0$, or $B_3(B_3-B_2) \neq 0$, or $B_1(B_1+B_2) \neq 0$.

Then there exists a small neighbourhood U of the singular point (r_1,r_2,r_3) such that for any function $\psi_1 \in C_0^{\infty}(U)$ the maximal operator $\mathcal{M}^{\varphi} f$ is bounded in $L^p(\mathbb{R}^3)$ for $p > \max\{p'(S), 2\}$. Moreover, if $\psi_1(0,0) = \psi(r_1,r_2,r_3) > 0$ and $p'(S) > 2$, then the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ whenever $2 < p \leqslant p'(S)$.

Proof. Suppose that condition 1 is satisfied and at least one of the numbers r_1, r_2 is not equal to zero. Let us consider the boundedness of the maximal operator $\mathcal{M}^{\varphi} f$ at non-singular points of surface (3) (see Remark 1).

Let us consider the partition of the unity $\sum\limits_{k=0}^{\infty} \chi_k(s) = 1$ on the interval $0 < s \leqslant 1$, where $\chi_k(s) := \chi(2^k s)$, $\chi \in C_0^{\infty}(\mathbb{R})$ supported on the interval $[0.5; 2]$ and $\chi_{j_1,j_2}(u_1,u_2) = \chi_{j_1}(u_1)\chi_{j_2}(u_2)$, $j_1, j_2 \in N$. Then averaging operator $A_t^{\varphi} f$ is decomposed as follows

$$\mathcal{A}_t^{\varphi,j_1,j_2} f(y) = \int\limits_{\mathbb{R}^2_+} f\Big(y_1 - t\big(r_1 + u_1^{a_1}u_2^{a_2} g_1(u_1,u_2)\big), y_2 - t\big(r_2 + u_1^{b_1}u_2^{b_2} g_2(u_1,u_2)\big),$$
$$y_3 - t\big(r_3 + u_1^{c_1}u_2^{c_2} g_3(u_1,u_2)\big)\Big)\psi_1(u_1,u_2)\chi_{j_1,j_2}(u_1,u_2) u_1^{d_1}u_2^{d_2}\varphi(u_1,u_2) du_1 du_2.$$

Next, by applying the change of variables $u_1 = 2^{-j_1}v_1$, $u_2 = 2^{-j_2}v_2$, one can obtain

$$A_t^{\varphi,j_1,j_2} f(y) = 2^{-(j_1+j_2)-(j_1d_1+j_2d_2)} \int\limits_{\mathbb{R}^2_+} f\Big(y_1 - t\big(r_1 + 2^{-(j_1a_1+j_2a_2)} v_1^{a_1}v_2^{a_2} \times$$
$$\times g_1(2^{-j_1}v_1, 2^{-j_2}v_2)\big), y_2 - t\big(r_2 + 2^{-(j_1b_1+j_2b_2)} v_1^{b_1}v_2^{b_2} g_2(2^{-j_1}v_1, 2^{-j_2}v_2)\big),$$
$$y_3 - t\big(r_3 + 2^{-(j_1c_1+j_2c_2)} v_1^{c_1}v_2^{c_2} g_3(2^{-j_1}v_1, 2^{-j_2}v_2)\big)\Big)\psi_1(2^{-j_1}v_1, 2^{-j_2}v_2)\chi(v_1)\chi(v_2)\times$$
$$\times v_1^{d_1}v_2^{d_2}\varphi(2^{-j_1}v_1, 2^{-j_2}v_2) dv_1 dv_2,$$

where $0.5 \leqslant v_1 \leqslant 2$, $0.5 \leqslant v_2 \leqslant 2$, $j_1, j_2 \geqslant j_0$, j_0 is a large number such that implies from the smallness of the support of ψ_1.

Let us change the variables as follows

$$\begin{cases} w_1 = v_1^{a_1}v_2^{a_2} g_1(2^{-j_1}v_1, 2^{-j_2}v_2) \\ w_2 = v_1^{b_1}v_2^{b_2} g_2(2^{-j_1}v_1, 2^{-j_2}v_2), \end{cases} \tag{5}$$

and assume that $g_1(0,0) = g_2(0,0) = 1$. Then in the first quadrant $\mathbb{R}^2_+$ the system

$$\begin{cases} w_1 = v_1^{a_1} v_2^{a_2} \\ w_2 = v_1^{b_1} v_2^{b_2} \end{cases},$$

yields

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \end{cases}. \tag{6}$$

In particular, relations (6) are valid in the set $\{(w_1, w_2) \in \mathbb{R}^2_+ : 2^{-(a_1+a_2)} \leqslant w_1 \leqslant 2^{a_1+a_2},\ 2^{-(b_1+b_2)} \leqslant w_2 \leqslant 2^{b_1+b_2}\}$.

Consequently, the change of variables

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1 \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2 \end{cases}, \tag{7}$$

is introduced, where $\hat{g}_1, \hat{g}_2$ are new variables and it is supposed that $\hat{g}_1 \sim 1$, $\hat{g}_2 \sim 1$. As a result system (5) implies

$$\begin{aligned} &(\hat{g}_1)^{a_1} (\hat{g}_2)^{a_2} g_1\Big(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1,\ 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2\Big) = 1, \\ &(\hat{g}_1)^{b_1} (\hat{g}_2)^{b_2} g_2\Big(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1,\ 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2\Big) = 1. \end{aligned} \tag{8}$$

According to the implicit function theorem, system (8) has a unique smooth solutions with respect to $\hat{g}_1, \hat{g}_2$ in a sufficiently small neighbourhood of the point $(0,0,1,1)$

$$\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1} \tilde{h}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2} \tilde{h}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2),$$

$$\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1} \tilde{\rho}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2} \tilde{\rho}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2).$$

Here $\tilde{h}_1, \tilde{h}_2, \tilde{\rho}_1, \tilde{\rho}_2$ are smooth functions. It is assumed that $\tilde{g}_1(0,0,1,1) = 1$, $\tilde{g}_2(0,0,1,1) = 1$.

Then taking into account (7), one can obtain

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2) \end{cases}. \tag{9}$$

Applying relations (5) and (9) to the last integral, we obtain

$$\mathcal{A}_t^{\varphi, j_1, j_2} f(y) = 2^{-(j_1+j_2)-(j_1 d_1 + j_2 d_2)} \int_{\mathbb{R}^2_+} f\Big(y_1 - t\big(r_1 + 2^{-(j_1 a_1 + j_2 a_2)} w_1\big),$$

$$y_2 - t\big(r_2 + 2^{-(j_1 b_1 + j_2 b_2)} w_2\big),\ y_3 - t\big(r_3 + 2^{-(j_1 c_1 + j_2 c_2)} \alpha(w_1, w_2)\big)\Big) \beta(w_1, w_2) dw_1 dw_2,$$

where $\alpha(w_1, w_2) = w_1^{\frac{-B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1, w_2)$,

$$g(w_1, w_2) = \big(\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2)\big)^{c_1} \big(\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2)\big)^{c_2} \times$$

$$\times g_3\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\beta(w_1,w_2)=\tilde{\psi}_1(w_1,w_2)\tilde{\chi}_1(w_1,w_2)\tilde{\chi}_2(w_1,w_2)(\varphi_1(w_1,w_2))^{d_1}(\varphi_2(w_1,w_2))^{d_2}\tilde{\varphi}(w_1,w_2)J(w_1,w_2),$$

$$\tilde{\psi_1}(w_1,w_2)=\psi_1\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\tilde{\chi}_1(w_1,w_2)=\chi\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\tilde{\chi}_2(w_1,w_2)=\chi\Big(2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big),$$

$$\varphi_1(w_1,w_2)=w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ \varphi_2(w_1,w_2)=w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2),$$

$$\tilde{\varphi}(w_1,w_2)=\varphi\Big(2^{-j_1}w_1^{\frac{b_2}{B_1}}w_2^{\frac{-a_2}{B_1}}\tilde{g}_1(2^{-j_1},2^{-j_2},w_1,w_2),\ 2^{-j_2}w_1^{\frac{-b_1}{B_1}}w_2^{\frac{a_1}{B_1}}\tilde{g}_2(2^{-j_1},2^{-j_2},w_1,w_2)\Big)$$

are fractional power series, $J(w_1,w_2)$ is the Jacobian of the change of variables (9). *** The dilation operators

$$T_1^{j_1,j_2}f(y):=2^{\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}f\Big(2^{j_1a_1+j_2a_2}y_1,\ 2^{j_1b_1+j_2b_2}y_2,\ 2^{j_1c_1+j_2c_2}y_3\Big)$$

are isometric in $L^p(\mathbb{R}^3)$ and they transform the averaging operators $\mathcal{A}_t^{\varphi,j_1,j_2}f$ into new ones

$$\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f(y)=2^{-(j_1+j_2)-(j_1d_1+j_2d_2)+\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}\times$$

$$\times\int\limits_{\mathbb{R}^2_+}f\Big(2^{j_1a_1+j_2a_2}\big(y_1-tr_1-t\cdot 2^{-(j_1a_1+j_2a_2)}w_1\big),2^{j_1b_1+j_2b_2}\big(y_2-tr_2-t\cdot 2^{-(j_1b_1+j_2b_2)}w_2\big),$$

$$2^{j_1c_1+j_2c_2}\big(y_3-tr_3-t\cdot 2^{-(j_1c_1+j_2c_2)}\alpha(w_1,w_2)\big)\Big)\beta(w_1,w_2)dw_1dw_2.$$

Also, the dilation operators

$$T_2^{-j_1,-j_2}f(y):=2^{-\frac{j_1a_1+j_2a_2+j_1b_1+j_2b_2+j_1c_1+j_2c_2}{p}}f\Big(2^{-j_1a_1-j_2a_2}y_1,\ 2^{-j_1b_1-j_2b_2}y_2,\ 2^{-j_1c_1-j_2c_2}y_3\Big)$$

are isometric in space $L^p(\mathbb{R}^3)$ and they turn operators $\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f$ into new operators

$$T_2^{-j_1,-j_2}\mathcal{A}_t^{\varphi,j_1,j_2}T_1^{j_1,j_2}f(y)=2^{-(j_1+j_2)-(j_1d_1+j_2d_2)}\int\limits_{\mathbb{R}^2_+}\times$$

$$\times f\Big(y_1-t\big(s_1+w_1\big),\ y_2-t\big(s_2+w_2\big),y_3-t\big(s_3+\alpha(w_1,w_2)\big)\Big)\beta(w_1,w_2)dw_1dw_2,$$

where $s_1=2^{j_1a_1+j_2a_2}r_1, s_2=2^{j_1b_1+j_2b_2}r_2,\ s_3=2^{j_1c_1+j_2c_2}r_3$.

Suppose that $\max\{|s_1|,|s_2|,|s_3|\}=|s_3|$ and define the following rotation operator

$$R^{\theta}f(y):=f(e_{11}x_1+e_{12}x_2+e_{13}x_3,\ e_{21}x_1+e_{22}x_2+e_{23}x_3,\ e_{31}x_1+e_{32}x_2+e_{33}x_3)$$

which is isometric in space $L^p(\mathbb{R}^3)$. Rotation orthogonal matrix $(e_{ij})_{i,j=1}^3$ is

$$\begin{pmatrix}\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2\cos\theta_3 & \sin\theta_1\sin\theta_3\\ \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2\cos\theta_3 & -\cos\theta_1\sin\theta_3\\ \sin\theta_2\sin\theta_3 & -\sin\theta_3\cos\theta_2 & \cos\theta_3\end{pmatrix},$$

where $\theta_1, \theta_2, \theta_3$ are Euler angles, θ_3 is the angle between vectors $(0,0,d)$ and (s_1, s_2, s_3), $d = \sqrt{s_1^2 + s_2^2 + s_3^2}$ (see [16], pp. 288–289).

The rotation operator $R^{\theta} f$ and its inverse $R^{-\theta} f$ turn operators $T_2^{-j_1,-j_2} \mathcal{A}_t^{\varphi, j_1, j_2} T_1^{j_1, j_2} f$ into the following new operators

$$R^{-\theta} T_2^{-j_1,-j_2} \mathcal{A}_t^{\varphi, j_1, j_2} T_1^{j_1, j_2} R^{\theta} f(y) = 2^{-(j_1+j_2)-(j_1 d_1 + j_2 d_2)} \times$$
$$\times \int\limits_{\mathbb{R}_+^2} f\Big(y_1 - t\alpha_1(w_1, w_2),\, y_2 - t\alpha_2(w_1, w_2), y_3 - t\big(d + \alpha_3(w_1, w_2)\big)\Big)\beta(w_1, w_2) dw_1 dw_2, \qquad (10)$$

where

$$\alpha_1(w_1, w_2) = e_{11} w_1 + e_{12} w_2 + e_{13} \alpha(w_1, w_2),$$
$$\alpha_2(w_1, w_2) = e_{21} w_1 + e_{22} w_2 + e_{23} \alpha(w_1, w_2),$$
$$\alpha_3(w_1, w_2) = e_{31} w_1 + e_{32} w_2 + e_{33} \alpha(w_1, w_2).$$

It is well-known that the second fundamental form of the surface given by parametric equations

$$\bar{r}(w_1, w_2) = \bar{r}\big(\alpha_1(w_1, w_2),\, \alpha_2(w_1, w_2),\, \alpha_3(w_1, w_2)\big) \qquad (11)$$

has the following form

$$L = L_{11} dw_1^2 + 2L_{12} dw_1 dw_2 + L_{22} dw_2^2,$$

where

$$L_{11} = (\bar{r}_{11}, \bar{n}), \quad L_{12} = (\bar{r}_{12}, \bar{n}), \quad L_{22} = (\bar{r}_{22}, \bar{n}), \qquad (12)$$
$$\bar{r}_{11} = \frac{\partial^2 \bar{r}}{\partial w_1^2}, \quad \bar{r}_{12} = \frac{\partial^2 \bar{r}}{\partial w_1 \partial w_2}, \quad \bar{r}_{22} = \frac{\partial^2 \bar{r}}{\partial w_2^2},$$

$\bar{n} = \bar{N} \cdot |\bar{N}|^{-1}$ is the unit normal vector. A normal vector $\bar{N}$ in any point of surface (11) defined by

$$barN = \begin{vmatrix} i & j & k \\ \dfrac{\partial \alpha_1}{\partial w_1} & \dfrac{\partial \alpha_2}{\partial w_1} & \dfrac{\partial \alpha_3}{\partial w_1} \\ \dfrac{\partial \alpha_1}{\partial w_2} & \dfrac{\partial \alpha_2}{\partial w_2} & \dfrac{\partial \alpha_3}{\partial w_2} \end{vmatrix}.$$

Coefficients L_{11}, L_{22}, L_{12} in (12) are

$$L_{11} = \frac{\partial^2 \alpha}{\partial w_1^2} = C w_1^{-\frac{B_2}{B_1}-2} w_2^{\frac{B_3}{B_1}} \Big(B_2(B_2+B_1) g(w_1, w_2) - B_2 B_1 w_1 \frac{\partial g(w_1, w_2)}{\partial w_1} + B_1^2 w_1^2 \frac{\partial^2 g(w_1, w_2)}{\partial w_1^2}\Big),$$
$$L_{22} = \frac{\partial^2 \alpha}{\partial w_2^2} = C w_1^{-\frac{B_2}{B_1}} w_2^{\frac{B_3}{B_1}-2} \Big(B_3(B_3-B_1) g(w_1, w_2) + B_3 B_1 w_2 \frac{\partial g(w_1, w_2)}{\partial w_2} + B_1^2 w_2^2 \frac{\partial^2 g(w_1, w_2)}{\partial w_2^2}\Big),$$
$$L_{12} = \frac{\partial^2 \alpha}{\partial w_1 \partial w_2} = -C w_1^{-\frac{B_2}{B_1}-1} w_2^{\frac{B_3}{B_1}-1} \Big(B_2 B_3 g(w_1, w_2) - B_3 B_1 w_1 \frac{\partial g(w_1, w_2)}{\partial w_1} +$$
$$+ B_2 B_1 w_2 \frac{\partial g(w_1, w_2)}{\partial w_2} - B_1^2 w_1 w_2 \frac{\partial^2 g(w_1, w_2)}{\partial w_1 w_2}\Big),$$

where $C = \dfrac{|\bar{N}|^{-1}}{B_1^2}$.

It follows from condition 1 of the Theorem that at least one of the numbers $B_2(B_2 + B_1)$, $B_3(B_3 - B_1)$, $B_2 B_3$ is not equal to zero. Therefore, at least one of the coefficients L_{11}, L_{12}, L_{22} is not equal to zero for sufficiently large j_0.

Hence, surface (11) satisfies the assumptions of Proposition 4.5 in [9]. Applying this proposition to integral (10) for $p > 2$, we obtain

$$\| \sup_{t>0} |R^{-\theta} T_2^{-j_1,-j_2} \mathcal{A}_t^{\varphi,j_1,j_2} T_1^{j_1,j_2} R^{\theta} f| \|_{L^p} \leqslant D_p \left(\frac{d}{|e_{33}|} \right)^{\frac{1}{p}} 2^{-\frac{p(j_1 d_1 + j_2 d_2) + p(j_1 + j_2)}{p}} \| f \|_{L^p},$$

where $\max\{|s_1|, |s_2|, |s_3|\} = |s_3|$. Taking into account this inequality and isometry of operators $T_1^{j_1,j_2} f$, $T_2^{-j_1,-j_2} f$, $R^{\theta} f$, $R^{-\theta} f$ and considering condition $\max\{|s_1|, |s_2|, |s_3|\} = |s_3|$, we obtain

$$\| \sup_{t>0} |\mathcal{A}_t^{\varphi,j_1,j_2} f| \|_{L^p} \leqslant D_p 2^{\frac{j_1 c_1 + j_2 c_2 - p(j_1 d_1 + j_2 d_2) - p(j_1 + j_2)}{p}} \| f \|_{L^p} .$$

Consequently, we have

$$\sum_{j_1, j_2 \geqslant j_0} \| \mathcal{M}^{\varphi, j_1, j_2} f \|_{L^p} \leqslant D_p \sum_{j_1, j_2 \geqslant j_0} 2^{\frac{j_1 c_1 + j_2 c_2 - p(j_1 d_1 + j_2 d_2) - p(j_1 + j_2)}{p}} \| f \|_{L^p} .$$

The series on the right side of the last inequality converges for all p satisfying the condition $p > \max\left\{ \frac{c_1}{d_1+1}, \frac{c_2}{d_2+1} \right\}$. Therefore, for such p the following inequalities

$$\| \mathcal{M}^{\varphi} f \|_{L^p} \leqslant \sum_{j_1, j_2 \geqslant j_0} \| \mathcal{M}^{\varphi, j_1, j_2} f \|_{L^p} \leqslant C_p \| f \|_{L^p}$$

hold true, where C_p is some positive number.

Analogously, one can show that if $\max\{|s_1|, |s_2|, |s_3|\} = |s_1|$ or $\max\{|s_1|, |s_2|, |s_3|\} = |s_2|$ then the maximal operator $\mathcal{M}^{\varphi} f$ is bounded in $L^p(\mathbb{R}^3)$ for $p > \max\left\{ \frac{a_1}{d_1+1}, \frac{a_2}{d_2+1} \right\}$ or for $p > \max\left\{ \frac{b_1}{d_1+1}, \frac{b_2}{d_2+1} \right\}$ respectively.

Thus, the proof of the positive result of Theorem is completed.

Let us prove now the negative result. For this reason suppose that $\max\left\{ \frac{c_1}{d_1+1}, \frac{c_2}{d_2+1} \right\} > 2$. Then following [1] consider the function

$$f(x_1, x_2, x_3) = \frac{\eta_1(x_1, x_2) \eta_2(x_3)}{|x_3|^{\frac{1}{p}} \, |\ln |x_3||^{\frac{1}{p}}},$$

where η_1, η_2 are smooth functions satisfying the following condition

$$\eta_1(x_1, x_2) \eta_2(x_3) = \begin{cases} 1, & |x| \leqslant \frac{\kappa}{2} \\ 0, & |x| \geqslant \kappa. \end{cases}$$

Here $\kappa > 0$ is some sufficiently small number. Taking into account relations (2) and (3) the averaging operator corresponding to function $f(x_1, x_2, x_3)$ is represented as follows

$$\mathcal{A}_t^{\varphi} f(y) = \int_{\mathbb{R}_+^2} \frac{\eta_1\big(y_1 - t x_1(u_1, u_2), y_2 - t x_2(u_1, u_2)\big) \eta_2(y_3 - t x_3(u_1, u_2))}{|y_3 - t x_3(u_1, u_2)|^{\frac{1}{p}} \left| \ln |y_3 - t x_3(u_1, u_2)| \right|^{\frac{1}{p}}} \times$$

$$\times \psi_1(u_1, u_2) u_1^{d_1} u_2^{d_2} \varphi(u_1, u_2) du_1 du_2.$$

Let us assume that $\psi_1(0,0) > 0,\ t = \dfrac{y_3}{r_3} > 0$. Since κ is sufficiently small number consider (y_1, y_2) that lies in a small neighbourhood of the point $\left(\dfrac{r_1 y_3}{r_3}, \dfrac{r_2 y_3}{r_3}\right)$. Then one can obtain

$$\sup_{t>0} |\mathcal{A}_t^{\varphi} f(y)| \geqslant C \frac{1}{|\frac{y_3}{r_3}|^{\frac{1}{p}}} \int\limits_{|u| \leqslant \frac{\kappa}{2}} \frac{|u_1|^{d_1 - \frac{c_1}{p}} |u_2|^{d_2 - \frac{c_2}{p}}}{|\ln|\frac{y_3}{r_3} u_1^{c_1} u_2^{c_2} g_3(u_1, u_2)||^{\frac{1}{p}}} du_1 du_2,$$

where C is some positive number. The last integral diverges for all p satisfying $2 < p \leqslant \max\left\{\dfrac{c_1}{d_1+1}, \dfrac{c_2}{d_2+1}\right\}$. Hence, the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ for these p.

Analogously, one can show that if $\max\{|s_1|, |s_2|, |s_3|\} = |s_1|$ or $\max\{|s_1|, |s_2|, |s_3|\} = |s_2|$ then the maximal operator $\mathcal{M}^{\varphi} f$ is not bounded in $L^p(\mathbb{R}^3)$ whenever $2 < p \leqslant \max\left\{\dfrac{a_1}{d_1+1}, \dfrac{a_2}{d_2+1}\right\}$ or $2 < p \leqslant \max\left\{\dfrac{b_1}{d_1+1}, \dfrac{b_2}{d_2+1}\right\}$, respectively.

Thus, making similar arguments under conditions 2 or 3, the proof of Theorem 3.1 is completed.

Consider now a number of corollaries in connection with Theorem 3.1.

Corollary 1. *Let $\varphi(u_1, u_2)$, $\{g_k(u_1, u_2)\}_{k=1}^3$ be fractional power series defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ such that $\varphi(0,0) \neq 0$, $g_k(0,0) \neq 0$ and $d_1 > -1$, $d_2 > -1$. Then the following assertions hold true*

1. If $r_1 = 0, r_2 = 0, r_3 \neq 0, B_1 \neq 0$ and either $B_2 B_3 \neq 0$ or $B_2(B_2 + B_1) \neq 0$ or $B_3(B_3 - B_1) \neq 0$ then $p'(S) = \max\left\{\dfrac{c_1}{d_1+1}, \dfrac{c_2}{d_2+1}\right\}$.

2. If $r_1 = 0, r_3 = 0, r_2 \neq 0, B_3 \neq 0$ and either $B_2 B_1 \neq 0$ or $B_2(B_2 - B_3) \neq 0$ or $B_1(B_1 - B_3) \neq 0$ then $p'(S) = \max\left\{\dfrac{b_1}{d_1+1}, \dfrac{b_2}{d_2+1}\right\}$.

3. If $r_2 = 0, r_3 = 0, r_1 \neq 0, B_2 \neq 0$ and either $B_1 B_3 \neq 0$ or $B_3(B_3 - B_2) \neq 0$ or $B_1(B_1 + B_2) \neq 0$ then $p'(S) = \max\left\{\dfrac{a_1}{d_1+1}, \dfrac{a_2}{d_2+1}\right\}$.

Corollary 2. *Let us assume that $\varphi(u_1, u_2)$, $\{g_k(u_1, u_2)\}_{k=1}^3$ are real analytic functions defined in a small neighbourhood of the origin of coordinate system of $\mathbb{R}^2$ and they satisfy the following conditions: $\varphi(0,0) \neq 0$, $g_k(0,0) \neq 0$. Then under the assumptions of Theorem 3.1 its assertions are true.*

Corollary 3. *If conditions 1–3 of Theorem 3.1 are replaced with the relations*

$$r_3 \neq 0,\ B_1 \neq 0,\ A_1^{-1}\bar{c} \neq (1,0),\ A_1^{-1}\bar{c} \neq (0,1),\ A_1^{-1}\bar{c} \neq (0,0);$$

$$r_1 \neq 0,\ B_2 \neq 0,\ A_2^{-1}\bar{a} \neq (1,0),\ A_2^{-1}\bar{a} \neq (0,1),\ A_2^{-1}\bar{a} \neq (0,0);$$

$$r_2 \neq 0,\ B_3 \neq 0,\ A_3^{-1}\bar{b} \neq (1,0),\ A_3^{-1}\bar{b} \neq (0,1),\ A_3^{-1}\bar{b} \neq (0,0)$$

, respectively, and other conditions are satisfied then assertions of Theorem hold true. Here A_1, A_2, A_3 are matrices B_1, B_2, B_3, respectively, and $\bar{a} = (a_1, a_2)$, $\bar{b} = (b_1, b_2)$, $\bar{c} = (c_1, c_2)$.

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О максимальных операторах, ассоциированных с семейством сингулярных поверхностей

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Аннотация. В этой статье рассматривается максимальный оператор, ассоциированный с сингулярными поверхностями. Доказываем ограниченность этого оператора в пространстве суммируемых функций, когда сингулярные поверхности задаются параметрическими уравнениями. А также найден показатель ограниченности максимального оператора для таких пространств.

Ключевые слова: максимальный оператор, оператор усреднения, дробно-степенной ряд, сингулярная поверхность, показатель ограниченности.