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A Note on the Diophantine Equation $(4^q-1)^u + \left(2^{q+1}\right)^v = w^2$

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Abstract. Let a, b and c be positive integers such that $a^2+b^2=c^2$ with $\gcd(a,b,c)=1$, a even. Terai's conjecture claims that the Diophantine equation $x^2+b^y=c^z$ has only the positive integer solution $(x,y,z)=(a,2,2)$. In this short note, we prove that the equation of the title, has only the positive integer solution $(u,v,w)=(2,2,4^q+1)$, where q is a positive integer.

Keywords: Terai's conjecture, Pythagorean triple.

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1. Introduction and preliminaries

In 1956, Sierpinski [2] studied the equation

$$3^u + 4^v = 5^w$$

and proved that it only possesses $(u,v,w)=(2,2,2)$ as a solution in integers. In turn, Jésmanowicz [1] showed that the only positive solution in integers of any of the following equations

$$5^u+12^v=13^w, \quad 7^u+24^v=25^w, \quad 9^u+40^v=41^w, \quad 11^u+60^v=61^w$$

is $(u,v,w)=(2,2,2)$, and posed the following Conjecture 1.1 (see [3]).

Recall that when positive integers a,b,c satisfy $a^2+b^2=c^2$ we say that (a,b,c) is a Pythagorean triple, and if in addition $\gcd(a,b,c)=1$ it is said a primitive Pythagorean triple.

Historically, Euclid of Alexandria (323–300 BC) was the first mathematician who proved that (a,b,c) is a primitive Pythagorean triple with a odd, if and only if, there exists a pair of numbers $(\alpha,\beta)\in\mathbb{N}^{*2}$ with $\alpha>\beta$, α and β are coprime and of different parity, such that

$$a=\alpha^2-\beta^2, \quad b=2\alpha\beta \quad \text{and} \quad c=\alpha^2+\beta^2.$$

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Conjecture 1.1. *If* (a, b, c) *is Pythagorean triple, then the equation*

$$a^u + b^v = c^w$$

has the only solution $(u, v, w) = (2, 2, 2)$.

In 2013, Z. Xinwen and Z. Wenpeng [6] showed that, for any positive integers n and m the exponential Diophantine equation

$$\left((2^{2m} - 1)n\right)^x + \left(2^{m+1}n\right)^y = ((2^{2m} + 1)n)^z$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

Recently, Hai Yang and Ruiqin Fu [7] by combining Baker's method with an elementary approach, have proven that if $\alpha\beta \equiv 2 \pmod 4$ and $\alpha > 17.8\beta$, then the Conjecture 1.1 is true, this is for $(a, b, c) = (2\alpha\beta, \alpha^2 - \beta^2, \alpha^2 + \beta^2)$.

Thirty years before, Terai had conjectured [4]

Conjecture 1.2. *Let* α, β *be positive integers such that* $\alpha > \beta$, $\gcd(\alpha, \beta) = 1$ *and* $\alpha \not\equiv \beta \pmod 2$, *then the equation*

$$x^2 + (\alpha^2 - \beta^2)^m = (\alpha^2 + \beta^2)^n$$

has the only positive solution in integers $(x, m, n) = (2\alpha\beta, 2, 2)$.

In 2020, M. Le and G. Soydan [5] studied Conjecture 1.2 in the case $\alpha = 2^r s$ and $\beta = 1$, where r, s are positive integers satisfying $2 \nmid s, r \geqslant 2$ and $s < 2^{r-1}$.

First Terai conjecture is "Let a, b, c be relatively prime positive integers such that $a^p + b^q = c^r$ for fixed integers $p, q, r \geqslant 2$. Terai conjectured that The equation $a^x + b^y = c^z$ in positive integers has only the solution $(x, y, z) = (p, q, r)$ except for some specific cases".

There are many results and studies related to this conjecture we can cite among them: Nobuhiro Terai [12, 13] and Takafumi Miyazaki [8–11].

In this short note we prove

Theorem 1.3. *Let* q *be a positive integer. Then the Diophantine equation*

$$(4^q - 1)^u + \left(2^{q+1}\right)^v = w^2$$

has only the positive integer solution $(u, v, w) = (2, 2, 4^q + 1)$.

2. Proof of the main result

Proof. Suppose that there are positive integers u, v and w such that

$$(4^q - 1)^u + \left(2^{q+1}\right)^v = w^2 \tag{1}$$

then w is odd and

$$w^2 \equiv 1 \pmod{4}.$$

Reducing equation (1) modulo 4, we get

$$(4^q - 1)^u \equiv 1 \pmod{4},$$

or equivalently

$$(-1)^u \equiv 1 \pmod{4}.$$

This implies $u = 2t$ for some positive integer t.
Thus,

$$2^{(q+1)v} = \left(2^{q+1}\right)^v = w^2 - \left((4^q-1)^t\right)^2 = \left(w + (4^q-1)^t\right)\left(w - (4^q-1)^t\right)$$

Hence,

$$w + (4^q-1)^t = 2^s$$

and

$$w - (4^q-1)^t = 2^r,$$

with $s > r$ and $s + r = (q+1)\,v$. Solving for w and $(4^q-1)^t$, we get

$$w = 2^{r-1}\left(2^{s-r}+1\right) \quad \text{and} \quad (4^q-1)^t = 2^{r-1}\left(2^{s-r}-1\right).$$

Since the left side of both previous equalities is odd, r must be equal to 1. Let $x = s - r$. Then the equation

$$(4^q-1)^t = 2^{r-1}\left(2^{s-r}-1\right)$$

becomes

$$(4^q-1)^t = 2^x - 1.$$

The reduction modulo 3 gives

$$0 \equiv (-1)^x - 1 \pmod{3},$$

and so x is even, say $x = 2k$ for some positive integer k. Thus,

$$(4^q-1)^t = \left(2^k\right)^2 - 1$$

by the Mihailescu's Theorem $t = 0$ or $t = 1$. Consequently, $t = 1$, and so $x = 2q$. This gives us the unique solution $(u, v, w) = (2, 2, 4^q + 1)$. □

If we maintain the same conditions as before we believe in the validity of the following:

Conjecture 2.1. *If* $a^2 + b^2 = c^2$ *with* $(a, b, c) = 1$*, then the Diophantine equation*

$$a^u + b^v = w^2.$$

has only the positive integer solutions $(u, v, w) = (2, 2, c)$.

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Заметка о диофантовом уравнении $(4^q-1)^u + \big(2^{q+1}\big)^v = w^2$

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Аннотация. Пусть a, b и c — натуральные числа такие, что $a^2+b^2=c^2$ с $\gcd(a,b,c)=1$, a четным. Гипотеза Тераи утверждает, что диофантово уравнение $x^2+b^y=c^z$ имеет только натуральное решение $(x,y,z)=(a,2,2)$. В этой короткой заметке мы доказываем, что уравнение заголовка имеет только положительное целочисленное решение $(u,v,w)=(2,2,4^q+1)$, где q положительное целое число.

Ключевые слова: гипотеза Тераи, тройка Пифагора.