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Sharply Doubly Transitive Groups with Saturation Conditions

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Abstract. A number of conditions were found under which a sharply doubly transitive permutation group has an abelian normal divider.

Keywords: exactly doubly transitive group, Frobenius groups, saturation condition, finite and generalized finite the elements.

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Recall that the permutation group G of the set F ($|F| \geqslant k$) is called *exactly k-transitive* on F if for any two ordered sets $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ elements from F such that $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for $i \neq j$, there is exactly one element of the group G taking α_i to β_i ($i = 1, \ldots, k$).

As C. Jordan [1] proved, in a finite sharply doubly transitive group T regular permutations (moving each element of the set F) together with the identity substitution constitute an abelian normal subgroup. Is Jordan's theorem true for infinite groups in the general case is unknown (see, for example, [2, questions 11.52, 12.48]). To particular solutions of this central question in the theory of near-fields and near-domains [3] dozens of works by famous authors are devoted. We especially note that in 2014 in [4,5] exactly doubly transitive groups without abelian normal subgroups were constructed, but only in characteristic 2. [6] and [7] constructed examples of exactly 2-transitive groups and simple exactly 2-transitive groups, respectively, that have characteristic 0 and do not contain non-trivial Abelian normal subgroups. And in 2008 [8] studied the question for exactly 2-transitive groups of characteristic 3. For other characteristics the question remains open. Sharply 2-transitive groups are closely related to algebraic structures such as near-fields, near-areas, KT-fields (Kerby-Tits fields), projective planes, etc. (see [3, Ch. V], [9, chap. 20]).

We continue to investigate infinite exactly doubly transitive groups and related near-domains [10, 11]. In this paper, a number of conditions are found under which a group has an abelian normal divisor (see [2, questions 11.52, 12.48]), and the corresponding near-domain [3] is a near-field. All necessary definitions are given in Section 1. We only recall that the group G is saturated with groups of some set of finite groups $\mathfrak{X}$ if every finite subgroup of G is contained in a subgroup of the group G isomorphic to some group from $\mathfrak{X}$.

Theorem 1. *A sharply doubly transitive group T saturated with finite Frobenius groups of substitutions of the set F of odd characteristic has a regular abelian normal subgroup and the near-domain $F(+, \cdot)$ is a near-field if at least one of the following conditions on the stabilizer T_α of the point $\alpha \in F$ is satisfied:*

1. *T_α is a Shunkov group;*
2. *T_α is a periodic group and T_α does not contain conjugate dense subgroups;*

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3. T_α is (locally) a finite normal subgroup of order greater than 2;

4. T_α contains a finite element a of prime odd order not equal to five.

Theorem 2. *Let in a sharply triply transitive permutation group G odd characteristic the point stabilizer have a regular abelian normal subgroup and each of its finite subgroups is contained in a finite unsolvable subgroup of the group G. Then G is locally finite.*

1. Results and concepts used

Let T be a sharply doubly transitive group of permutations of the set F. According to M. Hall's theorem [3], we assume that F is a near-domain with operations of addition $+$ and multiplication $\cdot$, and T is its affine transformation group $x \to a + bx$ $(b \neq 0)$, and $T_\alpha \simeq F^*(\cdot)$ is the stabilizer of the point $\alpha \in F$. Let T_α be the stabilizer of the point $\alpha \in F$, J denote the set of involutions of the group T. For each involution $k \in J$ by N_k we denote the set $kJ = \{kj \mid j \in J\}$. The result of a substitution action $t \in T$ to an element (point) $\gamma \in F$ is denoted by γ^t. The following statements are true.

1. *(T, T_α) is a Frobenius pair, i.e. $T_\alpha \cap T_\alpha^t = 1$ for any $t \in T \setminus T_\alpha$.*

2. *The group T contains involutions, and all involutions in T are conjugate. The set $J \setminus T_\alpha$ is nonempty and T_α is transitive on $J \setminus T_\alpha$. Product of two different involutions from T is a regular substitution, i.e. acting on F without fixed points.*

3. *If T_α contains an involution j, then it is unique in T_α, $T = T_\alpha N_j$ and T acts by conjugation on the set J of all its involutions exactly twice transitively. All products kv, where $k, v \in J$, $k \neq v$ in the group T are conjugate and have the same either prime odd order p or infinite order. In the first case $\mathrm{Char}\, T = p$, in the second case $\mathrm{Char}\, T = 0$ (if there are no involutions in T_α, then $\mathrm{Char}\, T = 2$ by definition).*

4. *The following result was proved in [10] Let $\mathrm{Char}\, T = p > 2$, b is strictly real with respect to j is an element from $T \setminus T_\alpha$, $A = C_T(b)$ and $V = N_T(A)$. Then:*

1) the subgroup A is periodic, abelian, and is inverted by the involution j and is strongly isolated in T;

2) the subgroup V acts exactly twice transitively on the orbit $\Delta = \alpha^A$, moreover, A is an elementary abelian regular normal subgroup of V, $H = V \cap T_\alpha$ is a point stabilizer and $V = A\lambda H$. If in addition $|\bigcap_{x \in T_\alpha} H^x| > 2$, then A is a regular abelian normal subgroup of T, and the near-domain $F(+, \cdot)$ is a near-field.

The next proposition shows that the saturation condition by finite Frobenius groups partially holds in an arbitrary exactly doubly transitive group of odd characteristic.

5. *When $\mathrm{Char}\, T = p > 2$, each dihedral subgroup of T is contained in a finite Frobenius subgroup of T with kernel of order p and a cyclic complement of order $p - 1$.*

Recall that (a, b)-*finiteness condition* in the group G means that a and b are its nontrivial elements, and in the group G all subgroups $\langle a, b^x \rangle$ $(x \in G)$ are finite. The elements a and b are called *generalized finite*, and if $a = b$, then a is called a *finite* element in the group G. The following proposition was proved in [11, Lemmas 1, 2 and Theorem]:

6. *Let a sharply doubly transitive group T of characteristics $\mathrm{Char}\, T \neq 2$ contain elements a, b with (a, b)-finitness condition. Then*

1. *If for any $x \in T$ the subgroup $\langle a^x \rangle$ and $\langle b \rangle$ are not incident, then at least one of the elements a, b belongs to the stabilizer of some point of the near-domain F;*

2. *If $|a| = |b| > 2$, T has a regular abelian normal subgroup;*

3. *If $|a| \cdot |b| = 2k > 4$, T has a regular abelian normal subgroup.*

7 ([12], Theorem 2)**.** *If the group T has is a finite element of order > 2, then T has a regular abelian normal subgroup and the near-domain F is a near-field.*

8. *If T contains a locally finite subgroup containing a regular substitution and intersecting with T_α by a normal subgroup, consisting of more than two elements, then T has a regular abelian normal subgroup, and the near-domain $F(+, \cdot)$ is a near-field.*

9. *If* $\mathrm{Char}\, T \neq 2$, *T contains a Frobenius group V with involution and the complement $H = V \cap T_\alpha$, where T_α is the stabilizer of the point α, and in H there is a normal in T_α subgroup of order > 2, then the group T has a regular abelian normal subgroup.*

10. *In the complement of a finite Frobenius group, each cyclic subgroup of prime order $q > 5$ is normal.*

2. Proof of Theorem 1

Let the group T and the near-domain F satisfy the conditions of Theorem 1. The set of Frobenius subgroups of the group T containing finite subgroup L we denote by $\mathfrak{X}(L)$. For any Frobenius group $M \leqslant T$, we denote its kernel by F_M, and the appropriate to the context complement by H_M. The notation $L \leqslant H_M$ often used in what follows for the subgroup $L \leqslant M \in \mathfrak{X}(L)$ means that L is contained in some complement H_M group M.

Remark. *It is well known that a near-domain $F(+, \cdot)$ is a near-field if and only if T has a regular abelian normal subgroup. Therefore, to prove Theorem 1 it suffices to prove the existence in the infinite group T of a regular abelian normal subgroup.*

Lemma 1. *We can assume that* $\mathrm{Char}\, T \neq 3$.

Proof. In the case of $\mathrm{Char}\, T = 3$, the group T has a regular abelian normal subgroup and without the additional saturation condition (see, for example, [13, Lemma 2.7]. The lemma is proved. □

Lemma 2. *When T_α has a finite element of prime order $q > 5$, the theorem is true.*

Proof. Let Char $T = p > 3$, j be an involution from T_α and a is a finite in T_α element of prime order $q > 5$. Choose an arbitrary element $t \in T_\alpha$. Due to the finiteness of the element a and item 3, the subgroup $L_t = \langle a, a^t, j \rangle$ is finite, and $L_t \leqslant M \in \mathfrak{X}(K)$. In view of item 1 and properties of finite Frobenius groups $M \cap T_\alpha \leqslant H_M$, $F_M \cap T_\alpha = 1$ and F_M is an elementary abelian p-group, consisting of all products jk, where $k \in J \cap M$. According to the structure of complements in finite Frobenius groups $L_t = \langle a, a^t \rangle = \langle a \rangle$. Since the element $t \in T_\alpha$ is arbitrary, we conclude that the subgroup $\langle a \rangle$ is normal in T_α. Therefore, the subgroup $L = \langle a, j \rangle$ is contained in the complement of a finite Frobenius group $M^x \in \mathfrak{X}(A)$ for any $x \in T_\alpha$, and the set-theoretic union of the kernels of all such groups (for $x \in T_\alpha$), contains the set N_j. From this and the equality $T = T_\alpha N_j$ (item 3) it follows that all subgroups $L_g = \langle a, a^g \rangle$ are finite for any $g \in T$, and by item 6 T possesses a regular abelian normal subgroup. The lemma is proved.

□

Lemma 3. *If T_α is a Shunkov group and* $\operatorname{Char} T \neq 2$*, then T_α has a local finite periodic part and the subgroup $\Omega_1(T_\alpha)$ generated by all elements of prime orders from T_α is a group of one of the following types: 1) a (locally) cyclic group; $\Omega_1(T_\alpha) = C \times L$, where C is a (locally) cyclic $\{2,3\}'$-group, and $L \simeq SL_2(3)$; 3) $\Omega_1(T_\alpha) = \times L$, where C is a (locally) cyclic $\{2,3,5\}'$-group, $L \simeq SL_2(5)$. In any case, in T_α is a finite normal subgroup of order greater than 2.*

Proof. By virtue of the saturation condition and the condition $\operatorname{Char} T \neq 2$, each finite subgroup K of T_α is contained in the complement of a finite Frobenius group $M \in \mathfrak{X}(K)$, and according to the proposition [14, Proposition 11], its subgroup $\Omega_1(K)$ has the structure indicated in the lemma. By [12, Theorem 1], the same structure has the subgroup $\Omega_1(T_\alpha)$, and T_α possesses a local finite periodic part, we denote it by S. If S is not a 2-group, then $\Omega_1(T_\alpha)$ obviously contains a finite normal subgroup in T_α of order, greater 2. If S is a 2-group, then by [15, Theorem 2] it is either a quasicyclic group, either locally quaternionic and also contains a finite normal in T_α subgroup of order 2^n for any $n > 1$. The lemma is proved. □

Lemma 4. *The theorem is true if T_α contains a finite normal subgroup L, $|L| > 2$.*

Proof. Since the involution of j in T_α is unique, we can assume that $j \in L$. By the saturation condition $M \in \mathfrak{X}(L)$, and the subgroup $M \cap T_\alpha$ is strongly isolated in M. Therefore, the subgroup L is the complement of some finite Frobenius group $M \in \mathfrak{X}(L)$. From the normality of L in T_α it follows that $M^t \in \mathfrak{X}(L)$ and $H_M = L$. Hence it follows that the set-theoretic union of the kernels of all such groups (for $t \in T_\alpha$), contains the set N_j. Let a be an element of order greater than 2 from L (since $|L| > 2$, such an element exists by virtue of item 3). It follows from what has been proved that all subgroups $L_g = \langle a, a^g \rangle$ are finite for any $g \in T$, and by item 6 T possesses a regular abelian normal subgroup. The lemma is proved. □

Lemma 5. *When T_α contains a finite element a of order* 3*, the theorem is also true.*

Proof. As in Lemma 2, we prove that for any $t \in T_\alpha$ the finite subgroup $L_t = \langle a, a^t \rangle$ is contained in the complement of a finite Frobenius group M from $\mathfrak{X}(L)$. As in Lemma 3, we conclude that either $L_t = \langle a \rangle$, or L_t is isomorphic to one of the groups $SL_2(3)$, $SL_2(5)$. By the main theorem in [16], the normal closure $L = \langle a^{T_\alpha} \rangle$ of a in T_α is locally finite. As follows from the proof of Lemma 3 either $L = \langle a \rangle$, or L is isomorphic to one of the groups $SL_2(3)$, $SL_2(5)$. By Lemma 4, the theorem is true. The lemma is proved. □

A proper subgroup H of a group G is called *conjugate dense*, if H has a non-empty intersection with every conjugacy class in G elements.

Lemma 6. *The theorem is true when T is a periodic group and T_α has no conjugate dense subgroups.*

Proof. By item 3, T_α has a unique involution j, therefore, for an arbitrary element a of finite order from T_α the subgroup $L = \langle a, j \rangle$ is finite. By the saturation condition $L \leqslant M \in \mathfrak{X}(L)$, and we can assume that $H_M = L$, $F_M \subseteq N_j$. By item 4, for any non-identity element $b \in N_j$, the subgroup $A = C_T(b)$ is periodic, contained in N_j and strongly isolated in T, in this case, $N_T(A) = A \leftthreetimes H$, where $H = T_\alpha \cap N_T(A)$. Since in view of items 2, 3 T_α acts transitively by conjugation on the set N_j, $F_M^x \leqslant A$ for some $x \in T_\alpha$. This implies that $a^x \in H$ and H is the conjugate dense subgroup of the group T_α. According to the conditions of the lemma, $H = T_\alpha$, by item 3, $A = N_j$, and the lemma is proved. □

We now complete the proof of the theorem. The first statement of the theorem follows from Lemmas 3, 4. Statement 2 is proved in Lemma 6. Statement 3 of Theorem coincides with Lemma 4. Statement 4 follows from Lemmas 2 and 5. The theorem is proved.

3. Proof of Theorem 2

Let G be an infinite sharply triply transitive permutation group of $X = F \cup \{\infty\}$ and satisfy the conditions of Theorem 2. As in [17], by B we denote the stabilizer G_α of the point $\alpha \in X$ and by H the stabilizer $G_{\alpha\beta} = G_\alpha \cap G_\beta$ of two points $\alpha = \infty \in X$, $\beta \in F$. Let also J be the set of involutions of the group G, and J_m be the set of involutions stabilizing exactly m points, $m = 0, 1, 2$.

Proof of the theorem. By Lemma 1 $B = U \leftthreetimes H$ is a Frobenius group, H contains an involution z and $N = N_G(H) = H \leftthreetimes \langle v \rangle$, where $v \in J$. For $b \in U^\#$ there is an element $a \in H$ of order $p-1$ such that $\langle a, b\rangle = \langle b \rangle \leftthreetimes \langle a \rangle$ is a sharply twice transitive group of order $p(p-1)$. Let S be an arbitrary finite subgroup from U containing an arbitrary element c from $U \setminus \langle b \rangle$ and $K = \langle b, S, a\rangle$. The subgroup K is obviously finite and by the saturation condition $K \leqslant M$, where M is a finite unsolvable group. Let $L = L(M)$ be the layer of the group M [18, Proposition 1.4, p. 53]. It is clear that $Z(L) = 1$ and since the 2-rank of the group G is 2, then L is a simple group. By virtue of Lemma 6 [17] L is isomorphic to $L_2(p^n)$ and $P = U \cap L$ is a Sylow p-subgroup of the group L. As known, all cyclic subgroups of P are conjugate in the subgroup $N_L(P)$, and $P^\#$ splits into two conjugate classes. Since $\langle a, b\rangle \leqslant M$ and $\langle a \rangle$ acts transitively to $\langle b \rangle^\#$, obviously $|M : L| = 2$ and $M \simeq PGL_2(p^n)$. The subgroup $N_M(P)$ acts transitively on $P^\#$, therefore $c = b^h$ for some $h \in H \cap M$. Since the element $c \in P^\#$ we conclude that H is a periodic group. By [19, Theorem 2], the group G is locally finite. The theorem is proved. □

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О точно дважды транзитивных группах с условиями насыщенности

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Аннотация. Найден ряд условий, при которых точно дважды транзитивная группа подстановок обладает абелевым нормальным делителем.

Ключевые слова: точно дважды транзитивная группа, группы Фробениуса, условие насыщенности, конечные и обобщенно конечные элементы.