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Common Coupled Fixed Point Theorems for a Pair of S_b-metric Spaces

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Abstract. In this work, we investigate the existence of common coupled fixed point and coupled coincidence points in a setting of two S_b-metric spaces. Here we use a pair of w-compatible mappings. Various results are also given in the form of corollaries.

Keywords: common coupled fixed point, coupled coincidence point, S_b-metric spaces, w-compatible mappings.

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1. Introduction and preliminaries

S. Sedghi, N. Shobe and A. Aliouche [1] introduced S-metric space as a generalisation of metric space. They also claimed that S-metric space is a generalisation of G-metric space. But some researchers commented that the claim is not true. Further it is claimed that the class of S-metric and the class of G-metric are all distinct. For detail results in this claim and more about S-metric space one can see research papers in [2–4] and references there in.

Bakhtin [5] introduced the concept of b-metric space. The concept of Bakhtin is extensively used by S. Czerwick [6, 7]. Nizar and Nabil [8] introduced the concept of S_b-metric space by using the concept of both S-metric and b-metric. Y. Rohen, T. Došenović and S. Radenović [9]

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also gave a more general definition of S_b-metric space. For more results on S_b-metric space one can see research papers in [10–13].

In this paper we prove a common coupled fixed point and coupled coincidence point theorem for a pair of w-compatible mappings in the setting of two S_b-metric spaces in the line of the results obtained by Feng Gu [16].

Following definitions and properties will be needed in order to start the main result.

Definition 1 ([8]). *Let W be a nonempty set and let $b \geqslant 1$ be a given number. A function $S : W^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $\theta, \phi, \lambda, \mu \in X$, the following conditions hold:*

(i) $S(\theta, \phi, \lambda) = 0$ *if and only if* $\theta = \phi = \lambda$

(ii) $S(\theta, \phi, \lambda) \leqslant b[S(\theta, \theta, \mu) + S(\phi, \phi, \mu) + S(\lambda, \lambda, \mu)]$

(iii) $S(\theta, \theta, \phi) = S(\phi, \phi, \theta)$ *for all* $\theta, \phi \in W$.

The pair (W, S) is called a symmetric S_b-metric space. If the pair (W, S) does not fulfil (**iii**) *then it is called an S_b-metric space.*

Example 1 ([8]). *Let W be a nonempty set and card $(W) \geqslant 5$. Suppose $W = W_1 \cup W_2$ a partition of W such that card $(W_1) \geqslant 4$. Let $b \geqslant 1$. Then*

$$S_b(\theta, \phi, \lambda) = \begin{cases} 0, & \text{if } \theta = \phi = \lambda = 0 \\ 3b, & \text{if } (\theta, \phi, \lambda) \in W^3 \\ 1, & \text{if } (\theta, \phi, \lambda) \notin W^3 \end{cases}.$$

Definition 2 ([14, 15]). *Let W be a non-empty set, $P : W \times W \to W$ and $q : W \to W$ are two mappings then*

(i) *an element $(\theta, \phi) \in W \times W$ satisfying $P(\theta, \phi) = \theta$ and $P(\phi, \theta) = \phi$ is called a coupled fixed point of P.*

(ii) *an element $(\theta, \phi) \in W \times W$ satisfying $P(\theta, \phi) = q\theta$, $P(\phi, \theta) = q\phi$ is called a couple coincidence point of P and q. The point $(q\theta, q\phi)$ is called a coupled point of coincidence.*

(iii) *an element $(\theta, \phi) \in W \times W$ satisfying $P(\theta, \phi) = q(\theta) = \theta$, $P(\phi, \theta) = q\phi = \phi$ is called a common coupled fixed point of P and q.*

(iv) *the pair of mappings P and q is said to be w-compatible if $P(\theta, \phi) = q\theta$ and $P(\phi, \theta) = q\phi$ implies $qP(\theta, \phi) = P(q\theta, q\phi)$.*

2. Main results

We prove the following theorems.

Theorem 2.1. *Let S_1, S_2 be two S_b metric spaces in a non-empty set W satisfying $S_2(\theta, \theta, \phi) \leqslant S_1(\theta, \theta, \phi)$ for all $\theta, \phi \in W$ and $b \geqslant 1$ is a real number. Let $P : W \times W \to W$ and $q : W \to W$ be two mappings satisfying the following conditions*

(i)
$$\begin{aligned} & S_1(P(\theta, \phi), P(\theta, \phi), P(\xi, \eta)) + S_2(P(\phi, \theta), P(\phi, \theta), P(\eta, \xi)) \leqslant \\ & \leqslant k_1[S_2(q\theta, q\theta, q\xi) + S_2(q\phi, q\phi, q\eta)]] + k_2[S_2(q\theta, q\theta, P(\theta, \phi)) + S_2(q\phi, q\phi, P(\phi, \theta))] + \\ & + k_3[S_2(q\theta, q\theta, P(\xi, \eta)) + S_2(q\phi, q\phi, P(\eta, \xi))] \end{aligned}$$

where $(\theta, \phi), (\xi, \eta) \in W \times W$ and k_1, k_2, k_3 in $[0, 1]$ such that $0 \leqslant k_1 + k_2 + k_3 \leqslant \dfrac{1}{b^2}$

(ii) *$P(W \times W) \subset q(W)$*

(iii) *$q(W)$ is S_1 complete*

then P and q have a coupled coincidence point. Further, if P and q are w-compatible then P and q have a unique common coupled fixed point.

Proof. Let $(\theta_0, \phi_0) \in W \times W$. By (ii) there exists $(\theta_1, \phi_1) \in W$ such that $q\theta_1 = P(\theta_0, \phi_0)$, $q\phi_1 = P(\phi_0, \theta_0)$. Similarly, $(\theta_2, \phi_2) \in W$ such that $q\theta_2 = P(\theta_1, \phi_1)$, $q\phi_2 = P(\phi_1, \theta_1)$.

Continuing in this way sequences $\{\theta_n\}$ and $\{\phi_n\}$ can be constructed as

$$q\theta_{n+1} = P(\theta_n, \phi_n), q\phi_{n+1} = P(\phi_n, \theta_n), \quad for\ all\ n \geqslant 0.$$

From (i) we have,

$$\begin{aligned}
& S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_2(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2}) = \\
= \ & S_1(P((\theta_n, \phi_n), P(\theta_n, \phi_n), P(\theta_{n+1}, \phi_{n+1})) + S_2(P((\phi_n, \theta_n), P(\phi_n, \theta_n), P(\phi_{n+1}, \theta_{n+1})) \leqslant \\
\leqslant \ & k_1[S_2(q\theta_n, q\theta_n, q\theta_{n+1}) + S_2(q\phi_n, q\phi_n, q\phi_{n+1})] + \\
& + k_2[S_2(q\theta_n, q\theta_n, P(\theta_n, \phi_n)) + S_2(q\phi_n, q\phi_n, P(\phi_n, \theta_n))] + \\
& + k_3[S_2(q\theta_{n+1}, q\theta_{n+1}, P(\theta_{n+1}, \phi_{n+1})) + S_2(q\phi_{n+1}, q\phi_{n+1}, P(\phi_{n+1}, \theta_{n+1}))] = \\
= \ & k_1[S_2(q\theta_n, q\theta_n, q\theta_{n+1}) + S_2(q\phi_n, q\phi_n, q\phi_{n+1})] + \\
& + k_2[S_2(q\theta_n, q\theta_n, q\theta_{n+1}) + S_2(q\phi_n, q\phi_n, q\phi_{n+1})] + \\
& + k_3[S_2(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_2(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2})] \leqslant \\
\leqslant \ & k_1[S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})] + \\
& + k_2[S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})] + \\
& + k_3[S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2})] \,. \qquad (1)
\end{aligned}$$

It follows from (1) that

$$\begin{aligned}
& S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2}) \leqslant \\
& \leqslant \frac{k_1 + k_2}{1 - k_3}[S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})] = \\
& = k[S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})], \qquad (2)
\end{aligned}$$

where $k = \dfrac{k_1 + k_2}{1 - k_3}$, by the condition $0 \leqslant k_1 + k_2 + k_3 < \dfrac{1}{b^2}$, then we have $0 \leqslant k < \dfrac{1}{b^2}$. By taking

$$\delta_n = S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1}),$$

thus,

$$\delta_{n+1} \leqslant k\delta_n \leqslant k^2\delta_{n-1} \leqslant \quad \ldots \quad \leqslant k^{n+1}\delta_0 \,. \qquad (3)$$

Next, we show that $\{q\theta_n\}$ and $\{q\phi_n\}$ are Cauchy sequences in $q(W)$. For this, we consider $S_1(\theta_n, \theta_n, \theta_{n+p})$ into two cases.
Firstly, considering $p = 2l + 1$

$$\begin{aligned}
S_1(q\theta_n, q\theta_n, q\theta_{n+p}) \ &= \ S_1(q\theta_n, q\theta_n, q\theta_{n+2l+1}) \leqslant \\
&\leqslant \ 2bS_1(q\theta_n, q\theta_n, q\theta_{n+1}) + b^2 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2l+1}) \leqslant
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2bS_1(q\theta_n, q\theta_n, q\theta_{n+1}) + 2b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
&\quad + b^4 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+2l+1}) \leqslant \\
&\leqslant \dots \\
&\leqslant 2bS_1(q\theta_n, q\theta_n, q\theta_{n+1}) + 2b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
&\quad + 2b^5 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + \dots + \\
&\quad + 2b^{2(2l-1)+1} S_1(q\theta_{n+2l-1}, q\theta_{n+2l-1}, q\theta_{n+2l}) + \\
&\quad + b^{2(2l-1)+2} S_1(q\theta_{n+2l}, q\theta_{n+2l}, q\theta_{n+2l+1}) \leqslant \\
&\leqslant 2\{b(S_1(q\theta_n, q\theta_n, q\theta_{n+1})) + b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
&\quad + b^5 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + \dots + \\
&\quad + b^{2(2l-1)+1} S_1(q\theta_{n+2l-1}, q\theta_{n+2l-1}, q\theta_{n+2l}) + \\
&\quad + b^{2(2l-1)+3} S_1(q\theta_{n+2l}, q\theta_{n+2l}, q\theta_{n+2l+1})\} \,. \qquad (4)
\end{aligned}
$$

We can similarly prove the following result

$$
\begin{aligned}
S_1(q\phi_n, q\phi_n, q\phi_{n+p}) &= S_1(q\phi_n, q\phi_n, q\phi_{n+2l+1}) \leqslant \\
&\leqslant 2\{bS_1(q\phi_n, q\phi_n, q\phi_{n+1}) + b^3 S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2}) + \\
&\quad + b^5 S_1(q\phi_{n+2}, q\phi_{n+2}, q\phi_{n+3}) + \dots + \\
&\quad + b^{2(2l-1)+1} S_1(q\phi_{n+2l-1}, q\phi_{n+2l-1}, q\phi_{n+2l}) + \\
&\quad + b^{2(2l-1)+3} S_1(q\phi_{n+2l}, q\phi_{n+2l}, q\phi_{n+2l+1})\} \,. \qquad (5)
\end{aligned}
$$

Adding (4) and (5), we have

$$
\begin{aligned}
&S_1(q\theta_n, q\theta_n, q\theta_{n+p}) + S_1(q\phi_n, q\phi_n, q\phi_{n+p}) = \\
= \; &S_1(q\theta_n, q\theta_n, q\theta_{n+2l+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+2l+1}) \leqslant \\
\leqslant \; &2[b\{S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})\} + \\
&+ b^3\{S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2})\} + \\
&+ b^5\{S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + S_1(q\phi_{n+2}, q\phi_{n+2}, q\phi_{n+3})\} + \\
&+ \dots + \\
&+ b^{2(2l-1)+3}\{S_1(q\theta_{n+2l}, q\theta_{n+2l}, q\theta_{n+2l+1}) + S_1(q\phi_{n+2l}, q\phi_{n+2l}, q\phi_{n+2l+1})\}] = \\
= \; &2[b\delta_n + b^3\delta_{n+1} + b^5\delta_{n+2} + \dots + b^{2(2l-1)+3}\delta_{n+2l}] = \\
= \; &2[bk^n\delta_0 + b^3k^{n+1}\delta_0 + b^5k^{n+2}\delta_0 + \dots + b^{2(2l-1)+3}k^{n+2l}\delta_0] = \\
= \; &2bk^n\delta_0\{1 + b^2k + b^4k^2 + \dots + b^{4l}k^{2l}\} = \\
= \; &2bk^n \frac{1}{1 - b^2k}\delta_0 \,. \qquad (6)
\end{aligned}
$$

Secondly, considering $p = 2l$

$$
\begin{aligned}
S_1(q\theta_n, q\theta_n, q\theta_{n+p}) &= S_1(q\theta_n, q\theta_n, q\theta_{n+2l}) \leqslant \\
&\leqslant 2bS_1(q\theta_n, q\theta_n, q\theta_{n+1})) + b^2 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2l}) \leqslant \\
&\leqslant 2b(S_1(q\theta_n, q\theta_n, q\theta_{n+1})) + 2b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
&\quad + b^4 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+2l}) \leqslant \\
&\leqslant \dots \\
&\leqslant 2bS_1(q\theta_n, q\theta_n, q\theta_{n+1}) + 2b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
&\quad + 2b^5 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + \dots +
\end{aligned}
$$

$$
\begin{aligned}
& +2\, b^{2(2l-2)+1} S_1(q\theta_{n+2l-2}, q\theta_{n+2l-2}, q\theta_{n+2l-1}) + \\
& + b^{2(2l-2)+2} S_1(q\theta_{n+2l-1}, q\theta_{n+2l-1}, q\theta_{n+2l}) \leqslant \\
\leqslant \;& 2\{bS_1(q\theta_n, q\theta_n, q\theta_{n+1}) + b^3 S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + \\
& + b^5 S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + \cdots + \\
& + b^{2(2l-2)+1} S_1(q\theta_{n+2l-2}, q\theta_{n+2l-2}, q\theta_{n+2l-1}) + \\
& + b^{2(2l-2)+3} S_1(q\theta_{n+2l-1}, q\theta_{n+2l-1}, q\theta_{n+2l})\}\, . \qquad (7)
\end{aligned}
$$

By similar arguments as above,

$$
\begin{aligned}
S_1(q\phi_n, q\phi_n, q\phi_{n+p}) \;=\;& S_1(q\phi_n, q\phi_n, q\phi_{n+2l}) \leqslant \\
\leqslant \;& 2\{bS_1(q\phi_n, q\phi_n, q\phi_{n+1}) + b^3 S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2}) + \\
& + b^5 S_1(q\phi_{n+2}, q\phi_{n+2}, q\phi_{n+3}) + \cdots + \\
& + b^{2(2l-2)+1} S_1(q\phi_{n+2l-2}, q\phi_{n+2l-2}, q\phi_{n+2l-1}) + \\
& + b^{2(2l-2)+3} S_1(q\phi_{n+2l-1}, q\phi_{n+2l-1}, q\phi_{n+2l})\}\, . \qquad (8)
\end{aligned}
$$

Adding (7) and (8) we get

$$
\begin{aligned}
& S_1(q\theta_n, q\theta_n, q\theta_{n+p}) + S_1(q\phi_n, q\phi_n, q\phi_{n+p}) = \\
= \;& S_1(q\theta_n, q\theta_n, q\theta_{n+2l}) + S_1(q\phi_n, q\phi_n, q\phi_{n+2l}) \leqslant \\
\leqslant \;& 2[b\{S_1(q\theta_n, q\theta_n, q\theta_{n+1}) + S_1(q\phi_n, q\phi_n, q\phi_{n+1})\} + \\
& + b^3\{S_1(q\theta_{n+1}, q\theta_{n+1}, q\theta_{n+2}) + S_1(q\phi_{n+1}, q\phi_{n+1}, q\phi_{n+2})\} + \\
& + b^5\{S_1(q\theta_{n+2}, q\theta_{n+2}, q\theta_{n+3}) + S_1(q\phi_{n+2}, q\phi_{n+2}, q\phi_{n+3})\} + \\
& + \ldots + \\
& + b^{2(2l-2)+3}\{S_1(q\theta_{n+2l-1}, q\theta_{n+2l-1}, q\theta_{n+2l}) + S_1(q\phi_{n+2l-1}, q\phi_{n+2l-1}, q\phi_{n+2l})\}] = \\
= \;& 2[b\delta_n + b^3\delta_{n+1} + b^5\delta_{n+2} + \cdots + b^{2(2l-2)+3}\delta_{n+2l-1}] = \\
= \;& 2[bk^n\delta_0 + b^3k^{n+1}\delta_0 + b^5k^{n+2}\delta_0 + \cdots + b^{2(2l-2)+3}k^{n+2l-1}\delta_0] = \\
= \;& 2bk^n\delta_0\{1 + b^2k + b^4k^2 + \cdots + b^{4l-2}k^{2l-1}\} = \\
= \;& 2bk^n\frac{1}{1-b^2k}\delta_0]\, . \qquad (9)
\end{aligned}
$$

Since $k \in [0, \frac{1}{b^2})$, so $k^n \to 0$ when n tends to infinity. From (6) and (9), we have

$$\lim_{n\to\infty}[S_1(q\theta_n, q\theta_n, q\theta_{n+p}) + S_1(q\phi_n, q\phi_n, q\phi_{n+p})] = 0$$

which implies that $\{q\theta_n\}$ and $\{q\phi_n\}$ are Cauchy sequences in $q(W)$. Since $q(W)$ is complete, then there exists $\theta, \phi \in W$ such that

$$\lim_{n\to\infty} q\theta_n = q\theta \text{ and } \lim_{n\to\infty} q\phi_n = q\phi.$$

It follows from (i) and (3) that

$$
\begin{aligned}
& S_1(q\theta_{n+1}, q\theta_{n+1}, P(\theta,\phi)) + S_1(q\phi_{n+1}, q\phi_{n+1}, P(\phi,\theta)) = \\
= \;& S_1(P(\theta_n,\phi_n), P(\theta_n,\phi_n), P(\theta,\phi)) + S_1(P(\phi_n,\theta_n), P(\phi_n,\theta_n), P(\phi,\theta)) \leqslant \\
\leqslant \;& k_1[S_2(q\theta_n, q\theta_n, q\theta) + S_2(q\phi_n, q\phi_n, q\phi)] +
\end{aligned}
$$

$$
\begin{aligned}
&+k_2[S_2(q\theta_n,q\theta_n,P(\theta_n,\phi_n))+S_2(q\phi_n,q\phi_n,P(\phi_n,\theta_n))]+\\
&+k_3[S_2(q\theta,q\theta,P(\theta,\phi))+S_2(q\phi,q\phi,P(\phi,\theta))]=\\
=\;& k_1[S_2(q\theta_n,q\theta_n,q\theta)+S_2(q\phi_n,q\phi_n,q\phi)]+\\
&+k_2[S_2(q\theta_n,q\theta_n,q\theta_{n+1})+S_2(q\phi_n,q\phi_n,q\phi_{n+1})]+\\
&+k_3[S_2(q\theta,q\theta,P(\theta,\phi))+S_2(q\phi,q\phi,P(\phi,\theta))]\leqslant\\
\leqslant\;& k_1[S_1(q\theta_n,q\theta_n,q\theta)+S_1(q\phi_n,q\phi_n,q\phi)]+\\
&+k_2[S_1(q\theta_n,q\theta_n,q\theta_{n+1})+S_1q\phi_n,q\phi_n,q\phi_{n+1})]+\\
&+k_3[S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))]=\\
=\;& k_1[S_1(q\theta_n,q\theta_n,q\theta)+S_1(q\phi_n,q\phi_n,q\phi)]+\\
&+k_2\delta_n+k_3[S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))]\leqslant\\
\leqslant\;& k_1[S_1(q\theta_n,q\theta_n,q\theta)+S_1(q\phi_n,q\phi_n,q\phi)]+\\
&+k_2k^n\delta_0+k_3[S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))]\,. \qquad (10)
\end{aligned}
$$

From (10) and (3), we have

$$
\begin{aligned}
&S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))\leqslant\\
\leqslant\;& 2b\{S_1(q\theta,q\theta,q\theta_{n+1})+b^2S_1(q\theta_{n+1},q\theta_{n+1},P(\theta,\phi))\}+\\
&+2b\{S_1(q\phi,q\phi,q\phi_{n+1})+b^2S_1(q\phi_{n+1},q\phi_{n+1},P(\phi,\theta))\}=\\
=\;& 2b\{S_1(q\theta,q\theta,q\theta_{n+1})+S_1(q\phi,q\phi,q\phi_{n+1})\}+\\
&+b^2\{S_1(P(\theta_n,\phi_n),P(\theta_n,\phi_n),P(\theta,\phi))+S_1(P(\phi_n,\theta_n),P(\phi_n,\theta_n),P(\phi,\theta))\}=\\
=\;& 2b\{S_1(q\theta,q\theta,q\theta_{n+1})+S_1(q\phi,q\phi,q\phi_{n+1})\}+\\
&+b^2[k_1\{S_2(q\theta_n,q\theta_n,q\theta)+S_2(q\phi_n,q\phi_n,q\phi)\}+\\
&+k_2\{S_2(q\theta_n,q\theta_n,P(\theta_n,\phi_n))+S_2(q\phi_n,q\phi_n,P(\phi_n,\theta_n))\}+\\
&+k_3\{S_2(q\theta,q\theta,P(\theta,\phi))+S_2(q\phi,q\phi,P(\phi,\theta))\}]\leqslant\\
\leqslant\;& 2b\{S_1(q\theta,q\theta,q\theta_{n+1})+S_1(q\phi,q\phi,q\phi_{n+1})\}+\\
&+b^2k_1\{S_1(q\theta_n,q\theta_n,q\theta)+S_1(q\phi_n,q\phi_n,q\phi)\}+\\
&+b^2k_2\{S_1(q\theta_n,q\theta_n,P(\theta_n,\phi_n))+S_1(q\phi_n,q\phi_n,P(\phi_n,\theta_n))\}+\\
&+b^2k_3\{S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))\}=\\
=\;& 2b\{S_1(q\theta,q\theta,q\theta_{n+1})+S_1(q\phi,q\phi,q\phi_{n+1})\}+\\
&+b^2k_1\{S_1(q\theta_n,q\theta_n,q\theta)+S_1(q\phi_n,q\phi_n,q\phi)\}+\\
&+b^2k_2\{S_1(q\theta_n,q\theta_n,q\theta_{n+1})+S_1(q\phi_n,q\phi_n,q\phi_{n+1})\}+\\
&+b^2k_3\{S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))\}\,. \qquad (11)
\end{aligned}
$$

Taking limit as $n\to\infty$ we have

$$
\begin{aligned}
&(1-b^2k_3)\{S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))\}\leqslant 2b\times 0+b^2k_1\times 0+b^2k_2\times 0\\
\Rightarrow\;& S_1(q\theta,q\theta,P(\theta,\phi))+S_1(q\phi,q\phi,P(\phi,\theta))=0\\
&[\text{because } 1-b^2k_3\geqslant 0,\ b^2(k_1+k_2)\geqslant 0]\,. \qquad (12)
\end{aligned}
$$

So, $q\theta=P(\theta,\phi)$ and $q\phi=P(\phi,\theta)$ which shows that (θ,ϕ) is the coupled coincidence point of P and q.
In order to prove the uniqueness of coupled coincidence point, let (θ^*,ϕ^*) be the second coupled coincidence point of P and q.

From (i),

$$\begin{aligned}
& S_1(q\theta, q\theta, q\theta^*) + S_1(q\phi, q\phi, q\phi^*) = \\
= \; & S_1(P(\theta,\phi), P(\theta,\phi), P(\theta^*,\phi^*)) + S_1(P(\phi,\theta), P(\phi,\theta), P(\phi^*,\theta^*)) \leqslant \\
\leqslant \; & k_1\{S_2(q\theta, q\theta, q\theta^*) + S_2(q\phi, q\phi, q\phi^*)\} + \\
& + k_2\{S_2(q\theta, q\theta, P(\theta,\phi)) + S_2(q\phi, q\phi, P(\phi,\theta))\} + \\
& + k_3\{S_2(q\theta^*, q\theta^*, P(\theta^*,\phi^*)) + S_2(q\phi^*, q\phi^*, P(\phi^*,\theta^*))\} = \\
= \; & k_1\{S_2(q\theta, q\theta, q\theta^*) + S_2(q\phi, q\phi, q\phi^*)\} \leqslant \\
\leqslant \; & k_1\{S_1(q\theta, q\theta, q\theta^*) + S_1(q\phi, q\phi, q\phi^*)\} \, . \qquad (13)
\end{aligned}$$

But $0 \leqslant k_1 + k_2 + k_3 < \dfrac{1}{b^2} \Rightarrow k_1 < 1$. We have

$$S_1(q\theta, q\theta, q\theta^*) + S_1(q\phi, q\phi, q\phi^*) = 0.$$

Thus, $q\theta = q\theta^*$ and $q\phi = q\phi^*$, which shows that coupled point of coincidence of P and q is unique.

Now, we need to show $q\theta = q\phi$. By (1), we have

$$\begin{aligned}
& S_1(q\theta, q\theta, q\phi) + S_1(q\phi, q\phi, q\theta) = \\
= \; & S_1(P(\theta,\phi), P(\theta,\phi), P(\phi,\theta)) + S_1(P(\phi,\theta), P(\phi,\theta), P(\theta,\phi)) \leqslant \\
\leqslant \; & k_1\{S_2(q\theta, q\theta, q\phi) + S_2(q\phi, q\phi, q\theta)\} + \\
& + k_2\{S_2(q\theta, q\theta, P(\theta,\phi)) + S_2(q\phi, q\phi, P(\phi,\theta))\} + \\
& + k_3\{S_2(q\phi, q\phi, P(\phi,\theta)) + S_2(q\theta, q\theta, P(\theta,\phi))\} = \\
= \; & k_1\{S_2(q\theta, q\theta, q\phi) + S_2(q\phi, q\phi, q\theta)\} \leqslant \\
\leqslant \; & k_1\{S_1(q\theta, q\theta, q\phi) + S_1(q\phi, q\phi, q\theta)\} \, .
\end{aligned}$$

As $0 \leqslant k_1 \leqslant k_1 + k_2 + k_3 < \dfrac{1}{b^2} < 1$ and (10), we deduce

$$S_1(q\theta, q\theta, q\phi) + S_1(q\phi, q\phi, q\theta) = 0.$$

Hence, $q\theta = q\phi$.

By w-compatibility of P and q, we get $q(P(\theta,\phi)) = P(q\theta, q\phi)$. Taking $a = q\theta$, we get $a = q\theta = P(\theta,\phi) = q\phi = P(\phi,\theta)$,
therefore $qa = qq\theta = q(P(\theta,\phi)) = P(q\theta, q\phi) = P(a,a)$.
Hence, (qa, qa) is a coupled point of coincidence of q and P. Due to uniqueness, $qa = q\theta$,
therefore $P(a,a) = qa = a$,
therefore (a,a) is the unique common coupled fixed point of q and P. □

Corollary 2.2. *Let S_1, S_2 be two S_b metric spaces in a non-empty set W satisfying $S_2(\theta,\theta,\phi) \leqslant S_1(\theta,\theta,\phi)$ for all $\theta,\phi \in W$ and $b \geqslant 1$ is a real number. Let $P : W \times W \to W$ and $q : W \to W$ be two mappings satisfying the following conditions*

(i)
$$\begin{aligned}
& S_1(P(\theta,\phi), P(\theta,\phi), P(\xi,\eta)) \leqslant \\
& \leqslant k_1[S_2(q\theta, q\theta, q\xi) + S_2(q\phi, q\phi, q\eta)] + k_2[S_2(q\theta, q\theta, P(\theta,\phi)) + S_2(q\phi, q\phi, P(\phi,\theta))] + \\
& + k_3[S_2(q\xi, q\xi, P(\xi,\eta)) + S_2(q\eta, q\eta, P(\eta,\xi))]
\end{aligned}$$

where $(\theta,\phi), (\xi,\eta) \in W \times W$ and k_1, k_2, k_3 in $[0,1)$ such that $0 \leqslant 2(k_1 + k_2 + k_3) < \dfrac{1}{b^2}$

(ii) $P(W \times W) \subset q(W)$

(iii) *$q(W)$ is S_1 complete.*

Then, P and q have a coupled coincidence point. Further, if P and q are w-compatible then P and q have a unique common coupled fixed point.

Proof. It follows from (11) that

$$\begin{aligned} & S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta)) \leqslant \\ \leqslant \ & k_1[S_2(q\theta,q\theta,q\xi)+S_2(q\phi,q\phi,q\eta)]+k_2[S_2(q\theta,q\theta,P(\theta,\phi))+S_2(q\phi,q\phi,P(\phi,\theta))]+ \\ & +k_3[S_2(q\xi,q\xi,P(\xi,\eta))+S_2(q\eta,q\eta,P(\eta,\xi))] \end{aligned} \tag{14}$$

and

$$\begin{aligned} & S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \\ \leqslant \ & k_1[S_2(q\phi,q\phi,q\eta)+S_2(q\theta,q\theta,q\xi)]+k_2[S_2(q\phi,q\phi,P(\phi,\theta))+S_2(q\theta,q\theta,P(\theta,\phi))]+ \\ & +k_3[S_2(q\eta,q\eta,P(\eta,\xi))+S_2(q\xi,q\xi,P(\xi,\eta))] \ . \end{aligned} \tag{15}$$

Adding (14) and (15) we have

$$\begin{aligned} & S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta))+S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \\ \leqslant \ & 2k_1[S_2(q\theta,q\theta,q\xi)+S_2(q\phi,q\phi,q\eta)]+2k_2[S_2(q\theta,q\theta,P(\theta,\phi))+S_2(q\phi,q\phi,P(\phi,\theta))]+ \\ & +2k_3[S_2(q\xi,q\xi,P(\xi,\eta))+S_2(q\eta,q\eta,P(\eta,\xi))] \ . \end{aligned}$$

By the Theorem 2.1, we get the conclusion. □

Taking $S_1(\theta,\theta,\phi)=S_2(\theta,\theta,\phi)=S(\theta,\theta,\phi)$ for all $\theta,\phi \in W$, in Theorem 2.1, we have,

Corollary 2.3. *Let (W,S) be a complete S_b metric space and $b \geqslant 1$ is a real number. Let $P: W\times W \to W$ and $q: W \to W$ be two mappings satisfying the following conditions*

(i)
$$\begin{aligned} & S(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta))+S(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \\ & \leqslant k_1[S(q\theta,q\theta,q\xi)+S(q\phi,q\phi,q\eta)]+k_2[S(q\theta,q\theta,P(\theta,\phi))+S(q\phi,q\phi,P(\phi,\theta))]+ \\ & +k_3[S(q\xi,q\xi,P(\xi,\eta))+S(q\eta,q\eta,P(\eta,\xi))] \end{aligned}$$
where $(\theta,\phi),(\xi,\eta) \in W \times W$ and k_1,k_2,k_3 in $[0,1)$ such that $0 \leqslant 2(k_1+k_2+k_3) < \dfrac{1}{b^2}$

(ii) $P(W\times W) \subset q(W)$.

Then, P and q have a coupled coincidence point. Further, if P and q are w-compatible then P and q have a unique common coupled fixed point.

Corollary 2.4. *Let S_1, S_2 be two S_b metric spaces in a non-empty set W satisfying $S_2(\theta,\theta,\phi) \leqslant S_1(\theta,\theta,\phi)$ for all $\theta,\phi \in W$ and $b \geqslant 1$ is a real number. Let $P: W\times W\to W$ and $q: W \to W$ be two mappings satisfying the following conditions*

(i)
$$\begin{aligned} & S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta)) \leqslant \\ & \leqslant a_1S_2(q\theta,q\theta,q\xi)+a_2S_2(q\phi,q\phi,q\eta)+a_3S_2(q\theta,q\theta,P(\theta,\phi))+a_4S_2(q\phi,q\phi,P(\phi,\theta))+ \\ & +a_5S_2(q\xi,q\xi,P(\xi,\eta))+a_6S_2(q\eta,q\eta,P(\eta,\xi)) \end{aligned}$$

where $(\theta,\phi),(\xi,\eta) \in W\times W$ and $a_i (i=1,2,\dots,6)$ in $[0,1)$ such that $0 \leqslant a_1+a_2+a_3+\cdots+a_6 < \dfrac{1}{b^2}$ and $0 \leqslant b(a_5+a_6) < 1$

(ii) *$P(W \times W) \subset q(W)$*

(iii) *$q(W)$ is S_1 complete.*

Then, P and q have a coupled coincidence point. Further, if P and q are w-compatible then P and q have a unique common coupled fixed point.

Proof. Since $(\theta, \phi), (\xi, \eta) \in W \times W$, we have from (15) that

$$\begin{aligned} & S_1(P(\theta,\phi), P(\theta,\phi), P(\xi,\eta)) \leqslant \\ \leqslant\ & a_1 S_2(q\theta, q\theta, q\xi) + a_2 S_2(q\phi, q\phi, q\eta) + a_3 S_2(q\theta, q\theta, P(\theta,\phi)) + a_4 S_2(q\phi, q\phi, P(\phi,\theta)) + \\ & + a_5 S_2(q\xi, q\xi, P(\xi,\eta)) + a_6 S_2(q\eta, q\eta, P(\eta,\xi)) \end{aligned} \tag{16}$$

and

$$\begin{aligned} & S_1(P(\phi,\theta), P(\phi,\theta), P(\eta,\xi)) \leqslant \\ \leqslant\ & a_1 S_2(q\phi, q\phi, q\eta) + a_2 S_2(q\theta, q\theta, q\xi) + a_3 S_2(q\phi, q\phi, P(\phi,\theta)) + a_4 S_2(q\theta, q\theta, P(\theta,\phi)) + \\ & + a_5 S_2(q\eta, q\eta, P(\eta,\xi)) + a_6 S_2(q\xi, q\xi, P(\xi,\eta))\ . \end{aligned} \tag{17}$$

Adding (16) and (17), we have

$$\begin{aligned} & S_1(P(\theta,\phi), P(\theta,\phi), P(\xi,\eta)) + S_1(P(\phi,\theta), P(\phi,\theta), P(\eta,\xi)) \leqslant \\ \leqslant\ & (a_1 + a_2)\{S_2(q\theta, q\theta, q\xi) + S_2(q\phi, q\phi, q\eta)\} + \\ & + (a_3 + a_4)\{S_2(q\theta, q\theta, P(\theta,\phi)) + S_2(q\phi, q\phi, P(\phi,\theta))\} + \\ & + (a_5 + a_6)\{S_2(q\xi, q\xi, P(\xi,\eta)) + S_2(q\eta, q\eta, P(\eta,\xi))\}\ . \end{aligned}$$

By Theorem 2.1, required result follows. □

Remark 1. *Taking $S_1(\theta,\theta,\phi) = S_2(\theta,\theta,\phi) = S(\theta,\theta,\phi)$ for all $\theta, \phi \in W$, where S is an S_b-metric on W, in Corollary 2.3 we can have another result.*

We have following corollary from Theorem 2.1.

Corollary 2.5. *Let S_1, S_2 be two S_b metric spaces in a non-empty set W satisfying $S_2(\theta,\theta,\phi) \leqslant S_1(\theta,\theta,\phi)$ for all $\theta, \phi \in W$ and $b \geqslant 1$ is a real number. Let $P : W \times W \to W$ and $q : W \to W$ be two mappings satisfying the following conditions*

(i) $$\begin{aligned} & S_1(P(\theta,\phi), P(\theta,\phi), P(\xi,\eta)) + S_1(P(\phi,\theta), P(\phi,\theta), P(\eta,\xi)) \leqslant \\ & \leqslant k\{S_2(q\theta, q\theta, q\xi) + S_2(q\phi, q\phi, q\eta)\} \end{aligned}$$

where $(\theta,\phi), (\xi,\eta) \in W \times W$ and $k \in [0, \frac{1}{b^2})$

(ii) *$P(W \times W) \subset q(W)$*

(iii) *$q(W)$ is S_1 complete.*

Then, P and q have a coupled coincidence point. Further, if P and q are w-compatible then P and q have a unique common coupled fixed point.

Corollary 2.6. *Let S_1, S_2 be two S_b metric spaces in a non-empty set W satisfying $S_2(\theta,\theta,\phi) \leqslant S_1(\theta,\theta,\phi)$ for all $\theta, \phi \in W$ and $b \geqslant 1$ is a real number. Let $P : W \times W \to W$ and $q : W \to W$ be two mappings satisfying the following conditions*

(i) $$\begin{aligned} & S_1(P(\theta,\phi), P(\theta,\phi), P(\xi,\eta)) + S_1(P(\phi,\theta), P(\phi,\theta), P(\eta,\xi)) \leqslant \\ & \leqslant k\{S_2(q\xi, q\xi, P(\xi,\eta)) + S_2(q\eta, q\eta, P(\eta,\xi))\} \end{aligned}$$

where $(\theta,\phi), (\xi,\eta) \in W \times W$ and $k \in [0, \frac{1}{b^2})$

(ii) $P(W \times W) \subset q(W)$

(iii) $q(W)$ *is* S_1 *complete.*

Then, P *and* q *have a coupled coincidence point. Further, if* P *and* q *are* w*-compatible then* P *and* q *have a unique common coupled fixed point.*

Corollary 2.7. *Let* S_1, S_2 *be two* S_b *metric spaces in a non-empty set* W *satisfying* $S_2(\theta,\theta,\phi) \leqslant S_1(\theta,\theta,\phi)$ *for all* $\theta, \phi \in W$ *and* $b \geqslant 1$ *is a real number. Let* $P : W \times W \to W$ *and* $q : W \to W$ *be two mappings satisfying the following conditions*

(i)
$$S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta)) + S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \\ \leqslant k\{S_2(q\xi,q\xi,P(\xi,\eta)) + S_2(q\eta,q\eta,P(\eta,\xi))\}$$

where $(\theta,\phi),(\xi,\eta) \in W \times W$ *and* $k \in [0, \frac{1}{b^2})$

(ii) $P(W \times W) \subset q(W)$

(iii) $q(W)$ *is* S_1 *complete.*

Then, P *and* q *have a coupled coincidence point. Further, if* P *and* q *are* w*-compatible then* P *and* q *have a unique common coupled fixed point.*

Remark 2. **(i)** *Replacing* q *by identity mapping in the above results, we have corresponding coupled fixed point results.*

(ii) *Taking* $b = 1$ *in the above results, we can have corresponding results in* S *metric space.*

3. Applications

Example 2. *Let* $W = \mathbb{R}$*, define* $S : W \times W \times W \to \mathbb{R}^+$ *as*

$$S(\theta,\phi,\lambda) = |\theta + \phi - 2\lambda|^k, \theta,\phi,\lambda \in W,$$

where $k \geqslant 1$*. Here,* (W,S) *is an* S_b*-metric space.*

Example 3. *Let* $W = \mathbb{R}$ *and* S_1*,* S_2 *are two* S_b*-metrics in* W *such that*

$$S_1(\theta,\phi,\lambda) = (\theta + \phi - 2\lambda)^2, S_2(\theta,\phi,\lambda) = \Big(\frac{\theta + \phi - 2\lambda}{2}\Big)^2, \quad \theta,\phi,\lambda \in W$$

Let us define $P : W \times W \to W$ *and* $q : W \to W$ *by*

$$P(\theta,\phi) = \frac{\theta - \phi}{3}, \quad q\theta = 2\theta, \text{ for all } \theta,\phi \in W.$$

We have $P(W \times W) \subset q(W)$*,* $q(W)$ *is* S_1*-complete. Also,* P *and* q *are* w*-compatible. Now,*

$$\begin{aligned} S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta)) &= \{P(\theta,\phi) + P(\theta,\phi) - 2P(\xi,\eta)\}^2 = \\ &= 4\{P(\theta,\phi) - P(\xi,\eta)\}^2 = \\ &= 4\Big\{\frac{\theta - \phi}{3} - \frac{\xi - \eta}{3}\Big\}^2 = 4\Big\{\frac{\theta - \xi}{3} + \frac{\eta - \phi}{3}\Big\}^2 \leqslant \\ &\leqslant 2 \times 4\Big\{(\frac{\theta - \xi}{3})^2 + (\frac{\phi - \eta}{3})^2\Big\} = \end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{9}\left\{\frac{(2\theta-2\xi)^2}{2}+\frac{(2\phi-2\eta)^2}{2})\right\}= \\
&= \frac{4}{9}\left\{\frac{(q\theta-q\xi)^2}{2}+\frac{(q\phi-q\eta)^2}{2})\right\}= \\
&= \frac{1}{9}\left\{\frac{(2q\theta-2q\xi)^2}{2}+\frac{(2q\phi-2q\eta)^2}{2})\right\}= \\
&= \frac{1}{9}\left\{\frac{(q\theta+q\theta-2q\xi)^2}{2}+\frac{(q\phi+q\phi-2q\eta)^2}{2})\right\}= \\
&= \frac{1}{9}\left\{S_2(q\theta,q\theta,q\xi)+S_2(q\phi,q\phi,q\eta)\right\}.
\end{aligned}
$$

Similarly,

$$S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \frac{1}{9}\{S_2(q\phi,q\phi,q\eta)+S_2(q\theta,q\theta,q\xi)\}.$$

Further, we have

$$S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta))+S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \frac{2}{9}\left\{S_2(q\theta,q\theta,q\xi)+S_2(q\phi,q\phi,q\eta)\right\}.$$

Then, by corollary 2.6, $(0,0)$ *is the unique common coupled fixed point of* P *and* q.

Now, let $W=C[c,d]$ *is the set of all continuous functions.*

$$
\begin{aligned}
\text{Let}\quad S_1(\theta,\phi,\lambda) &= \max_{\mu\in[a,b]}|\theta(\mu)+\phi(\mu)-2\lambda(\mu)|^k \\
S_2(\theta,\phi,\lambda) &= \frac{\max_{\mu\in[a,b]}|\theta(\mu)+\phi(\mu)-2\lambda(\mu)|^k}{2} \quad \text{for all } \theta,\phi\in W,\ (k\geqslant 1).
\end{aligned}
$$

Also, let $b=3^{k-1}$. *Consider*

$$
\begin{aligned}
\theta(r) &= K(r)+\int_c^d G(r,\mu)\{f(\mu,\theta(\mu))+q(\mu,\phi(\mu))\}d\mu, \\
\phi(r) &= K(r)+\int_c^d G(r,\mu)\{f(\mu,\phi(\mu))+q(\mu,\theta(\mu))\}d\mu\,. \qquad (18)
\end{aligned}
$$

Next, we will analyse (18) under the following condition

(i) $f,q:[c,d]\times\mathbb{R}\to\mathbb{R}$ *be two continuous functions.*

(ii) $K:[c,d]\to\mathbb{R}$ *is a continuous function.*

(iii) $G:[c,d]\times\mathbb{R}\to[0,\infty)$ *is a continuous function.*

(iv) *There exists* $u,v>0$ *such that for all* $\theta,\phi\in\mathbb{R}$,

$$
\left\{
\begin{array}{l}
|f(\mu,\theta(\mu))-f(\mu,\phi(\mu))|\leqslant u|\theta-\phi|, \\
|q(\mu,\theta(\mu))-q(\mu,\phi(\mu))|\leqslant v|\theta-\phi|.
\end{array}
\right.
$$

(v) $\max_{r\in[c,d]}\left(\int\limits_c^d|G(r,\mu)|d\mu\right)^k<\frac{1}{2^{k+1}L^k}$, *with* $L=\max\{u,v\}$.

Theorem 1. *Under the condition (i)–(v), the integral equation (18) has a unique common solution on* $[c,d]$.

Proof. Let $P : W \times W \to W$ and $q : W \to W$

$$P(\theta,\phi)(r) = K(r) + \int_c^d G(r,\mu)|f(\mu,\theta(\mu)) + q(\mu,\phi(\mu))|d\mu,$$

$$q\theta = 2\theta, \forall \theta \in W, \mu \in [c,d], \theta, \phi \in W.$$

$$S_1(P(\theta,\phi), P(\xi,\eta), P(\lambda,w)) = \\ = \max_{r\in[c,d]} |P(\theta,\phi)(r) + P(\xi,\eta)(r) - 2P(\lambda,w)(r)|^k, \text{ for all } \theta,\phi,\xi,\eta,w,\lambda \in W,$$

$$S_2(P(\theta,\phi), P(\xi,\eta), P(\lambda,w)) = \\ = \frac{\max_{r\in[c,d]} |P(\theta,\phi)(r) + P(\xi,\eta)(r) - 2P(\lambda,w)(r)|^k}{2}, \text{ for all } \theta,\phi,\xi,\eta,w,\lambda \in W\,.$$

Also, $P(W \times W) \subset q(W)$, $q(W)$ is S_1-complete, and P and q are w-compatible.

Now,

$$\begin{aligned}
&|P(\theta,\phi)(r) + P(\theta,\phi)(r) - 2P(\xi,\eta)(r)|^k = \\
=\;& 2^k|P(\theta,\phi)(r) - P(\xi,\eta)(r)|^k = \\
=\;& 2^k\left[\Big|\int_c^d G(r,\mu)\{f(\mu,\theta(\mu)) - f(\mu,\xi(\mu))\}d\mu + \int_c^d G(r,\mu)\{q(\mu,\phi(\mu)) - q(\mu,\eta(\mu))\}d\mu\Big|^k\right] \leqslant \\
\leqslant\;& 2^{k-1}\times 2^k\Big|\int_c^d G(r,\mu)\{f(\mu,\theta(\mu)) - f(\mu,\xi(\mu))\}d\mu\Big|^k + \\
&+ 2^k \times 2^{k-1}\Big|\int_c^d G(r,\mu)\{q(\mu,\phi(\mu)) - q(\mu,\eta(\mu))\}d\mu\Big|^k \leqslant \\
\leqslant\;& 2^k\left[\Big|\int_c^d G(r,\mu)\{f(\mu,\theta(\mu)) - f(\mu,\xi(\mu))\}d\mu\Big|^k + \right. \\
&\left. + \Big|\int_c^d G(r,\mu)\{q(\mu,\phi(\mu)) - q(\mu,\eta(\mu))\}d\mu\Big|^k\right] \times 2^{k-1} \leqslant \\
\leqslant\;& 2^{k-1}\times 2^k\left[p^k(\max_{\mu\in[c,d]}|\theta(\mu)-\xi(\mu)|)^k + q^k(\max_{\mu\in[c,d]}|\phi(\mu)-\eta(\mu)|)^k\right]\Big(\int_c^d G(r,\mu)d\mu\Big)^k \leqslant \\
\leqslant\;& 2^{k-1}\times 2^k \times L^k\left[\max_{\mu\in[c,d]}|\theta(\mu)-\xi(\mu)|^k + \max_{\mu\in[c,d]}|\phi(\mu)-\eta(\mu)|^k\right]\frac{1}{2^{k+1}L^k} \leqslant \\
\leqslant\;& \frac{2^k}{2^{k+2}}\left[\max_{\mu\in[c,d]}|2\theta(\mu)-2\xi(\mu)|^k + \max_{\mu\in[c,d]}|2\phi(\mu)-2\eta(\mu)|^k\right] = \\
=\;& \frac{1}{2^{k+2}}\left[\max_{\mu\in[c,d]}|2\theta(\mu)+2\theta(\mu)-2(2\xi(\mu))|^k + \max_{\mu\in[c,d]}|2\phi(\mu)+2\phi(\mu)-2(2\eta(\mu))|^k\right] = \\
=\;& \frac{1}{2^{k+1}}\left[\frac{\max_{\mu\in[c,d]}|2\theta(\mu)+2\theta(\mu)-2(2\xi(\mu))|^k}{2} + \frac{\max_{\mu\in[c,d]}|2\phi(\mu)+2\phi(\mu)-2(2\eta(\mu))|^k}{2}\right] = \\
=\;& \frac{1}{2^{k+1}}[S_2(q\theta,q\theta,q\xi) + S_2(q\phi,q\phi,q\eta)].
\end{aligned}$$

Thus,

$$\begin{aligned}
S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta)) \;&=\; \max_{r\in[c,d]}|P(\theta,\phi)(r)+P(\theta,\phi)(r)-2P(\xi,\eta)(r)|^k \leqslant \\
&\leqslant\; \frac{1}{2^{k+1}}[S_2(q\theta,q\theta,q\xi) + S_2(q\phi,q\phi,q\eta)]\,. \qquad (19)
\end{aligned}$$

Similarly,

$$S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \frac{1}{2^{k+1}}[S_2(q\phi,q\phi,q\eta)+S_2(q\theta,q\theta,q\xi)] \tag{20}$$

It follows from (29) and (30) that

$$S_1(P(\theta,\phi),P(\theta,\phi),P(\xi,\eta))+S_1(P(\phi,\theta),P(\phi,\theta),P(\eta,\xi)) \leqslant \frac{1}{2^k}[S_2(q\theta,q\theta,q\xi)+S_2(q\phi,q\phi,q\eta)].$$

Consequently, all the conditions of Corollary 2.6 are satisfied. It follows from the result of Corollary 2.6 that P and q have a unique common coupled fixed point and hence integral equation in equation (18) has a unique solution. □

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Общие связанные теоремы о неподвижной точке для пары S_b-метрических пространств

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Аннотация. В данной работе мы исследуем существование общих связанных неподвижных точек и связанных точек совпадения в сеттинге двух S_b-метрических пространств. Здесь мы используем пару w-совместимых отображений. Различные результаты приводятся также в виде следствий.

Ключевые слова: общая связанная неподвижная точка, связанная точка совпадения, S_b-метрические пространства, w-совместимые отображения.