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Comparative Analysis of the Analytical and Numerical Solution of the Problem of Thermocapillary Convection in a Rectangular Channel

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Abstract. The paper compares the exact solution of one-dimensional and two-dimensional stationary convective flow equations with a free boundary for a flat liquid channel. Constant temperature gradient is set on the bottom solid wall. On the upper free boundary the surface tension coefficient is linearly dependent on temperature. Zero heat flux and velocities are set on the side walls of the two-dimensional problem. The deviation of the one-dimensional exact solution is determined for different aspect ratio and Marangoni number.

Keywords: thermocapillary, interface, Marangoni number.

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Introduction

Analytical solutions of convection problems with a free boundary can be used in engineering calculations of heat exchangers, electrical and electronic devices, metallurgical installations and in other fields. However, analytical solutions are usually obtained for relatively simple problems in a geometric or physical sense. Nevertheless, the application of exact solutions to evaluate more complex physical phenomena is a commonly used practice for engineers. In this regard, the estimation of errors and, thereby, the admissibility of the application of exact solutions in practice is an actual task.

In this paper, the exact stationary solution of the velocity distribution over the height of the horizontal layer is compared with a numerical two-dimensional one. The flows with a constant temperature gradient at the bottom boundary of the flat channel were considered. A boundary condition of the third kind on a free non-deformable upper boundary is set. The surface tension coefficient is linear function of temperature. Fig. 1 shows the schemes of the studied flow areas.

The one-dimensional problem has an analytical solution under the assumption that the length of the flat layer is infinite. In the two-dimensional convection problem in a flat channel, it is assumed that the channel length is finite and equal to l. For the case of an isothermal free

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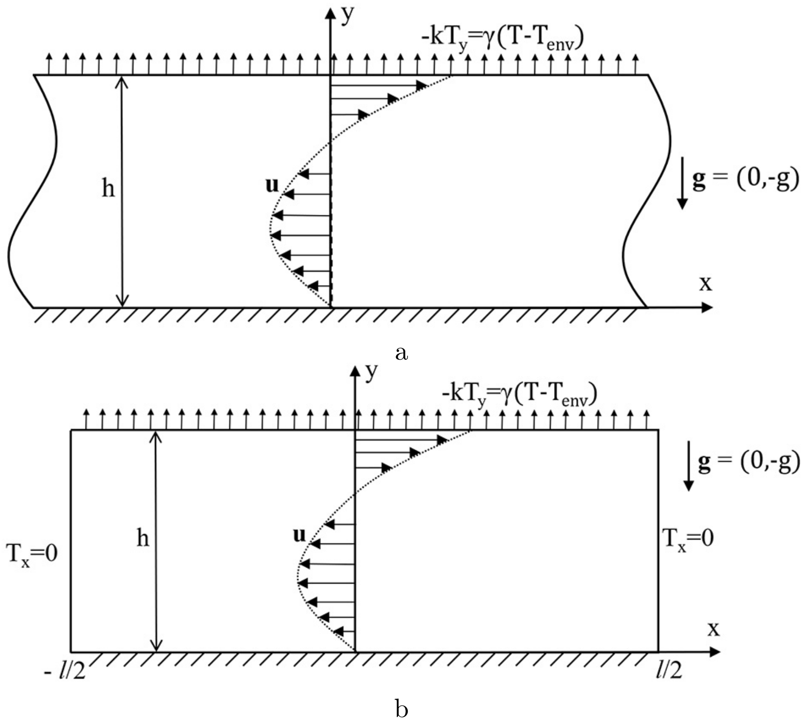


Fig. 1. To the problem statement: a) one-dimensional problem; b) two-dimensional problem

boundary ($\gamma = \infty$) an analytical solution of the problem of unidirectional convective motion was obtained in a widely known paper [1]. It is called the Birich's solution. For it, the flow of liquid through the cross section is zero. This makes it possible to interpret this solution as thermogravitational and thermocapillary convection in a horizontal channel whose length is much larger than its width, i.e., the flow in the core of a long cuvette away from its vertical walls ($h/l = \delta << 1$, Fig. 1). The possibility of such an interpretation of the Birich's solution is shown by experimental and numerical methods in [2]. For $\gamma = 0$ the possibility of experimental implementation of convection for $\delta << 1$ is established in [3–4].

In [5] , based on the asymptotic expansion of the solution by the parameter δ it is obtained the formula from the article [1], i.e., the flow in the core of a sufficiently long cuvette. At the same time, the effects of boundary conditions on the side walls of $x = \pm l/2$ can be neglected. A similar study for a liquid cylinder was carried out in [6].

1. Formulation and solution of a one-dimensional stationary problem

Let the thickness of the liquid layer be h, the y axis is directed vertically upwards, the x axis is from the hot wall to the cold, the origin is located at the base of the liquid layer. The motion does not depend on the coordinate z, perpendicular to the plane xy see Fig. 1.

Free convection equations for one-dimensional stationary flow $\mathbf{u} = (u(y), 0)$, $T = T(x, y)$:

$$P_x = \rho\nu u_{yy}, \tag{1}$$

$$P_y = \rho g\beta T, \tag{2}$$

$$uT_x = \chi(T_{xx} + T_{yy}). \tag{3}$$

Here $P(x, y) = p(x, y) + \rho g y$ is modified pressure, $p(x, y)$ is pressure, ρ is average liquid density, ν is kinematic viscosity, β and χ coefficients of thermal expansion and thermal conductivity, respectively, g is acceleration of gravity.

It is assumed that the flow rate of the liquid through the cross-section of the layer is zero:

$$\int_0^h u(y)dy = 0. \tag{4}$$

The boundary conditions on the bottom solid wall and the upper free surface are as follows

$$y = 0: \quad T = Ax; \quad u = 0; \tag{5}$$

$$y = h: \quad kT_y + \gamma\,(T - T_{env}) = 0; \quad \rho\nu\frac{\partial u}{\partial y} = -æ\frac{\partial T}{\partial x}. \tag{6}$$

where A is the temperature gradient set on the bottom, γ is coefficient of heat exchange with the environment, $T_{env} = T_g x + T_0$ is ambient temperature. Surface tension coefficient is linearly dependent on temperature $\sigma = \sigma_0 - æ(T - T_0)$ where $æ = -d\sigma/dT$ is temperature coefficient of surface tension ($æ > 0$), T_0 is some constant value of the average temperature.

The solution of the problem at $\gamma = \infty$ received in [1]. We have obtained a solution with a boundary condition of the third kind.

It follows from equations (1)–(3) that

$$u = C_1\frac{y^4}{4!} + C_2\frac{y^3}{3!} + C_3\frac{y^2}{2!} + C_4 y + C_5, \tag{7}$$

$$T = \frac{\nu}{g\beta}\,(C_1 y + C_2)\,x + f(y), \tag{8}$$

$$P = \rho_0 g\beta\left[\left(C_1\frac{y^2}{2} + C_2 y\right)x + \int_{y_0}^{y} f(y)dy\right] + \rho_0\nu C_3 x + C_6, \tag{9}$$

where ρ_0 is constant average density and the function $f(y)$ satisfies the equation $uT_x = \chi f_{yy}$. Further, only expression for velocity is used. Using conditions (4)-(6) the integration constants are defined.

We introduce dimensionless variables and numbers: $\eta = y/h, U = u/u^*$, $u^* = \chi/h$ is characteristic velocity, $\mathrm{Ma} = \dfrac{æ\mathrm{A}h^2}{\nu\rho\chi}$ is Marangoni number, $\mathrm{Ra} = \dfrac{g\beta\mathrm{A}h^4}{\nu\chi}$ is Rayleigh number, $\mathrm{Bi} = \dfrac{\gamma h}{k}$ is Bio number, $\theta = T_g/A$. Then the dimensionless velocity has the form

$$U = \mathrm{Ra}\left[\frac{\mathrm{Bi}\,(\theta-1)}{24\,(\mathrm{Bi}+1)}\eta^4 + \frac{1}{6}\eta^3 - \left(\frac{9\mathrm{Bi}\,(\theta-1)}{80\,(\mathrm{Bi}+1)} + \frac{15}{48}\right)\eta^2 + \left(\frac{7\mathrm{Bi}\,(\theta-1)}{120\,(\mathrm{Bi}+1)} + \frac{1}{8}\right)\eta\right] + \mathrm{Ma}\left[-\left(\frac{3\mathrm{Bi}\,(\theta-1)}{4\,(\mathrm{Bi}+1)} + \frac{3}{4}\right)\eta^2 + \left(\frac{\mathrm{Bi}\,(\theta-1)}{2\,(\mathrm{Bi}+1)} + \frac{1}{2}\right)\eta\right]. \tag{10}$$

2. Formulation of a two-dimensional stationary problem

The system of equations describing the two-dimensional stationary motion of a liquid inside a flat channel has the form

$$uu_x + vu_y + \frac{1}{\rho}P_x = \nu(u_{xx} + u_{yy}), \tag{11}$$

$$uv_x + vv_y + \frac{1}{\rho}P_y = \nu(v_{xx} + v_{yy}) + g\beta(T - T_0), \tag{12}$$

$$u_x + v_y = 0, \tag{13}$$

$$uT_x + vT_y = \chi(T_{xx} + T_{yy}). \tag{14}$$

To solve equations (11)–(14), the following boundary conditions were assumed:
on the side boundaries at $x = \pm l/2$

$$u(\pm\frac{l}{2}, y) = 0, \quad v(\pm\frac{l}{2}, y) = 0, \quad T_x(\pm\frac{l}{2}, y) = 0; \tag{15}$$

on the bottom boundary at $y = 0$

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad T(x, 0) = Ax; \tag{16}$$

on the upper boundary at $y = h$

$$-kT_y = \gamma(T - T_{env}), \quad v(x, h) = 0. \tag{17}$$

The first equality of (17) is a condition for the thermal contact of the free boundary with the environment. The second condition is the kinematic condition. The ambient temperature $T_{env} = T_g x + T_0$ is set similarly to a one-dimensional task. It is assumed that the dependency $\sigma(T)$ well approximated by linear dependence $\sigma(T) = \sigma_0 - æ(T - T_0)$. This assumption for most clean interface surfaces and not large values of the temperature gradient on the free surface is valid. In addition, we consider that the free boundary is non-deformable, i.e. the Weber number $\mathrm{We} = \dfrac{\sigma_0 h}{\nu\chi\rho} >> 1$ [7]. On the upper boundary at $y = h$ the tangent dynamic condition $\rho\nu(u_y + v_x) = -æT_x$, or taking into account the kinematic condition (17) we get $\rho\nu(u_y(x, h)) = -æT_x(x, h)$.

Let's write down a system of equations (11)–(14) and boundary conditions (15)–(17) in dimensionless form. We introduce dimensionless variables and numbers: $\xi = x/l$, $\eta = y/h$, $U = u/u^*$, $V = v/\delta u^*$, $\mathcal{P} = \dfrac{Ph^2}{\rho u^* \nu l}$, $\Theta = \dfrac{T - T_0}{\mathrm{A}l}$, $\delta = h/l$ is aspect ratio, $u^* = \chi/h$ is characteristic velocity, $\mathrm{Pr} = \dfrac{\nu}{\chi}$ is Prandtl number, $\mathrm{Ma} = \dfrac{æ\mathrm{A}h^2}{\nu\rho\chi}$ is Marangoni number, $\mathrm{Ra} = \dfrac{g\beta \mathrm{A} h^4}{\nu\chi}$ is Rayleigh number, $\mathrm{Bi} = \dfrac{\gamma h}{k}$ is Bio number.

Then the basic equations in dimensionless form can be written as follows

$$UU_\xi + VU_\eta + \mathrm{Pr}\mathcal{P}_\xi = \mathrm{Pr}(\delta U_{\xi\xi} + U_{\eta\eta}), \tag{18}$$

$$\frac{\delta^2}{\mathrm{Pr}}(UV_\xi + VV_\eta) + \frac{1}{\delta}\mathcal{P}_\eta = \delta^2 V_{\xi\xi} + \delta V_{\eta\eta} + \frac{\mathrm{Ra}\Theta}{\delta}, \tag{19}$$

$$U_\xi + V_\eta = 0, \tag{20}$$

$$U\Theta_\xi + V\Theta_\eta = \delta\Theta_{\xi\xi} + \frac{1}{\delta}\Theta_{\eta\eta}. \tag{21}$$

Dimensionless boundary conditions:
on the side boundaries at $\xi = \pm 1/2$, $0 \leqslant \eta \leqslant 1$

$$U(\pm\frac{1}{2}, \eta) = 0, \quad V(\pm\frac{1}{2}, \eta) = 0, \quad \Theta_\xi(\pm\frac{1}{2}, \eta) = 0, \tag{22}$$

on the bottom boundary at $-1/2 < \xi < 1/2$, $\eta = 0$

$$U(\xi, 0) = 0, \quad V(\xi, 0) = 0, \tag{23}$$

$$\Theta(\xi, 0) = -\xi, \tag{24}$$

on the upper boundary at $-1/2 < \xi < 1/2$, $\eta = 1$

$$-\frac{\partial \Theta}{\partial \eta} = \mathrm{Bi}\Theta, \quad V(\xi, 1) = 0; \tag{25}$$

$$\left(\frac{\partial U}{\partial \eta} + \delta^2 \frac{\partial V}{\partial \xi}\right) = \frac{\partial \Theta}{\partial \xi}. \tag{26}$$

3. Results of the computational experiment

To obtain the dependence of the deviation, which is obtained in a one-dimensional problem relative to a two-dimensional one, it is necessary to perform a computational experiment. The deviation is a function of the aspect ratio δ and Marangoni number. To do this, we performed accurate one-dimensional calculations with different layer heights h and Marangoni number. Also we performed numerical two-dimensional calculations with different aspect ratio δ and Marangoni number with the same other parameters. Two-dimensional numerical calculations were performed by the finite volume method in the Ansys Fluent program.

Two parameters were used to assess the discrepancy between the obtained results: the average absolute deviation $\sigma_1 = \left(\sum_{i=1}^{n} \sqrt{(U_2 - U)^2}\right)/n$ and the absolute value of the maximum velocity difference $\sigma_2 = \max|U_2 - U|$, U_2 is horizontal velocity component in a two-dimensional solution, U is the velocity obtained in the exact one-dimensional solution, n is the number of points equidistant in height.

The following values were taken for experiments $n = 22$, $\mathrm{Ma} = 0.1, 1, 10$, $\mathrm{Ra} = 0.01$, $\mathrm{Bi} = 0.01$. The results obtained below were compared with the results obtained at $n = 44$, as a result, it was determined that the accuracy did not significantly increase.

As an example, Fig. 2 shows graphs of unidirectional flow and the horizontal component of the velocity of a two-dimensional flow. It can be seen that the one-dimensional solution gives large velocity values relative to the two-dimensional one. Fig. 3 and Fig. 4 shows the velocity field and temperature field of a two-dimensional flow for $\delta = 0.5$.

As a result of processing the data of the computational experiment, two graphs $\sigma_1(\delta, \mathrm{Ma}), \sigma_2(\delta, \mathrm{Ma})$ (Fig. 5) showing the discrepancy between one-dimensional and two-dimensional solutions. As can be seen from these graphs, the character of the curves of the average and maximum deviation from the aspect ratio and the Marangoni number is the same. The deviations increase significantly at $\delta > 0.3$. It can be seen that with an increase the number of Ma, the deviation of the one-dimensional solution increases. For the numbers Ma=0.1 and Ma=1, the graphs turned out to be very close.

It should be noted that when comparing one-dimensional and two-dimensional solutions, the maximum discrepancy between the values of the horizontal component of the velocity was observed on a free surface, where the largest velocity values were also obtained. Therefore, Fig. 5b could be obtained from the velocity values on the free surface. In particular, this conclusion is well illustrated in Fig. 2, where it can be seen that the maximum discrepancy is observed on the free surface.

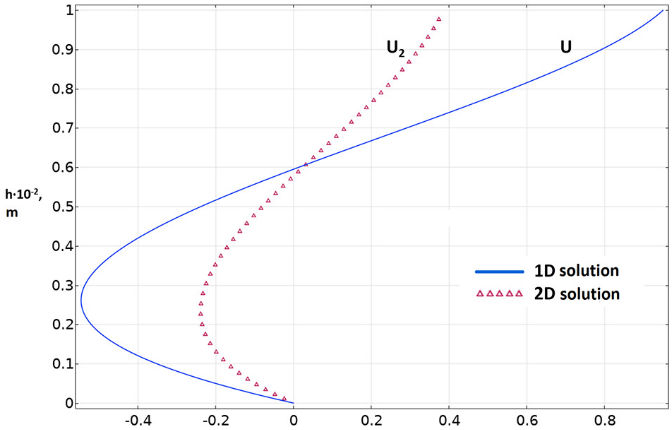


Fig. 2. Dimensionless velocities of one-dimensional and two-dimensional flow

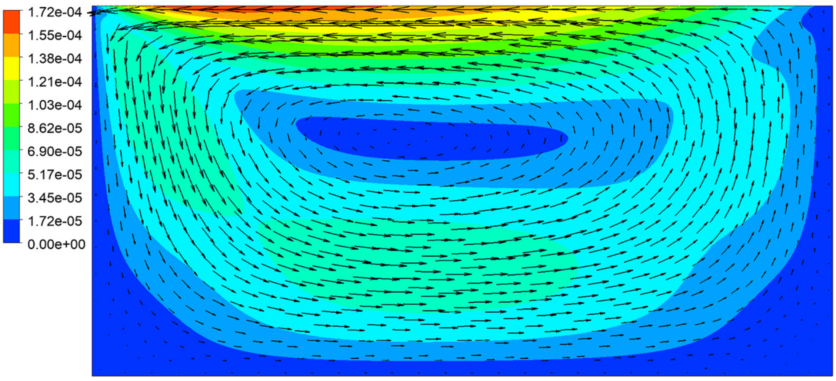


Fig. 3. Velocity field in a two-dimensional problem, m/s

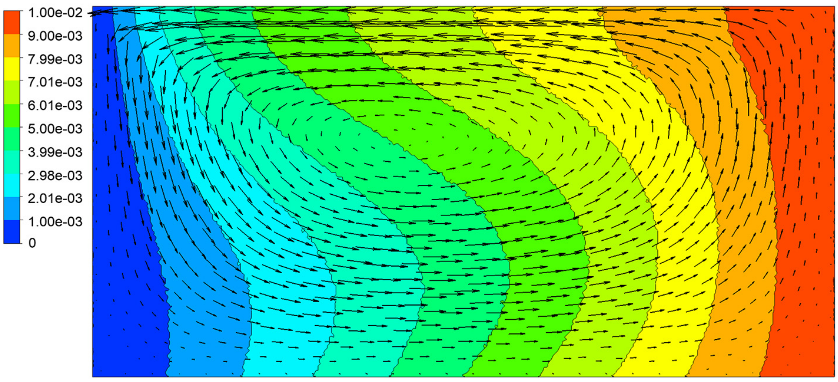
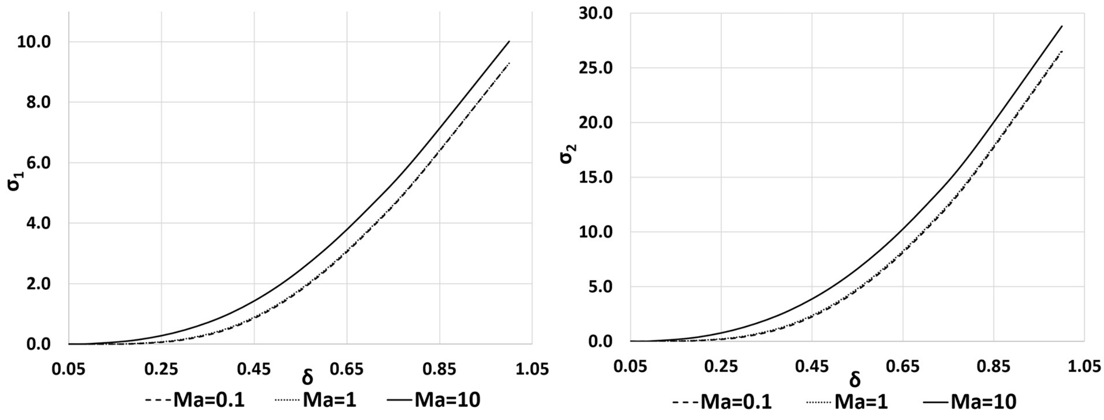


Fig. 4. Temperature field in a two-dimensional problem, °C

Conclusions

It is established that with an increase aspect ratio, the deviation of the exact solution of a one-dimansional (unidirectional) problem increases. The obtained graphs of the dependence of



a b

Fig. 5. Dependencies $\sigma_1(\delta, \mathrm{Ma}), \sigma_2(\delta, \mathrm{Ma})$: a — the average absolute deviation, b — is the absolute maximum deviation

the deviation on the aspect ratio, at Ma $= 0.1, 1, 10$, Ra $= 0.01$, Bi $= 0.01$ allow us to evaluate the possibility of using the formula for unidirectional flow in practice. For values of aspect ratio not exceeding 0.2 and the numbers Ma < 10, Ra $= 0.01$, Bi $= 0.01$ the value of the average absolute deviation of the exact unidirectional flow from the two-dimensional one does not differ significantly. Exceeding the aspect ratio of the value 0.25 within the specified criteria significantly increases the deviation of the unidirectional solution.

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Сравнительный анализ аналитического и численного решения задачи о термокапиллярной конвекции в прямоугольном канале

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Аннотация. В работе выполнено сравнение точного решения уравнений одномерного и двумерного стационарного конвективного течения со свободной границей для плоского горизонтального слоя жидкости с постоянным градиентом температуры на нижней границе слоя и свободной верхней границей с коэффициентом поверхностного натяжения, линейно зависящим от температуры. Определена погрешность одномерного точного решения при различной степени стеснения потока.

Ключевые слова: термокапиллярность, поверхность раздела, число Марангони.