EDN: TTFTGS
УДК 511

Explicit Formula for Sums Related to the Generalized Bernoulli Numbers

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Received 10.07.2022, received in revised form 16.09.2022, accepted 4.11.2022

Abstract. Let χ be a Dirichlet character modulo a prime number $p \geqslant 3$ and let $B_m(\chi)$ $(m = 1, 2, \ldots)$ be the generalized Bernoulli numbers associated with χ. Explicit formulas for the sums:

$$\sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi\neq\chi_0}} B_m(\chi)B_n(\overline{\chi}) \quad \text{and} \quad \sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} B_m(\chi)B_n(\overline{\chi})$$

are given in this paper.

Keywords: character sum, Dirichlet L-function, Bernoulli number, generalized Bernoulli number.

Citation: B. Mittou, Explicit Formula for Sums Related to the Generalized Bernoulli Numbers, J. Sib. Fed. Univ. Math. Phys., 2023, 16(1), 135–141. EDN: TTFTGS

1. Introduction and main result

Throughout this paper, for a prime $p \geqslant 3$, we let $G(p) = \left\{\overline{1}, \overline{x}, \ldots, \overline{x}^{p-2}\right\}$ and $\widehat{G}(p) = \{\chi_0, \chi_1, \ldots, \chi_{p-2}\}$ denote the group of reduced residue classes modulo p and the group of Dirichlet characters modulo p, respectively.

Let χ be a Dirichlet character modulo $k \geqslant 3$. Then the generalized Bernoulli numbers $B_m(\chi)$ $(m = 0, 1, 2, \ldots)$ are defined by using the generating function:

$$\sum_{a=1}^{k} \chi(a) \frac{ze^{az}}{e^{kz}-1} = \sum_{m=0}^{\infty} \frac{B_m(\chi)}{m!} z^m, \quad |z| < \frac{2\pi}{k}.$$

They can be expressed in terms of Bernoulli polynomials as:

$$B_m(\chi) = k^{m-1} \sum_{i=1}^{k} \chi(i) B_m\left(\frac{i}{k}\right),$$

where the Bernoulli polynomials $B_m(x)$ are the coefficients in the power series expansion:

$$\frac{ze^{xz}}{e^z-1} = \sum_{m=0}^{\infty} B_m(x) \frac{z^m}{m!}, \quad |z| < 2\pi.$$

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The Bernoulli numbers B_m are the values of the Bernoulli polynomials $B_m(x)$ at $x = 0$. The expression of the Bernoulli polynomials in terms of the Bernoulli numbers is given by:

$$B_m(x) = \sum_{j=0}^{\infty} \binom{m}{j} B_{m-j} x^j.$$

Consequently, the generalized Bernoulli numbers can be expressed in terms of Bernoulli numbers as:

$$B_m(\chi) = \sum_{i=1}^{k} \chi(i) \sum_{j=0}^{m} \binom{m}{j} B_j i^{m-j} k^{j-1}.$$

Many mathematicians have been studied sums and products related to the generalized Bernoulli numbers, for example Chen and Eie [3, Proposition 7] gave a closed expression for sums of products of generalized Bernoulli numbers. The author and Derbal [7, Theorem 3.8] proved, for a primitive Dirichlet character χ modulo $k \geqslant 3$, the following formulas:

1. If $\chi(-1) = +1$ and $r \geqslant 1$, then

$$\sum_{m=1}^{r} \frac{k^{2r-2m}}{2r-2m+1} \binom{2r}{2m} B_{2m}(\chi) = \frac{1}{k} \sum_{m=1}^{k-1} m^{2r} \chi(m).$$

2. If $\chi(-1) = -1$ and $r \geqslant 0$, then

$$\sum_{m=0}^{r} \frac{k^{2r-2m}}{2r-2m+1} \binom{2r+1}{2m+1} B_{2m+1}(\chi) = \frac{1}{k} \sum_{m=1}^{k-1} m^{2r+1} \chi(m).$$

It is the main purpose of this paper to prove the following formulas.

Theorem 1.1. *Let $p \geqslant 3$ be a prime and let m and n be positive integers. For $l \in \{1, 2, \ldots, m+n\}$, define*

$$r_{m,n,l} := B_{m+n-l} \sum_{\substack{a=0 \\ a+b \geqslant m+n-l}}^{m} \sum_{b=0}^{n} B_{m-a} B_{n-b} \frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}.$$

The following assertions hold:

1. *If $m \equiv n \equiv 0 \pmod 2$, then*

$$\sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} B_m(\chi) B_n(\overline{\chi}) =$$
$$= p^{m+n-1} \left((p-1) \sum_{l=1}^{m+n} r_{m,n,l} \left(1 - \frac{1}{p^l}\right) p^{l-m-n} - B_m B_n \left(1 - \frac{1}{p^m}\right)\left(1 - \frac{1}{p^n}\right) \right). \quad (1)$$

2. *If $m \equiv n \equiv 1 \pmod 2$, then*

$$\sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} B_m(\chi) B_n(\overline{\chi}) =$$
$$= (p-1) p^{m+n-1} \left(\sum_{l=1}^{m+n} r_{m,n,l} \left(1 - \frac{1}{p^l}\right) p^{l-m-n} - \frac{1}{p} B_m B_n \left(1 - \frac{1}{p^{m+n-1}}\right) \right). \quad (2)$$

Example 1.2. *Let $p \geqslant 3$ be a prime. Then*

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}} B_2(\chi)B_4(\overline{\chi}) = -\frac{(p-1)(p-2)(p-3)(p^2-1)(2p^2+3p+5)}{1260p},$$

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=-1}} B_1(\chi)B_3(\overline{\chi}) = -\frac{(p-1)(p^2-1)(p^2-4)}{120p},$$

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=-1}} B_1(\chi)B_1(\overline{\chi}) = \sum_{\substack{\chi \mod p\\ \chi(-1)=-1}} |B_1(\chi)|^2 = \frac{(p-1)^2(p-2)}{12p},$$

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}} B_2(\chi)B_2(\overline{\chi}) = \sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}} |B_2(\chi)|^2 = \frac{(p-1)(p-2)(p-3)(p^2-1)}{180p},$$

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=-1}} B_3(\chi)B_3(\overline{\chi}) = \sum_{\substack{\chi \mod p\\ \chi(-1)=-1}} |B_3(\chi)|^2 = \frac{(p-1)(p^2-1)(p^2-4)(p^2+5)}{840p}.$$

2. Proof of Theorem 1.1

Let $k \geqslant 3$ and l be integers with $\gcd(l,k) = 1$. Let χ be a Dirichlet character modulo k and let $L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$, $(\Re(s) > 1)$ be the Dirichlet L-function corresponding to χ. Set

$$M(k,l,m,n) := \frac{2}{\varphi(k)} \sum_{\substack{\chi \mod k\\ \chi(-1)=(-1)^m=(-1)^n}} \chi(l)L(m,\chi)L(n,\overline{\chi}),$$

where φ is the totient's Euler function.

Liu and Zhang [5] gave the following result.

Proposition 2.1. *Let $k > 2$, $m \geqslant 1$, and $n \geqslant 1$ be integers with $m \equiv n \pmod 2$. Set $\epsilon_{m,n} = 1$ if $m \equiv n \equiv 1 \pmod 2$ and $\epsilon_{m,n} = 0$ if $m \equiv n \equiv 0 \pmod 2$. Then*

$$M(k,1,m,n) = \frac{(-1)^{\frac{m-n}{2}}(2\pi)^{m+n}}{2m!n!} \times \\ \times \left(\sum_{l=0}^{m+n} r_{m,n,l}\varphi_l(k)k^{l-m-n} - \frac{\epsilon_{m,n}}{k}B_mB_n\varphi_{m+n-1}(k)\right),$$

where

$$r_{m,n,l} = B_{m+n-l} \sum_{\substack{a=0\\ a+b\geqslant m+n-l}}^{m} \sum_{b=0}^{n} B_{m-a}B_{n-b} \frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}$$

and

$$\varphi_l(k) = \prod_{p|k}\left(1 - \frac{1}{p^l}\right).$$

Let $p \geqslant 3$ be a prime. The following theorem gives explicit formulas for $M(p,l,m,n)$ by using Bernoulli and generalized Bernoulli numbers.

Theorem 2.2. *Let $p \geqslant 3$ be a prime. Let $m > 0, n > 0$, and l be integers with* $\gcd(p, l) = 1$. *If $m \equiv n \equiv 0 \pmod 2$, then*

$$M(p,l,m,n) = (-1)^{\frac{m+n}{2}} \frac{(2\pi)^{m+n}}{2(p-1)m!n!p^{m+n-1}} \times \\ \times \left(B_m B_n \frac{(p^m - 1)(p^n - 1)}{p} + \sum_{a=1}^{p-2} \chi_a(l) B_m(\chi_a) B_n(\chi_{a'}) \right), \quad (3)$$

and if $m \equiv n \equiv 1 \pmod 2$, then

$$M(p,l,m,n) = (-1)^{\frac{m-n}{2}} \frac{(2\pi)^{m+n}}{2(p-1)m!n!p^{m+n-1}} \times \sum_{a=1}^{p-2} \chi_a(l) B_m(\chi_a) B_n(\chi_{a'}), \quad (4)$$

where $\chi_a \in \widehat{G}(p)$ $(1 \leqslant a \leqslant p-2)$ are the non-principal characters modulo p and $a + a' = p - 1$.

In order to prove Theorem 2.2 we need the following lemma.

Lemma 2.3. *Let $p \geqslant 3$ be a prime. If $\chi_a, \chi_{a'} \in \widehat{G}(p)$ $(0 \leqslant a, a^{'} \leqslant p-2)$, then*

1. *For any $a \in \{0, 1, \ldots, p-2\}$ and any $n \in \mathbb{Z}$, the character χ_a is defined by:*
$$\chi_a(n) = [\chi_1(n)]^a = \begin{cases} \exp\left(i\frac{2a\nu\pi}{p-1}\right), & \text{if } \overline{n} = \overline{x}^\nu \in G(p); \\ 0, & \text{otherwise.} \end{cases}$$
2. *The character χ_a is odd if, and only if, a is odd.*
3. *The character χ_a conjugate to $\chi_{a'}$ if, and only if, $a + a' = p - 1$.*

Proof. 1. For the first item, see e.g., [1, p. 218].

2. For the second item, we have according to [1, Theorem 10.10] $\overline{-1} = \overline{x}^{(p-1)/2}$, so $\chi_1(-1) = \exp(i\pi) = -1$. Thus
$$\chi_a(-1) = [\chi_1(-1)]^a = (-1)^a,$$
from which χ_a is odd character if, and only if, a is odd.
3. Now, let us prove the third item. Let $n \in \mathbb{Z}$ such that $\gcd(n, p) = 1$ and $\overline{n} = \overline{x}^\nu \in G(p)$. Then
$$\chi_a(n)\chi_{p-1-a}(n) = \exp\left(i\frac{2a\nu\pi}{p-1}\right) \exp\left(i\frac{2(p-1-a)\nu\pi}{p-1}\right) = 1.$$
Hence $\overline{\chi_a} = \chi_{p-1-a} = \chi_{a'}$, i.e., $a + a' = p - 1$.

This completes the proof of the lemma. □

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Suppose that $m \equiv n \equiv 0 \pmod 2$. Then

$$M(p,l,m,n) = \frac{2}{p-1} \sum_{\substack{\chi \mod p \\ \chi(-1)=+1}} \chi(l) L(m,\chi) L(n,\overline{\chi}) =$$

$$= \frac{2}{p-1} \left(L(m,\chi_0) L(n,\chi_0) + \sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} \chi(l) L(m,\chi) L(n,\overline{\chi}) \right).$$

According to [1, p. 232] and to [1, Theorem 12.17], we have

$$L(m,\chi_0)=\left(1-\frac{1}{p^m}\right)\zeta(m)=(-1)^{\frac{m+2}{2}}\left(1-\frac{1}{p^m}\right)\frac{(2\pi)^m}{2m!}B_m,$$

from which

$$L(m,\chi_0)L(n,\chi_0)=(-1)^{\frac{m+n}{2}}(2\pi)^{m+n}\frac{B_mB_n}{4m!n!}\frac{(p^m-1)(p^n-1)}{p^{m+n}}. \tag{5}$$

On the other hand, it follows by using [2, Theorem 9.6] that:

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}}\chi(l)L(m,\chi)L(n,\overline{\chi})=(-1)^{\frac{m+n}{2}}\frac{p}{4\cdot m!\cdot n!}\left(\frac{2\pi}{p}\right)^{m+n}\times$$

$$\times\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}}\chi(l)B_m(\overline{\chi})B_n(\chi). \tag{6}$$

Next, it is well known (see e.g., [2, Proposition 4.5]) that:

$$\begin{cases}\text{If } \chi(-1)=+1 \text{ and } m\equiv 1 \pmod 2, \text{ then } B_m(\chi)=0.\\ \text{If } \chi(-1)=-1 \text{ and } m\equiv 0 \pmod 2, \text{ then } B_m(\chi)=0.\end{cases}$$

This facts and Lemma 2.3 allow us to write

$$\sum_{\substack{\chi \mod p\\ \chi(-1)=+1,\chi\neq\chi_0}}\chi(l)B_m(\overline{\chi})B_n(\chi)=\sum_{a=1}^{p-2}\chi_a(l)B_m(\chi_a)B_n(\chi_{a'}), \tag{7}$$

where $a+a'=p-1$. Finally, from (5), (6) and (7) we get Formula (3).
Now, suppose that $m\equiv n\equiv 1 \pmod 2$. Then, similarly we can get Formula (4). This proves the theorem. □

Example 2.4. *Let $p>3$ be a prime. Then from [4, Corollary 1.1] we have*

$$M(p,3,2,2)=\begin{cases}\dfrac{\pi^4}{810}\dfrac{(p-1)(p^3+p^2+166p+291)}{p^4}, & \text{if } p\equiv 1 \pmod 3;\\ \dfrac{\pi^4}{810}\dfrac{p^4+165p^2-35p-291}{p^4}, & \text{if } p\equiv -1 \pmod 3).\end{cases}$$

In particular, $M(5,3,2,2)=\dfrac{238}{9\cdot 5^5}\pi^4$ *and* $M(7,3,2,2)=\dfrac{41}{3\cdot 7^4}\pi^4$. *Louboutin [6] got* $M(5,3,2,2)=\dfrac{32}{5^5}\pi^4$ *and showed that the above formulas are not correct. On the other hand, by using the formulas of Theorem 2.2 we get*

$$M(5,3,2,2)=\frac{(2\pi)^4}{2\cdot 4\cdot 4\cdot 5^3}\left(B_2^2\cdot\frac{24^2}{5}+\chi_2^{(5)}(3)B_2^2(\chi_2^{(5)})\right)=\frac{32}{5^5}\pi^4,$$

$$M(7,3,2,2)=\frac{(2\pi)^4}{2\cdot 6\cdot 4\cdot 7^3}\left(B_2^2\cdot\frac{48^2}{7}+\chi_2^{(7)}(3)B_2(\chi_2^{(7)})B_2(\chi_4^{(7)})+\chi_4^{(7)}(3)B_2(\chi_4^{(7)})B_2(\chi_2^{(7)})\right)=$$
$$=\frac{16}{7^4}\pi^4,$$

where $\chi_2^{(5)}$ is the character modulo 5 such that $\chi_2^{(5)}(2)=-1$ and $\chi_2^{(7)}$ is the character modulo 7 such that $\chi_2^{(7)}(3)=\exp\left(i\frac{2\pi}{3}\right)$, and $\chi_4^{(7)}=\overline{\chi_2^{(7)}}$.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $p \geqslant 3$ be a prime. If we take $k = p$ in Proposition 2.1, we obtain
• For $m \equiv n \equiv 0 \pmod 2$,

$$M(p,1,m,n) = \frac{(-1)^{\frac{m-n}{2}}(2\pi)^{m+n}}{2m!n!} \times \sum_{l=0}^{m+n} r_{m,n,l}\left(1-\frac{1}{p^l}\right)p^{l-m-n}. \tag{8}$$

• For $m \equiv n \equiv 1 \pmod 2$,

$$M(p,1,m,n) = (-1)^{\frac{m-n}{2}}\frac{(2\pi)^{m+n}}{2\cdot m!\cdot n!}\times \\ \times\left(\sum_{l=0}^{m+n} r_{m,n,l}\left(1-\frac{1}{p^l}\right)p^{l-m-n} - \frac{1}{p}B_mB_n\left(1-\frac{1}{p^{m+n-1}}\right)\right). \tag{9}$$

On the other hand if we take $l = 1$ in Theorem 2.2, we get
• For $m \equiv n \equiv 0 \pmod 2$,

$$M(p,1,m,n) = (-1)^{\frac{m+n}{2}}\frac{(2\pi)^{m+n}}{2(p-1)\cdot m!\cdot n!\cdot p^{m+n-1}}\times \\ \times\left(B_mB_n\frac{(p^m-1)(p^n-1)}{p} + \sum_{\substack{\chi \mod p \\ \chi(-1)=+1,\chi\neq\chi_0}} B_m(\chi)B_n(\overline{\chi})\right). \tag{10}$$

• For $m \equiv n \equiv 1 \pmod 2$,

$$M(p,1,m,n) = \frac{(-1)^{\frac{m-n}{2}}(2\pi)^{m+n}}{2(p-1)m!n!p^{m+n-1}} \times \sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} B_m(\chi)B_n(\overline{\chi}). \tag{11}$$

Consequently, one can show that Formulas (8) and (10) imply Formula (1), while Formulas (9) and (11) imply Formula (2). This completes the proof. □

The author would like to thank the anonymous referee for his careful reading. Also, the author is thankful to Bakir Farhi for his help to proofread this paper.

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DOI: 10.20948/mathmontis-2021-50-2

Явная формула для сумм, относящихся к обобщенным числам Бернулли

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Аннотация. Пусть χ — характер Дирихле по модулю простого числа $p \geqslant 3$, а $B_m(\chi)$ $(m = 1, 2, \ldots)$ — обобщенные числа Бернулли, связанные с χ. Явные формулы для сумм:

$$\sum_{\substack{\chi \mod p \\ \chi(-1)=+1, \chi \neq \chi_0}} B_m(\chi)B_n(\overline{\chi}) \quad \text{и} \quad \sum_{\substack{\chi \mod p \\ \chi(-1)=-1}} B_m(\chi)B_n(\overline{\chi})$$

приведены в этой статье.

Ключевые слова: сумма характеров, L-функция Дирихле, число Бернулли, обобщенное число Бернулли.