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Solution of Convection Problem in a Rotating Tube by the Fourier Method

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Abstract. The non-stationary boundary value problem on the motion of a fluid in a rotating cylindrical pipe is studied in this paper,. The Oberbeck-Boussinesq equations are used to describe the motion of a fluid. From a mathematical point of view, the problem is inverse with respect to pressure gradient along the axis of the cylinder. The solution is found with the use of the method of separation of variables in the form of special Fourier series. Sufficient conditions are given for the solution of a non-stationary problem to reach a stationary regime with increasing time.

Keywords: convection, inverse problem, asymptotic behaviour, method of separation of variables , Bessel functions.

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Introduction

Thermal convection in rotating systems has been studied for various applications such as modelling heat exchangers, various large-scale circulations in the Earth atmosphere, and other phenomena [1–5]. However, convective flows were assumed to be stationary in these works, and the rotation angular velocity was constant.

In the present work, the angular velocity of a round tube depends on time, and the motion is rotationally symmetric and unsteady. In addition, the fluid rate through the pipe cross section is also the function of time. There are no mass forces. This takes place at a sufficiently large angular velocity of pipe rotation (centrifugal acceleration can be 10^6 times greater than gravity acceleration in practical vortex tubes) or in conditions close to weightlessness. The constant temperature gradient is applied along the pipe surface. From a mathematical point of view, an inverse initial-boundary value problem arises because it is also required to determine the non-stationary pressure gradient along the pipe axis, and the over determination condition for the fluid flow rate is set.

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1. Problem statement and formulation of basic equations

The fluid thermal convection problem is considered in a rotating cylindrical tube. Oberbeck-Boussinesq equations are used as a mathematical model. Here r, φ, z are cylindrical coordinates, t is time, a is the cylinder radius and $\omega(t)$ is the angular velocity of cylinder rotation around z axis. Further, u and w are radial and axial components of the velocity vector; v is the difference between the azimuthal velocity and the fluid rotation velocity $\omega(t)r$ as a solid body; p is the deviation of true pressure from the equilibrium state $\rho\omega^2(t)r^2/2$; Θ is the deviation of temperature from its average value $\overline{\Theta} = \text{const}$. The fluid is characterized by the following physical parameters: density ρ at temperature $\overline{\Theta}$, kinematic viscosity ν, thermal diffusion χ and the coefficient of thermal expansion β. These parameters are assumed to be constant and positive. There is no gravity acceleration. The last assumption is satisfied if the angular velocity $\omega(t)$ is sufficiently large.

Under given above assumptions the thermal convection equations have the form [6]

$$
\begin{gathered}
u_t + uu_r - \frac{v}{r}u_\varphi + \omega(t)u_\varphi + wu_z - 2\omega(t)v - \frac{v^2}{r} = -\frac{1}{\rho}p_r + \nu\left(Lu - \frac{2}{r^2}v_\varphi - \frac{u}{r^2}\right) - \omega^2(t)\beta r\Theta,\\
v_t + \omega_t(t)r + uv_r + \omega(t)v_\varphi + wv_\varphi + 2\omega(t)u + \frac{uv}{r} = -\frac{1}{\rho r}p_\varphi + \nu\left(Lv - \frac{2}{r^2}u_\varphi - \frac{v}{r^2}\right) - \omega^2(t)\beta r\Theta,\\
w_t + uw_r + \frac{v}{r}w_\varphi + \omega(t)w_\varphi + ww_z = -\frac{1}{\rho}p_z + \nu Lw,\\
u_r + \frac{u}{r} + \frac{1}{r}v_\varphi + w_z = 0,\\
\Theta_t + u\Theta_r + \frac{v}{r}\Theta_\varphi + \omega(t)\Theta_\varphi + w\Theta_z = \chi L\Theta,
\end{gathered}
\tag{1}
$$

where $L = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\varphi + \partial^2/\partial z$ is the Laplace operator.

We seek a solution of system (1) in the form

$$
\begin{gathered}
u = u(r,t), \quad v = v(r,t), \quad w = w(r,t),\\
p = \rho\left[A\beta\omega^2(t)\frac{r^2}{2} + f(t)\right]z + q(r,t),\\
\Theta = -Az + T(r,t), \quad A = \text{const}.
\end{gathered}
\tag{2}
$$

According to the classification given in [7], it is an invariant solution of system (1) with respect to the infinite Lie subgroup defined by operators

$$
\frac{\partial}{\partial\varphi}, \quad \frac{\partial}{\partial z} - A\frac{\partial}{\partial\Theta} + \left(f(t) + \frac{A\beta\omega^2(t)\rho r^2}{2}\right)\frac{\partial}{\partial p}.
$$

Substituting solution (2) into (1) and assuming that there are no sources or sinks on the axis z, we obtain $u = 0$ and the system of linear equations

$$
\begin{gathered}
w_t = \nu\left(w_{rr} + \frac{1}{r}w_r\right) - \frac{1}{2}A\beta\omega^2(t)r^2 - f(t)\\
T_t = \chi\left(T_{rr} + \frac{1}{r}T_r\right) + Aw,\\
v_t = \nu\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v\right) - \omega_t r, \quad q_r = \frac{\rho v^2}{r}.
\end{gathered}
\tag{3}
$$

The initial and boundary conditions for this system are as follows

$$
\begin{gathered}
w(r,0) = w_0(r), \quad T(r,0) = T_0(r), \quad v(r,0) = v_0(r), \quad 0 \leqslant r \leqslant a;\\
w(a,t) = 0, \quad T(a,t) = 0, \quad v(a,t) = 0, \quad 0 \leqslant t \leqslant t_1;\\
\mid w(0,t) \mid < \infty, \quad \mid T(0,t) \mid < \infty, \quad \mid v(0,t) \mid < \infty.
\end{gathered}
\tag{4}
$$

In addition, the fluid volumetric flow rate through the tube cross section $Q(t)$ is given as

$$\int_0^a rw(r,t)dr = \frac{Q(t)}{2\pi}. \tag{5}$$

In equations (1)–(3), ν is the fluid kinematic viscosity; $(-A)$ is the temperature gradient along the tube axis; β is the coefficient of thermal expansion; χ is the thermal diffusivity; a is the tube radius; $\omega(t)$ is rotation angular velocity of the tube around its axis; $\omega_t(t)$ is angular acceleration.

Given functions $\omega(t), Q(t)$, one can give the following interpretation of problem (3)–(5). The fluid fills a cylinder of radius a rotating with an angular velocity $\omega(t)$, and fluid is pumped with a flow rate $Q(t)$ through the cross section of the cylinder. The temperature gradient $-A$ along the axis is set on the cylinder surface. It is required to determine the resulting rotationally symmetric motion. If $Q(t) = 0$ then one can expect that the problem solution describes convection far from solid wall perpendicular to the axis of the cylinder, that is, motion in the core.

From a mathematical point of view, problem (3)–(5) is an inverse problem. Along with $w(r,t), v(r,t), T(r,t)$, it is necessary to find an additional pressure gradient along z axis, that is, function $f(t)$. Let us also note that problem for $v(r,t)$ is separated. The main problem is for $w(r,t) and f(t)$. For known $w(r,t)$ function $T(r,t)$ is found as a solution of the classical first initial-boundary value problem, and $q(r,t)$ is found by integrating the last equation of system (3).

2. Stationary problem solution

In the stationary case, all variables in (3)–(5) do not depend on time

$$w = w^s(r), \quad f = f^s = \text{const}, \quad T = T^s(r), \quad v = v^s(r), \tag{6}$$

and the angular velocity ω^s and flow rate Q^s are constant.

Simple mathematical treatment shows that solution of this problem is

$$\begin{gathered} w^s(\xi) = \frac{\chi}{a}\left[\frac{Ra}{96}\left(3\xi^4 - 4\xi^2 + 1\right) + \frac{2\overline{Q}}{\pi}\left(1-\xi^2\right)\right], \\ T^s(\xi) = -Aa\left[\frac{Ra}{1152}\left(\xi^6 - 3\xi^4 + 3\xi^2 - 1\right) + \frac{\overline{Q}}{8\pi}\left(\xi^3 - 4\xi^2 + 3\right)\right], \quad v^s = 0, \\ f^s = -\frac{\nu\chi}{a^3}\left(\frac{8\overline{Q}}{\pi} + \frac{Ra}{6}\right), \end{gathered} \tag{7}$$

where $Ra = (A\beta\omega^{s2}a^5)/(\nu\chi)$ is the analogue of the Rayleigh number, $\xi = r/a$, $\overline{Q} = Q^s/a\chi$, $0 \leqslant \xi \leqslant 1$.

It should be noted that in the vicinity of the solid wall, zones of the fluid return flow can arise. This is due to two types of flow mechanism that are present in the problem. The first one is related to the fluid rate, and the second one is related to thermal effects and fluid rotation. It is easy to find that the reverse flow occurs when $\overline{Q} < \pi Ra/96$, in particular, this always takes place when $Q^s = 0$.

3. Solution of the unsteady problem by the method of separation of variables

The method of separation of variables, or the Fourier method, is one of the widely used methods for solving linear partial differential equations. Its application leads to a solution in the series form.

In our case, the main problem is to find $w(r,t)$ and $f(t)$ from the first equation of system (3). In order to use the method of separation of variables it is necessary to reduce the problem to a homogeneous one with respect to the boundary conditions. This can be achieved by doing the following change of variables

$$w(r,t)=\overline{w}(r,t)+\frac{10}{\pi a^5}Q(t)(a-r)r^2+d(t)\left(r^3-\frac{6}{5}ar^2+\frac{1}{5}a^3\right),$$
$$d(t)=\frac{25}{3\pi a^5}\left[Q(t)-Q(0)\right]+\frac{5A\beta}{12a}\int_0^t\omega^2(\tau)d\tau. \tag{8}$$

Since function $f(t)$ is unknown the problem is inverse. Then we differentiate the equation with respect to r and introduce $\overline{w}_r(r,t)=W(r,t)$. Thus the pressure gradient function is excluded. As a result, we obtain the following direct problem with integral (non-classical) condition on function $W(r,t)$

$$W_t=\nu\left(W_{rr}+\frac{1}{r}W_r-\frac{1}{r^2}W\right)+U,$$
$$U(r,t)=-\nu\frac{90}{\pi a^5}Q(t)+9\nu d(t)+\left(\frac{30}{\pi a^5}Q_t(t)-3d_t(t)\right)r^2, \tag{9}$$
$$W(r,0)=W_0(r)=w_{0r}(r)-\frac{10}{\pi a^5}Q(0)\left(2ar-3r^2\right),\quad 0\leqslant r\leqslant a,$$
$$|W(0,t)|<\infty,\quad 0\leqslant t\leqslant t_1,$$

$$\int_0^a r^2W(r,t)dr=0,\quad r\in[0,a],\quad t\in[0,t_0]. \tag{10}$$

Integral condition (10) follows from (5).

Let us consider a homogeneous version of (9) ($U=0$) in order to find the Fourier expansion basis. To do this, it is necessary to separate the variables and solve a Sturm-Liouville type problem to find eigenfunctions and eigenvalues. After some mathematical treatment, we find that expansion basis contains the Bessel functions of the first kind and first order. Therefore, the solution of inhomogeneous problem is sought in the form

$$W(r,t)=\sum_{k=1}^{\infty}C_k(t)J_1\left(\sqrt{\lambda_k}r\right),\quad U(r,t)=\sum_{k=1}^{\infty}u_k(t)J_1\left(\sqrt{\lambda_k}r\right), \tag{11}$$

where $\lambda_k=\xi_k^2/a^2$, and ξ_k are roots of equation $J_2(\xi_k)=0$. They are $\xi_1=5.13562$, $\xi_2=8.41724$, $\xi_3=11.6198$, $\xi_4=14.796$, $\xi_5=17.9598$.

Remark 1. Series (11) includes the Bessel functions terms $J_1(\xi_k r/a)$, where ξ_k are roots of equation $J_2(\xi_k)=0$. However, taking into account relation $J_1(\xi)-\xi J_1'(\xi)=\xi J_2(\xi)$, we obtain that ξ_k- are roots of equation $J_1(\xi_k)-\xi_k J_1'(\xi_k)=0$. Therefore, expansion (11) makes sense [8]. In particular, the system of functions $\{J_1(\xi_k r/a)\}$ is orthogonal and complete in $L_2[(0,a);r]$ and

$$\int_0^a rJ_1\left(\frac{\xi_k r}{a}\right)J_1\left(\frac{\xi_n r}{a}\right)dr=0,\quad k\neq n;$$
$$\int_0^a rJ_1^2\left(\frac{\xi_k r}{a}\right)dr=\frac{a^2}{2}\left\{\left[J_1'(\xi_k)\right]^2+\left(1-\frac{1}{\xi_k^2}\right)J_1^2(\xi_k)\right\}=\frac{a^2}{2}J_1^2(\xi_k). \tag{12}$$

Substituting expressions (11) into (9), we obtain after simple transformations the system of ordinary differential equations, and find $C_k(t)$:

$$C_k(t)=\int_0^t u_k(\tau)e^{\nu\left(\frac{\xi_k}{a}\right)^2(\tau-t)}d\tau+C_{0k}e^{-\nu\left(\frac{\xi_k}{a}\right)^2t}. \tag{13}$$

Here coefficients C_{0k} is obtained from the expansion of initial conditions $W_0(r)$ from (9) into the Fourier–Bessel series.

Integrating $W(r,t)$ from (11), we return to the variable $\overline{w}$. Then, taking into account (8), we obtain the solution of the first equation (3)

$$w(r,t) = a\sum_{k=1}^{\infty}\frac{1}{\xi_k}\left[\int_0^t u_k(\tau)e^{\nu\left(\frac{\xi_k}{a}\right)^2(\tau-t)}d\tau + C_{0k}e^{-\nu\left(\frac{\xi_k}{a}\right)^2 t}\right]\left[J_0\left(\frac{\xi_k}{a}r\right) - J_0(\xi_k)\right] + \\ +\frac{10}{\pi a^5}Q(t)(a-r)r^2 + \left(\frac{25}{3\pi a^5}\left[Q(t)-Q(0)\right] + \frac{5A\beta}{12a}\int_0^t \omega^2(\tau)d\tau\right)\left(r^3 - \frac{6}{5}ar^2 + \frac{1}{5}a^3\right), \tag{14}$$

where

$$C_{0k} = \frac{2}{\left[aJ_1(\xi_k)\right]^2}\int_0^a r\left(w_{0r}(r) + \frac{10}{\pi a^5}Q(0)3r^2\right)J_1\left(\frac{\xi_k}{a}r\right)dr,$$

$$u_k(\tau) = \frac{2}{\left[aJ_1(\xi_k)\right]^2}\int_0^a r\left(-\nu\frac{90}{\pi a^5}Q(t) + 9\nu d(t) + \left(\frac{30}{\pi a^5}Q_t(t) - 3d_t(t)\right)r^2\right)J_1\left(\frac{\xi_k}{a}r\right)dr$$

are obtained using equalities (12). **Remark 2.** When finding coefficients in (14), the following integrals arise [8]

$$\int_0^a rJ_1\left(\frac{\xi_k}{a}r\right)dr = \frac{a^2}{\xi_k}\Gamma\left(\frac{3}{2}\right)\sqrt{\pi}\left[J_1(\xi_k)H_0(\xi_k) - J_0(\xi_k)H_1(\xi_k)\right],$$

$$\int_0^a r^3J_1\left(\frac{\xi_k}{a}r\right)dr = \frac{a^4}{\xi_k^3}\left[3J_1(\xi_k)G_{2,0}(\xi_k) - J_0(\xi_k)G_{3,1}(\xi_k)\right],$$

where $H_0(x)$, $H_1(x)$ are Struve functions, $G_{\mu,\nu}(x)$ are Lommel functions.

In addition, it is necessary to find the remaining unknown pressure gradient function $f(t)$. Multiplying the first equation (3) by r, integrating it over r from 0 to a and taking into account (4), (5), boundary and initial conditions, we obtain

$$f(t) = \frac{2\nu}{a}W(a,t) - \frac{1}{4}A\beta\omega^2(t)a^2 - \frac{1}{\pi a^2}Q_t(t), \tag{15}$$

where $W(r,t)$ is defined in (11).

It is interesting to find the initial value $f(0)$. We have from (15) that

$$f(0) = \frac{2\nu}{a}w_{0r}(a) - \frac{1}{4}A\beta\omega^2(0)a^2 - \frac{1}{\pi a^2}Q_t(0) \tag{16}$$

Functions $T(r,t)$ and $v(r,t)$ are found in the standard way. Considering second and third equations (3), we have [9]

$$T(r,t) = \sum_{k=1}^{\infty}\left[\int_0^t g_k(\tau)e^{\chi\left(\frac{\mu_k}{a}\right)^2(\tau-t)}d\tau + B_{0k}e^{-\chi\left(\frac{\mu_k}{a}\right)^2 t}\right]J_0\left(\frac{\mu_k}{a}r\right),$$

$$B_{0k} = \frac{2}{[aJ_0'(\mu_k)]^2}\int_0^a rT_0(r)J_0\left(\frac{\mu_k}{a}r\right)dr, \tag{17}$$

$$g_k(\tau) = \frac{2A}{[aJ_0'(\mu_k)]^2}\int_0^a rw(\tau,r)J_0\left(\frac{\mu_k}{a}r\right)dr,$$

$$v(r,t)=\sum_{k=1}^{\infty}\left[\int_0^t h_k(\tau)e^{\nu\left(\frac{\varepsilon_k}{a}\right)^2(\tau-t)}d\tau+N_{0k}e^{-\nu\left(\frac{\varepsilon_k}{a}\right)^2 t}\right]J_1\left(\frac{\varepsilon_k}{a}r\right),$$
$$h_k(\tau)=\frac{2\omega_t(\tau)}{[aJ_1'(\varepsilon_k)]^2}\int_0^a r^2 J_1\left(\frac{\varepsilon_k}{a}r\right)dr, \tag{18}$$
$$N_{0k}(\tau)=\frac{2}{[aJ_1'(\varepsilon_k)]^2}\int_0^a rv_0(r)J_1\left(\frac{\varepsilon_k}{a}r\right)dr,$$

where μ_k are roots of equation $J_0(\mu_k)=0$. They are $\mu_1=2.40483$, $\mu_2=5.52008$, $\mu_3=8.65373$, $\mu_4=11.7915$, $\mu_5=14.9309$. Parameters ε_k are roots of equation $J_1(\varepsilon_k)=0$. They are $\varepsilon_1=3.83171$, $\varepsilon_2=7.01559$, $\varepsilon_3=10.1735$, $\varepsilon_4=13.3237$, $\varepsilon_5=16.4706$.

4. Time evolution of the non-stationary solution

For simplicity, here we consider the practically important case $Q(t)=0$, $Q^s=0$. Let us show that solution of the non-stationary problem tends to the stationary regime as time increases, that is,

$$\lim_{t\to\infty}w(r,t)=w^s(r),\quad \lim_{t\to\infty}T(r,t)=T^s(r),\quad \lim_{t\to\infty}v(r,t)=v^s(r),\quad \lim_{t\to\infty}f(t)=f^s, \tag{19}$$

where $w^s(r)$, $T^s(r)$, $v^s(r)$, f^s are defined in (7) at $Q^s=0$. Let us introduce new functions

$$H(r,t)=w(r,t)-w^s(r),\quad \Omega(t)=\omega^2(t)-(\omega^s)^2,\quad F(t)=f(t)-f^s. \tag{20}$$

Due to linearity of the main inverse problem function $H(r,t)$ satisfies the same equation as $w(r,t)$ but with modified initial data: $H(0,r)=H_0(r)=w_0(r)-w^s(r)$, $\omega^2(t)$ is replaced with $\Omega(t)$, $f(t)$ is replaced with $F(t)$.

Substituting (20) into previously obtained solution (14), we find

$$H(r,t)=D(t)\left(r^3-\frac{6}{5}ar^2+\frac{1}{5}a^3\right)+a\sum_{k=1}^{\infty}\frac{W_k(t)}{\xi_k}\left[J_0\left(\frac{\xi_k}{a}r\right)-J_0(\xi_k)\right],$$
$$D(t)=\frac{5A\beta}{12a}\int_0^t\Omega(\tau)d\tau, \tag{21}$$

where

$$W_k(t)=e^{-\nu\frac{\xi_k^2}{a^2}t}C_{0k}+\int_0^t D_k(\tau)e^{-\nu\frac{\xi_k^2}{a^2}(t-\tau))}d\tau,$$
$$C_{0k}=\frac{2}{[aJ_1(\xi)]^2}\int_0^a rW_0(r)J_1\left(\frac{\xi_k}{a}r\right)dr, \tag{22}$$
$$D_k(t)=\frac{2}{[aJ_1(\xi)]^2}\left[9\nu D(t)\int_0^a rJ_1\left(\frac{\xi_k}{a}r\right)dr-3D_t(t)\int_0^a r^3J_1\left(\frac{\xi_k}{a}r\right)dr\right],$$

and C_{0k} are the Fourier series coefficients of $H_{0r}(r)$. Further, we have

$$F(t)=\frac{2\nu}{a}\sum_{k=1}^{\infty}W_k(t)J_1(\xi_k)-\frac{1}{4}A\beta\Omega(t)a^2+\frac{2}{5}a^2D(t). \tag{23}$$

Let us assume that $\Omega(t)$ behaves as follows

$$\Omega(t)\to 0,\quad \int_0^t\Omega(\tau)d\tau\to 0,\quad t\to\infty. \tag{24}$$

Let us take two functions that satisfy (24):

$$
\begin{gathered}
1)\ \Omega(t) = C_1 e^{-\alpha t}\left(1-\alpha t\right), \quad \alpha>0, \qquad \int_0^t \Omega(\tau)d\tau = C_1 t e^{-\alpha t};\\
2)\ \Omega(t) = C_1 e^{-\alpha t}\cos\alpha t, \qquad \int_0^t \Omega(\tau)d\tau = C_1\frac{e^{-\alpha t}}{2\alpha^2}\left[\alpha t\cos\alpha t + (\alpha t-1)\sin\alpha t\right] \to 0.
\end{gathered} \tag{25}
$$

In both cases we obtain

$$
|\Omega(t)| \leqslant C_2 t e^{-\alpha t}, \qquad \left|\int_0^t \Omega(\tau)d\tau\right| \leqslant C_2 t e^{-\alpha t} \tag{26}
$$

with positive constants C_2 and α.

Let us estimate $|H(r,t)|$ and $|F(t)|$. To do this, we first estimate $|W_k(t)|$. From (22) we obtain

$$
|D_k| \leqslant A\xi_k\left(|D(t)|+|D_t(t)|\right) \leqslant \frac{A_1}{\left[aJ_1(\xi)\right]^2}\left(\left|\int_0^t \Omega(\tau)d\tau\right| + \Omega(t)\right) \leqslant A_2 t e^{-\alpha t}, \tag{27}
$$

because $|J_1|\leqslant 1$, and $m_1/\xi_k \leqslant \int\limits_0^a rJ_1^2\left(\xi_k r/a\right)dr \leqslant m_2/\xi_k$ [10], where $A>0,\ A_{1,2}>0,\ m_{1,2}>0$ are constants.

Let us estimate the integral in the second relation (22)

$$
\begin{gathered}
\left|\int_0^t D_k(t)e^{-\nu\frac{\xi_k^2}{a^2}(t-\tau)}d\tau\right| \leqslant \frac{A_2}{m_1}\xi_k t e^{-\nu\frac{\xi_k^2}{a^2}t}\int_0^t e^{\left(\nu a^{-2}\xi_k^2-\alpha\right)\tau}d\tau = \\
= \frac{A_2}{m_1}\frac{\xi_k t}{\nu a^{-2}\xi_k^2-\alpha}\left[e^{-\alpha t} - e^{-\nu\frac{\xi_k^2}{a^2}t}\right] \leqslant \frac{A_2}{m_1}\xi_k t e^{-\alpha t}\begin{cases}\frac{1}{|\nu a^{-2}\xi_k^2-\alpha|}\left|1-e^{-\left(\nu a^{-2}\xi_k^2-\alpha\right)t}\right|,\\ t, \alpha=\nu\frac{\xi_{k0}2}{a^2}.\end{cases}
\end{gathered} \tag{28}
$$

The latter is true if for some $k=k_0$ we have $\alpha=\nu a^{-2}\xi_{k_0}^2$. For given ν, a there is a constant k_1, and at $k\geqslant k_1$ we have that $\alpha<\nu a^{-2}\xi_{k_1}^2$ because ξ_k increases with k (recall that $\xi_k\sim k\pi$ for $k\gg 1$).

Thus, the estimate for $|W_k(t)|$ is

$$
|W_k(t)| \leqslant |C_{0k}|e^{-\nu a^{-2}\xi_1^2 t}e^{-\nu a^{-2}(\xi_k^2-\xi_1^2)t} + \frac{A_2}{m_1}\xi_k t e^{-\alpha t}\begin{cases}\frac{1}{|\nu a^{-2}\xi_k^2-\alpha|}\left|1-e^{-\left(\nu a^{-2}\xi_k^2-\alpha\right)t}\right|,\\ t, \alpha=\nu\frac{\xi_{k0}2}{a^2}.\end{cases} \tag{29}
$$

Since $\xi_1<\xi_2,\ldots,\ |J_2(\xi_k)|\leqslant 1,\ |C_{0k}|<C_3,\ k\geqslant 1$ it follows from (21), (29) that

$$
\begin{gathered}
|H(r,t)| \leqslant |D(t)|\frac{12}{5}a^3 + C_3 a\,\frac{e^{-\nu a^{-2}\xi_1^2 t}}{\xi_1}\sum_{k=1}^{\infty} e^{-\nu a^{-2}(\xi_k^2-\xi_1^2)t} + \\
+\frac{A_2}{m_1}at\left\{\left(e^{-\alpha t}+e^{-\nu a^{-2}\xi_1^2 t}\right)\sum_{k=1}^{k_1}\frac{1}{|\nu a^{-2}\xi_k^2-\alpha|} + e^{-\alpha t}\sum_{k=k_1+1}^{\infty}\frac{1}{|\nu a^{-2}\xi_k^2-\alpha|}\right\}.
\end{gathered} \tag{30}
$$

It is known that the "large" roots $J_2(\xi)$ are approximately equal to $\xi_k\equiv\xi_k^{(2)}\approx 7\pi/4+k\pi$. Then, in curly brackets in the numerator the term t appears for $k=k_0$. Series (30) converges uniformly for $t\geqslant\varepsilon>0$ by the Dalamber criterion. Therefore, taking into account (26) $|H(r,t)|$ tends to zero as $te^{-\alpha t}$ when $t\to\infty$ uniformly for $r\in[0,a]$.

As for $F(t)$, the first part of its representation (23) is estimated as $|H(r,t)|$ in (30), and the estimation of the last terms follows from (26).

Estimates for $T^1(r,t)=T(r,t)-T^s(r)$ and $V(r,t)=v(r,t)-v^s(r)$ can be found in a similar way. Thus, if we additionally require $|\omega'(t)|\leqslant C_4 e^{-\alpha t}$ then the following theorem is true.

Theorem 1. *Let* $w_0(r) \in C^1[0,a]$, $\int\limits_0^a rw_0(r)dr = 0$, $T_0(r) \in C[0,r]$, $v_0(r) \in v[0,r]$ *and conditions (26) are satisfied. Then non-stationary solution of the inverse problem tends to zero with increasing time to stationary solution (7). To be more specific, when* $t \to \infty$, $r \in [0,a]$ *we obtain that*

$$\begin{gathered} |w(r,t) - w^s(r)| \leqslant C_5 te^{-\alpha_0 t}, \quad |f(t) - f^s| \leqslant C_6 te^{-\alpha_0 t}, \\ |T(r,t) - T^s(r)| \leqslant C_7 te^{-\alpha_1 t}, \quad |v(r,t)| \leqslant C_8 te^{-\alpha_2 t} \end{gathered} \tag{31}$$

with positive constants C_5, C_6, C_7, C_8, $\alpha_0 = \min(\alpha, \nu a^{-2}\xi_1^2)$, $\alpha_1 = \min\left(\alpha, \chi a^{-2}\mu_1^2\right)$, $\alpha_2 = \min\left(\alpha, \chi a^{-2}\varepsilon_1\right)$, *where* $J_2(\xi_k) = 0$, $J_1(\varepsilon_k) = 0$, $J_0(\mu_k) = 0$.

Remark 3. When $w^s = 0$, $f^s = 0$, $\omega^s = 0$, $Q^s = 0$, $T^s = 0$, $v^s = 0$ inequalities (31) are the a priori estimates of the formulated problem under conditions (24), (26).

Thus, it follows from estimates (31) that solution of the non-stationary problem exponentially tends to the stationary solution with increasing time. It means the asymptotic stability of the stationary solution (7).

Remark 4. The resulting solution is classical. This is proved by estimating the Fourier series with respect to the Bessel functions [10].

Conclusion

The following results were obtained in the work:

1. A stationary solution of the inverse problem is found in terms of polynomials in r.
2. The non-stationary solution of the problem is obtained by the method of separation of variables. In this case, the problem is reduced to the direct problem with an integral (nonclassical) condition.
3. With the help of the a priori estimates, it is established that non-stationary solution tends to the stationary mode with increasing time if conditions of the theorem are satisfied.

The obtained results are of theoretical and practical importance. They can be used to simulate rotationally symmetric convective motions of a viscous heat-conducting fluid in rotating tubes.

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References

[1] V.Barcilon, J.Pedlosky, On the steady motions produced by a stable stratification in a rapidly rotating fluid *Jour. Fluid Mech.*, **29**(1967), 673.

[2] G.M.Homsy, J. L.Hudson, Centrifugal covection and its effect on the asymtotic stability of a bounded rotating fluid heated from below, *Jour. Fluid Mech.*, **48**(1971), 605.

[3] J.EHart, On the influence of centrifugal buoyncy on rotating convection, *Jour. Fluid Mech.*, **403**(2000), 133. DOI: 10.1017/S0022112099006928

[4] F.H.Busse, Om the influence of centrifugal buoyncy on rotating convection, *Jour. Fluid Mech.* **44**(1970), 441.

[5] J.Herrmann, F.H.Busse, Convection in a rotating cylindrical annulus. Part 4. Modulation and transition to chaos at low Prandtl numbers, *Jour. Fluid Mech.*, **350**(1997), 209.

[6] G. Z.Gershuni, E.M.Zhukhovitsky, Convective stability of an incompressible fluid, Moscow, Nauka, 1972.

[7] L.V.Ovsyannikov, Group Analysis of Differential Equations, Moscow, Nauka, 1978.

[8] G.Bejtmen, A.Erdane, Higher transcendental functions. Vol. 2. Bessel Functions, Parabolic Cylinder Functions, Orthogonal Polynomials, Moscow, Nauka, 1974.

[9] A.D.Polyanin, Handbook of Linear Equations of Mathematical Physics, Moscow, FIZMATLIT, 2001.

[10] G.P.Tolstov, Fourier series, Moscow, FIZMATLIT, 1960.

Решение задачи о конвекции во вращающейся трубе методом Фурье

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Аннотация. Исследована нестационарная краевая задача о движении жидкости во вращающейся цилиндрической трубе. Для описания движения жидкости используются уравнения Обербека-Буссинеска. С математической точки зрения задача является обратной относительно градиента давления вдоль оси цилиндра. Методом разделения переменных решение найдено в виде специальных рядов Фурье. Даны достаточные условия выхода решения нестационарной задачи с ростом времени на стационарный режим.

Ключевые слова: конвекция, обратная задача, асимптотическое поведение, метод разделения переменных, функции Бесселя.