Unsteady Flow of two Binary Mixtures in a Cylindrical Capillary with Changes in the Internal Energy of the Interface

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Abstract. The problem of two-dimensional unsteady flow of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces is studied. The mixtures are contacted through a common interface on which the energy condition is taken into account. The temperature and concentration of mixtures are distributed according to the quadratic law. It is in good agreement with the velocity field of the Hiemenz type. The resulting conjugate boundary value problem is a non-linear problem. It is also an inverse problem with respect to the pressure gradient along the axis of the cylindrical tube. To solve the problem the tau-method is used. It was shown that with increasing time the solution of the non-stationary problem tends to a steady state. It was established that the effect of increments of the internal energy of the inter-facial surface significantly affects the dynamics of the flow of mixtures in the layers.

Keywords: non-stationary solution, binary mixture, interface, energy condition, internal energy, inverse problem, pressure gradient, tau-method, thermal diffusion.

Introduction

The energy exchange between volume phases and a transition layer between them can lead to the inhomogeneous temperature distribution along the inter-facial surface. The mechanism of formation of the Marangoni stresses has been known for quite a long time. It implies that temperature gradient along the interface can arise and be maintained due to local increments of the internal energy of the inter-facial surface [1]. The temperature gradient, in turn, leads to the concentration gradient in the liquid mixture (these are the effects of thermodiffusion) [2].

For many liquids the surface tension is well approximated by a linear function. In this case *energy condition* is simplified to the following form [3, 4]:

$$k_2 \frac{\partial \theta_2}{\partial \mathbf{n}} - k_1 \frac{\partial \theta_1}{\partial \mathbf{n}} = \text{æ} \theta \nabla_\Gamma \cdot \mathbf{u}, \tag{1}$$

where $\text{æ} = -\partial\sigma/\partial\theta$, $\sigma = \sigma(\theta, c)$ is the surface tension coefficient that depends on temperature and concentration; k_1, k_2 are the coefficients of thermal conductivity; $\mathbf{n}$ is a unit vector of the normal to the interface of liquids Γ directed into the second liquid; $\nabla_\Gamma = \nabla - (\mathbf{n} \cdot \nabla)\mathbf{n}$ is a surface gradient; $\theta = \theta_1 = \theta_2$, $\mathbf{u} = \mathbf{u_1} = \mathbf{u_2}$ are pairwise coincident values of temperatures and velocity vector of both liquids on Γ; $\nabla_\Gamma \cdot \mathbf{u} = \text{div}_\Gamma \mathbf{u}$ is a surface divergence of the velocity vector

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u. According to the condition (1), changes in the surface internal energy induce corresponding changes in heat flows through the interface Γ.

For ordinary liquids at room temperature the effect of changes in the internal energy of the inter-facial surface on the formation of heat fluxes, temperature fields and velocities in its vicinity is insignificant in relation to the viscous friction and heat transfer [1]. However, at sufficiently high temperatures when the viscosity and thermal conductivity of ordinary liquids decrease significantly and for liquids with low viscosity increments of the internal energy of the inter-facial surface can have a significant impact on the dynamics of the flow [5]. The influence of changes in the internal energy of the inter-facial surface on the movement of liquids was studied [6–9].

A mathematical model that describes the two-layer unsteady thermodiffusion motion of binary mixtures in a cylindrical capillary in the absence of mass forces is considered in this paper. The mixtures are in contact through a common interface on which the energy condition is taken into account. The mechanism of influence of changes in the surface internal energy on the dynamics of the flow of binary mixtures in layers is studied. A similar geometry in the case of steady motion of mixtures was studied [10]. The non-linear stationary problem was reduced to a system of non-linear algebraic equations which was solved by the Newton method.

1.   Statement of the problem

A two-dimensional unsteady axisymmetric flow of two immiscible incompressible binary mixtures in a cylindrical tube of radius R_2 is considered. The temperature of the system is kept constant. Binary mixtures occupy domains $\Omega_1 = \{0 \leqslant r \leqslant R_1, |z| < \infty\}$ and $\Omega_2 = \{R_1 \leqslant r \leqslant R_2, |z| < \infty\}$, where r, z are radial and axial cylindrical coordinates. Here, the common interface of binary mixtures is at $r = R_1 = \text{const}$, and the solid wall is at $r = R_2 = \text{const}$. Values related to the regions Ω_1 and Ω_2 are denoted by the indices 1 and 2, respectively. It is assumed that the characteristic transverse size of domain Ω_2 is small compared to the radius of domain Ω_1 so $R_2 - R_1 \ll R_1$ (Fig. 1).
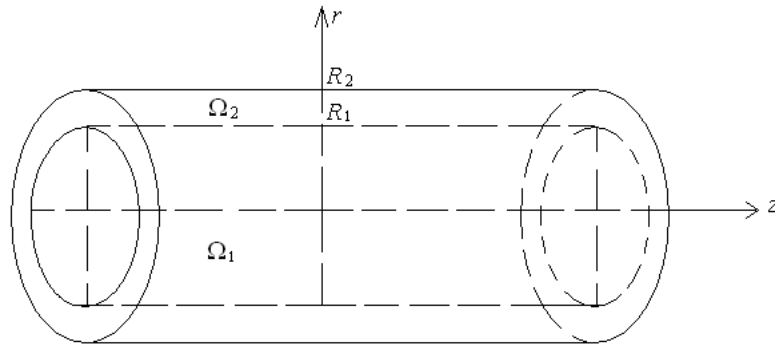

Fig. 1. Schematic of the flow domain

Binary mixture is characterized by constant conductivity k_j, specific heat capacity c_{pj}, dynamic viscosity μ_j, density ρ_j, coefficient of thermal conductivity $\chi_j = k_j/\rho_j c_{pj}$, kinematic viscosity $\nu_j = \mu_j/\rho_j$ (hereinafter, $j = 1, 2$). The influence of gravity is not taken into account. It can be justified for a narrow capillary.

Taking into account the effect of thermal diffusion, the defining equations of motion, heat and mass transfer have the form

$$u_{jt} + u_j u_{jr} + w_j u_{jz} + \frac{1}{\rho_j} p_{jr} = \nu_j \left(\Delta u_j - \frac{u_j}{r^2} \right),$$

$$w_{jt} + u_j w_{jr} + w_j w_{jz} + \frac{1}{\rho_j} p_{jz} = \nu_j \Delta w_j,$$

$$u_{jr} + \frac{u_j}{r} + w_{jz} = 0, \tag{2}$$

$$\theta_{jt} + u_j \theta_{jr} + w_j \theta_{jz} = \chi_j \Delta \theta_j,$$

$$c_{jt} + u_j c_{jr} + w_j c_{jz} = d_j \Delta c_j + \alpha_j d_j \Delta \theta_j,$$

where u_j, w_j are projections of the velocity vector on r, z axis of the cylindrical coordinate system, respectively; p_j is the pressure in the layers; θ_j, c_j are deviations of temperature and concentration from the average values θ_0, c_0; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the Laplace operator; d_j, α_j are diffusion and thermal diffusion coefficients, respectively. Generally speaking, these coefficients depend on temperature and concentration. However, using assumptions given above, one can consider that they have constant values corresponding to the average values of temperature and concentration. Let us note that the diffusion coefficient is always positive. The thermal diffusion coefficient can be either positive or negative. It depends on the type of a mixture and on the average temperature and average concentration of the selected component [11]. It is assumed that c in system (2) is the concentration of a light component.

It is assumed that inter-facial tension coefficient depends linearly on temperature and concentration:

$$\sigma(\theta, c) = \sigma_0 - \text{æ}_1(\theta - \theta_0) - \text{æ}_2(c - c_0). \tag{3}$$

Here $\text{æ}_1 > 0$ is the temperature coefficient, æ_2 is the concentration coefficient of the surface tension (usually $\text{æ}_2 < 0$, since the surface tension increases with increasing concentration).

The solution of the problem is taken in the special form:

$$u_j = u_j(r,t), \quad w_j = z v_j(r,t), \quad p_j = p_j(r,z,t),$$

$$\theta_j = a_j(r,t)z^2 + b_j(r,t), \quad c_j = h_j(r,t)z^2 + g_j(r,t). \tag{4}$$

In this representation, the velocity field corresponds to a solution of the Hiemenz type [12]. In this case, temperature θ_j takes an extreme value at the point $z = 0$: the maximum value when $a_j(r,t) < 0$ and the minimum value when $a_j(r,t) > 0$. A similar interpretation was obtained for the concentration of c_j. One should only consider function $h_j(r,t)$ instead of $a_j(r,t)$.

Substituting (4) into equations of motion (2) and separating the variable z, one can obtain the following system for unknown functions $u_j(r,t)$, $v_j(r,t)$, $p_j(r,t)$, $a_j(r,t)$, $b_j(r,t)$, $h_j(r,t)$, $g_j(r,t)$

$$u_{jt} + u_j u_{jr} + \frac{1}{\rho_j} p_{jr} = \nu_j (u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2}), \tag{5}$$

$$z(v_{jt} + u_j v_{jr} + v_j^2) + \frac{1}{\rho_j} p_{jz} = z \nu_j (v_{jrr} + \frac{1}{r} v_{jr}), \tag{6}$$

$$u_{jr} + \frac{u_j}{r} + v_j = 0, \tag{7}$$

$$a_{jt} + u_j a_{jr} + 2 v_j a_j = \chi_j (a_{jrr} + \frac{1}{r} a_{jr}), \tag{8}$$

$$b_{jt} + u_j b_{jr} = \chi_j (b_{jrr} + \frac{1}{r} b_{jr} + 2 a_j), \tag{9}$$

$$h_{jt} + u_j h_{jr} + 2 v_j h_j = d_j (h_{jrr} + \frac{1}{r} h_{jr}) + \alpha_j d_j (a_{jrr} + \frac{1}{r} a_{jr}), \tag{10}$$

$$g_{jt} + u_j g_{jr} = d_j (g_{jrr} + \frac{1}{r} g_{jr} + 2 h_j) + \alpha_j d_j (b_{jrr} + \frac{1}{r} b_{jr} + 2 a_j). \tag{11}$$

The pressure gradients (p_{jr}, p_{jz}) can be expressed from equations (5), (6):

$$p_{jr} = \rho_j \nu_j (u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2}) - \rho_j (u_{jt} + u_j u_{jr}), \tag{12}$$

$$p_{jz} = z[\rho_j \nu_j (v_{jrr} + \frac{1}{r} v_{jr}) - \rho_j (v_{jt} + u_j v_{jr} + v_j^2)]. \tag{13}$$

Conditions for the compatibility of equations (12), (13) are satisfied identically: $p_{jrz} = p_{jzr} = 0$. Hence it follows that pressure in the layers can be restored:

$$p_j = -\rho_j f_j(t) \frac{z^2}{2} + s_j(r, t), \tag{14}$$

where the derivative of function $s_j(r, t)$ with respect to r is exactly the right hand side of equation (12). Integrating this equation, we obtain the following representation of functions $s_j(r,t)$

$$s_j(r, t) = \rho_j \nu_j (u_{jr} + \frac{1}{r} u_j) - \rho_j \left(\frac{\partial}{\partial t} \int_0^r u_j dr + \frac{1}{2} u_j^2 \right) + s_j(t). \tag{15}$$

Functions $v_j(r, t)$ are determined from the equation:

$$v_{jt} + u_j v_{jr} + v_j^2 = \nu_j (v_{jrr} + \frac{1}{r} v_{jr}) + f_j(t). \tag{16}$$

It follows that flow in the layers is induced by longitudinal pressure gradients $f_j(t)$, $j = 1, 2$. These are unknown functions to be defined along with functions v_j, a_j, b_j, h_j, g_j. Therefore, we have an inverse problem.

Conditions on the solid wall $(r = R_2)$ are

$$u_2(R_2, t) = 0, \quad v_2(R_2, t) = 0, \quad a_2(R_2, t) = a_2(t), \quad b_2(R_2, t) = b_2(t); \tag{17}$$

$$h_{2r}(R_2, t) + \alpha_2 a_{2r}(R_2, t) = 0, \quad g_{2r}(R_2, t) + \alpha_2 b_{2r}(R_2, t) = 0, \tag{18}$$

with the specified functions $a_2(t)$, $b_2(t)$.

The inter-facial surface is assumed to be cylindrical and non-deformable. To do this, it is enough to require that Weber number We $\to \infty$. Then taking into account this requirement and relation (3), we have the following boundary conditions at $r = R_1$:

$$u_1(R_1, t) = u_2(R_1, t), \quad v_1(R_1, t) = v_2(R_1, t); \tag{19}$$

$$a_1(R_1, t) = a_2(R_1, t), \quad b_1(R_1, t) = b_2(R_1, t); \tag{20}$$

$$h_1(R_1, t) = h_2(R_1, t), \quad g_1(R_1, t) = g_2(R_1, t), \tag{21}$$

$$\mu_2 v_{2r}(R_1, t) - \mu_1 v_{1r}(R_1, t) = -2[æ_1 a_1(R_1, t) + æ_2 h_1(R_1, t)]; \tag{22}$$

$$d_1[h_{1r}(R_1, t) + \alpha_1 a_{1r}(R_1, t)] = d_2[h_{2r}(R_1, t) + \alpha_2 a_{2r}(R_1, t)]; \tag{23}$$

$$d_1[g_{1r}(R_1, t) + \alpha_1 b_{1r}(R_1, t)] = d_2[g_{2r}(R_1, t) + \alpha_2 b_{2r}(R_1, t)]; \tag{24}$$

$$k_2 a_{2r}(R_1, t) - k_1 a_{1r}(R_1, t) = æ_1 a_1(R_1, t) v_1(R_1, t); \tag{25}$$

$$k_2 b_{2r}(R_1, t) - k_1 b_{1r}(R_1, t) = æ_1 b_1(R_1, t) v_1(R_1, t). \tag{26}$$

Relations (25), (26) are *energy condition* on the interface of two binary mixtures. It can be interpreted as follows: a jump in the heat flow in the direction of the normal to the interface (at $r = R_1$) is compensated by the change in the internal energy of this surface. In turn, this change

is associated with both the change in temperature (specific internal energy) and the change in the area of the interface [13].

In addition, it is necessary to require the boundedness of functions on the axis of the cylindrical capillary ($r = 0$):

$$|u_1(0,t)| < \infty, \ |v_1(0,t)| < \infty, \ |s_1(0,t)| < \infty,$$
$$|a_1(0,t)| < \infty, \ |b_1(0,t)| < \infty, \ |h_1(0,t)| < \infty, \ |g_1(0,t)| < \infty. \tag{27}$$

Initial conditions at $t = 0$ are

$$u_j(r,0) = u_{j0}(r), \quad v_j(r,0) = v_{j0}(r), \quad a_j(r,0) = a_{j0}(r), \quad b_j(r,0) = b_{j0}(r);$$
$$h_j(r,0) = h_{j0}(r), \quad g_j(r,0) = g_{j0}(r), \quad s_j(r,0) = s_{j0}(r), \quad f_j(0) = f_{j0} \equiv \text{const.} \tag{28}$$

Let us note that functions u_{j0} and v_{j0} should be constrained according to continuity equation (7); functions h_{j0}, a_{j0} should be constrained according to conditions (18), (23); functions g_{j0}, b_{j0} should be constrained according to conditions (18), (24); functions v_{j0}, a_{10}, h_{10} should be constrained according to condition (22); functions v_{10}, a_{j0} should be constrained according to condition (25), and functions v_{10}, b_{j0} should be constrained according to condition (26). Thus, the compatibility conditions are fulfilled.

2.   Formulation of the problem in dimensionless variables

It should be noted that equations (7), (8), (10), (16) are independent of the others and they form a closed subsystem for defining functions $v_j(r,t)$, $a_j(r,t)$, $h_j(r,t)$ and $f_j(t)$ ($j = 1, 2$). After solving it, functions $b_j(r,t)$, $g_j(r,t)$ can be determined from equations (9), (11), and functions $s_j(r,t)$ can be uniquely determined from relation (15). Taking into account conditions (27) and adhesion on the solid wall (17), we integrate continuity equation (7) and exclude $u_j(r,t)$ in equations (8), (10), (16). Then one needs to find only functions $v_j(r,t)$, $a_j(r,t)$, $h_j(r,t)$ and $f_j(t)$. We introduce dimensionless variables and functions

$$\xi = \frac{r}{R_1}, \quad R = \frac{R_2}{R_1} > 1, \quad \tau = \frac{\nu_1}{R_1^2}, \quad V_j = \frac{R_1^2 v_j}{\text{Ma}\,\nu_1},$$
$$A_j = \frac{a_j}{\theta_0}, \quad H_j = \frac{h_j}{c_0}, \quad F_j = \frac{R_1^4 f_j}{\text{Ma}\,\nu_1^2}, \tag{29}$$

where θ_0, c_0 are the characteristic temperature and concentration.

The main similarity criteria in the problem under consideration are

$$\text{Ma} = \frac{æ_1 \theta_0 R_1^3}{\mu_2 \nu_1}, \quad \text{Mc} = \frac{æ_2 c_0 R_1^3}{\mu_2 \nu_1}, \quad \text{Pr}_j = \frac{\nu_j}{\chi_j}, \quad \text{Sc}_j = \frac{\nu_j}{d_j}, \quad \text{Sr}_j = \frac{\alpha_j \theta_0}{d_j c_0},$$
$$\mu = \frac{\mu_1}{\mu_2}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad k = \frac{k_1}{k_2}, \quad d = \frac{d_1}{d_2}, \quad \text{M} = \frac{\text{Mc}}{\text{Ma}} = \frac{æ_2 c_0}{æ_1 \theta_0}. \tag{30}$$

Here Ma is the thermal Marangoni number, Mc is the Marangoni concentration number, Pr_j is the Prandtl number, Sc_j is Schmidt number, Sr_j is the Soret number.

We obtain a nonlinear inverse initial-boundary value problem in the domain of the spatial variable ξ. When $j = 1$ it varies from 0 to 1, and when $j = 2$ it varies from 1 to R.

For $0 < \xi < 1$ we have

$$K_1(V_1, F_1) \equiv V_{1\xi\xi} + \frac{1}{\xi}V_{1\xi} - V_{1\tau} + \frac{\text{Ma}}{\xi}V_{1\xi}\int_0^\xi \xi V_1(\xi,\tau)\,d\xi - \text{Ma}V_1^2 + F_1(\tau) = 0, \tag{31}$$

$$S_1(V_1, A_1) \equiv \frac{1}{\mathrm{Pr}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) - A_{1\tau} + \frac{\mathrm{Ma}}{\xi} A_{1\xi} \int_0^\xi \xi V_1(\xi, \tau) \, d\xi - 2\mathrm{Ma} A_1 V_1 = 0; \quad (32)$$

$$T_1(V_1, A_1, H_1) \equiv \frac{1}{\mathrm{Sc}_1} \left(H_{1\xi\xi} + \frac{1}{\xi} H_{1\xi} \right) + \frac{\mathrm{Sr}_1}{\mathrm{Sc}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) - H_{1\tau} +$$

$$+ \frac{\mathrm{Ma}}{\xi} H_{1\xi} \int_0^\xi \xi V_1(\xi, \tau) \, d\xi - 2\mathrm{Ma} H_1 V_1 = 0. \quad (33)$$

For $1 < \xi < R$ we have

$$K_2(V_2, F_2) \equiv \frac{1}{\nu} \left(V_{2\xi\xi} + \frac{1}{\xi} V_{2\xi} \right) - V_{2\tau} - \frac{\mathrm{Ma}}{\xi} V_{2\xi} \int_\xi^R \xi V_2(\xi, \tau) \, d\xi - \mathrm{Ma} V_2^2 + F_2(\tau) = 0, \quad (34)$$

$$S_2(V_2, A_2) \equiv \frac{1}{\mathrm{Pr}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - A_{2\tau} - \frac{\mathrm{Ma}}{\xi} A_{2\xi} \int_\xi^R \xi V_2(\xi, \tau) \, d\xi - 2\mathrm{Ma} A_2 V_2 = 0; \quad (35)$$

$$T_2(V_2, A_2, H_2) \equiv \frac{1}{\mathrm{Sc}_2 \nu} \left(H_{2\xi\xi} + \frac{1}{\xi} H_{2\xi} \right) + \frac{\mathrm{Sr}_2}{\mathrm{Sc}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - H_{2\tau} -$$

$$- \frac{\mathrm{Ma}}{\xi} H_{2\xi} \int_\xi^R \xi V_2(\xi, \tau) \, d\xi - 2\mathrm{Ma} H_2 V_2 = 0. \quad (36)$$

Then the following conditions are satisfied on the solid wall ($\xi = R$)

$$V_2(R, \tau) = 0, \quad A_2(R, \tau) = \frac{a_2(\tau)}{\theta_0}, \quad H_{2\xi}(R, \tau) + \mathrm{Sr}_2 A_{2\xi}(R, \tau) = 0. \quad (37)$$

On the interface ($\xi = 1$) we have

$$V_1(1, \tau) = V_1(1, \tau), \quad \int_0^1 \xi V_1(\xi, \tau) \, d\xi = 0, \quad \int_1^R \xi V_2(\xi, \tau) \, d\xi = 0, \quad (38)$$

$$A_1(1, \tau) = A_2(1, \tau), \quad H_1(1, \tau) = H_2(1, \tau), \quad (39)$$

$$V_{2\xi}(1, \tau) - \mu V_{1\xi}(1, \tau) = -2[A_1(1, \tau) + \mathrm{M} H_1(1, \tau)], \quad (40)$$

$$d[H_{1\xi}(1, \tau) + \mathrm{Sr}_1 A_{1\xi}(1, \tau)] = H_{2\xi}(1, \tau) + \mathrm{Sr}_2 A_{2\xi}(1, \tau), \quad (41)$$

$$A_{2\xi}(1, \tau) - k A_{1\xi}(1, \tau) = E A_1(1, \tau) V_1(1, \tau), \quad (42)$$

where parameter $E = \ae_1^2 \theta_0 R_1^2 / \mu_2 k_2$ characterizes the significance of the process of release or absorption of heat at local increments of the area of the inter-facial surface for the development of convective motion near this surface. The mechanism of local change in the internal energy of the interface should be taken into account for most conventional liquids at elevated temperatures or for liquids with reduced viscosity, for example, for some cryogenic liquids. Calculations carried out for physical parameters of various liquids and phase interfaces showed that $E = O(1)$ is quite realistic [5].

The conditions of boundedness of functions are set on the axis of symmetry:

$$|V_1(0, \tau)| < \infty, \quad |A_1(0, \tau)| < \infty, \quad |H_1(0, \tau)| < \infty. \quad (43)$$

The initial conditions at $\tau = 0$ are

$$V_j(\xi, 0) = V_{j0}(\xi), \quad A_j(\xi, 0) = A_{j0}(\xi),$$

$$H_j(\xi, 0) = H_{j0}(\xi), \quad F_j(0) = F_{j0} \equiv \mathrm{const}. \quad (44)$$

Note that the integral redefinition conditions in (38) are used as additional ones when solving the inverse problem and they are nothing more than a closed flow condition. They play an important role in finding unknown longitudinal pressure gradients $F_j(\tau)$ in layers of binary mixtures.

Let us consider the creeping unsteady flow of binary mixtures (this is a linear problem).

Let us assume that the thermal Marangoni number Ma $\ll 1$ (a creeping motion) and Ma $\sim$ Mc, that is, thermal and concentration effects on the interface $\xi = 1$ are of the same order. Then the equations of momentum, energy and concentration transfer are simplified by discarding convective acceleration. As a result, the conjugate inverse initial-boundary value problem becomes linear. However, such problem cannot be solved consistently because of the non-linearity of energy condition (42).

3. Derivation of a finite-dimensional system of first-order ordinary differential equations

To solve non-linear problem (31)–(44) the tau-method is used. It is a modification of the Galerkin method [14]. For further consideration, it is essential to replace the variables: $\xi' = \xi$ when $j = 1$ and $\xi' = (\xi - R)/(1 - R)$ when $j = 2$ and re-assign $\xi' \leftrightarrow \xi$. Taking into account (43), an approximate solution is taken in the form

$$
V_j^n(\xi, \tau) = \sum_{k=0}^{n} V_j^k(\tau) R_k^{(0,1)}(\xi),
$$

$$
A_j^n(\xi, \tau) = \sum_{k=0}^{n} A_j^k(\tau) R_k^{(0,1)}(\xi), \tag{45}
$$

$$
H_j^n(\xi, \tau) = \sum_{k=0}^{n} H_j^k(\tau) R_k^{(0,1)}(\xi),
$$

where $R_k^{(0,1)}(\xi)$ are shifted Jacobi polynomials. In general, they are defined in terms of Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$ as follows $(\alpha > -1, \beta > -1)$ [15]

$$
R_k^{(\alpha,\beta)}(y) = P_k^{(\alpha,\beta)}(2y - 1), \quad y \in [0, 1]. \tag{46}
$$

It is known that shifted Jacobi polynomials $R_k^{(\alpha,\beta)}(y)$ are orthogonal on the segment $[0, 1]$ with the weight $(1 - y)^\alpha y^\beta$. Then

$$
\int_0^1 (1 - y)^\alpha y^\beta R_k^{(\alpha,\beta)}(y) R_m^{(\alpha,\beta)}(y) \, dy = \delta_{km} h_m, \tag{47}
$$

$$
h_m = \frac{\Gamma(\alpha + m + 1)\Gamma(\beta + m + 1)}{m!(\alpha + \beta + 2m + 1)\Gamma(\alpha + \beta + m + 1)}, \quad \delta_{km} = \left\{ \begin{array}{ll} 1, & k = m, \\ 0, & k \neq m, \end{array} \right.
$$

where $\Gamma(x)$ is the Euler gamma function.

In addition, polynomials $R_k^{(\alpha,\beta)}(y)$ form a basis in $L_2(0, 1)$ with the weight $(1 - y)^\alpha y^\beta$ and they satisfy the following properties [16]

$$
R_k^{(\alpha,\beta)}(0) = \frac{(-1)^k(\beta + k)!}{\beta! k!}, \quad R_k^{(\alpha,\beta)}(1) = \frac{(\alpha + k)!}{\alpha! k!}; \tag{48}
$$

$$
\frac{d^m}{dy^m} R_k^{(\alpha,\beta)}(y) = \frac{\Gamma(k + m + \alpha + \beta + 1)}{\Gamma(k + \alpha + \beta + 1)} R_{k-m}^{(\alpha+m,\beta+m)}(y). \tag{49}
$$

Functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$, $F_j(\tau)$ are determined from the system of Galerkin approximations

$$\int_0^1 K_j(V_j^n, F_j) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \tag{50}$$

$$\int_0^1 S_j(V_j^n, A_j^n) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \tag{51}$$

$$\int_0^1 T_j(V_j^n, A_j^n, H_j^n) R_m^{(0,1)}(\xi)\, \xi\, d\xi = 0, \quad m = 0, \ldots, n-3, \quad j = 1, 2. \tag{52}$$

Taking into account conditions (38) and property (47), we obtain that $V_1^0(\tau) = V_2^0(\tau) = 0$.

Taking into account properties (48) and (49), boundary conditions take the form

$$\sum_{k=0}^n (-1)^k (k+1) V_2^k(\tau) = 0, \quad \sum_{k=0}^n (-1)^k (k+1) A_2^k(\tau) = \frac{a_2(\tau)}{\theta_0}, \tag{53}$$

$$\sum_{k=1}^n (-1)^{k-1} k(k+1)(k+2)[H_2^k(\tau) + \mathrm{Sr}_2 A_2^k(\tau)] = 0. \tag{54}$$

$$\sum_{k=0}^n V_1^k(\tau) = \sum_{k=0}^n V_2^k(\tau), \quad \sum_{k=0}^n A_1^k(\tau) = \sum_{k=0}^n A_2^k(\tau),$$

$$\sum_{k=0}^n H_1^k(\tau) = \sum_{k=0}^n H_2^k(\tau), \tag{55}$$

$$\frac{1}{1-R} \sum_{k=1}^n k(k+2)(V_2^k(\tau) - \mu V_1^k(\tau)) = -2 \sum_{k=0}^n (A_1^k(\tau) + \mathrm{M}H_1^k(\tau)). \tag{56}$$

$$d \sum_{k=1}^n k(k+2)[H_1^k(\tau) + \mathrm{Sr}_1 A_1^k(\tau)] = \sum_{k=1}^n k(k+2)[H_2^k(\tau) + \mathrm{Sr}_2 A_2^k(\tau)], \tag{57}$$

$$\frac{1}{1-R} \sum_{k=1}^n k(k+2)(A_2^k(\tau) - k A_1^k(\tau)) = E \sum_{k=0}^n A_1^k(\tau) \sum_{k=0}^n V_1^k(\tau). \tag{58}$$

The finite-dimensional system of Galerkin approximations for functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$ $(k = 0, \ldots, n, j = 1, 2)$ and the calculation of the resulting definite integrals over various products of shifted Jacobi polynomials are described in detail in [17].

The system of integro-differential equations can be transformed to a closed system of first-order ordinary differential equations with respect to unknown functions $V_j^k(\tau)$, $A_j^k(\tau)$, $H_j^k(\tau)$, $F_j(\tau)$ $(k = 0, \ldots, n-3, j = 1, 2)$. It involves rather cumbersome treatment and it is not given here. The initial conditions follows from (44) and (45):

$$V_j^k(0) = V_{j0}^k, \quad A_j^k(0) = A_{j0}^k, \quad H_j^k(0) = H_{j0}^k, \quad F_j(0) = F_{j0} \equiv \text{const}, \tag{59}$$

where constants $V_{j0}^k, A_{j0}^k, H_{j0}^k$ are the coefficients of the expansions of functions $V_{j0}(\xi), A_{j0}(\xi), H_{j0}(\xi)$ in terms of the shifted Jacobi polynomials $R_k^{(0,1)}(\xi)$.

4.   Numerical solution of the non-linear problem

The system of ordinary differential equations of the first order was integrated numerically using the Runge-Kutta method of the fourth order of accuracy. Note that when using the tau method in order to ensure the exact fulfilment of the boundary conditions it is necessary to use a sufficient number of coefficients in the trial solution. In this case calculations were performed for $n = 10, 11, 12$ in Galerkin approximations. At the same time, with an increase in n a rapid increase in the accuracy of the solution is observed.

Some results of numerical solution are presented for the model system that consists of an aqueous solution of formic acid (mixture 1) – transformer oil (mixture 2). According to tabular data , the dimensionless parameters of the specified system are as follows

$$\mu = 0.0898, \quad \nu = 0.0649, \quad \chi = 1.4, \quad k = 2.41, \quad d = 0.0152,$$

$$\mathrm{Pr}_1 = 13.8, \quad \mathrm{Pr}_2 = 298, \quad \mathrm{Sc}_1 = 963, \quad \mathrm{Sc}_2 = 225,$$

$$\mathrm{Sr}_1 = 6, \quad \mathrm{Sr}_2 = 7, \quad \mathrm{Ma} = 20, \quad \mathrm{Mc} = 15.$$

The following parameter values were also used: $R = 1.5$, $E = 0.7$.

Fig. 2 shows the results of calculations of the velocity field. Function $V_j(\xi, \tau)$ and the radial velocity profile $U_j(\xi, \tau)$ are shown at various points in time. Analysing this result, we came to the conclusion that solution of a non-stationary problem with increasing time tends to the stationary mode obtained by solving the non-linear problem by the Newton method [10]. In turn, the pressure gradients $F_j(\tau)$ in the layers are stabilized with time and they converge to the values $F_1 = -1.78305$, $F_2 = -71.22054$. Calculations show that the pressure gradient in the second layer significantly exceeds the pressure gradient in the first layer in absolute value. This is because transformer oil is more viscous compared to the aqueous solution of formic acid. The greater is viscosity of the liquid the smaller is its mobility. Consequently, much greater pressure forces are required to cause movement in the second layer.
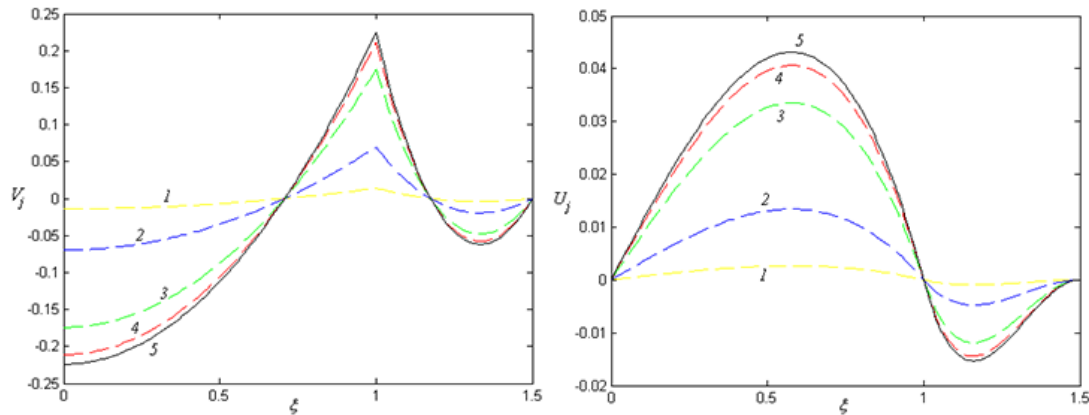

Fig. 2. Plots of functions $V_j(\xi, \tau)$ and $U_j(\xi, \tau)$ at various points in time: $1 - \tau = 0.08$, $2 - \tau = 0.4$, $3 - \tau = 1.21$, $4 - \tau = 1.42$, $5 - \tau = \infty$

It is of interest to consider how the change in the internal energy of the interface affects the flow pattern of binary mixtures. As a result of calculations it was found that with an increase in the energy parameter E at a fixed τ the absolute values of function $V_j(\xi, \tau)$ decrease (see Fig. 3). One can be concluded that the effects associated with the heat of formation of the inter-facial surface contribute to a decrease in the intensity and laminarization of the flow near this surface.

Note that function $V_j(\xi, \tau)$ when passing through zero on the intervals $0 < \xi < 1$ and $1 < \xi < 1.5$ changes sign. This means that flows of binary mixtures change the direction of
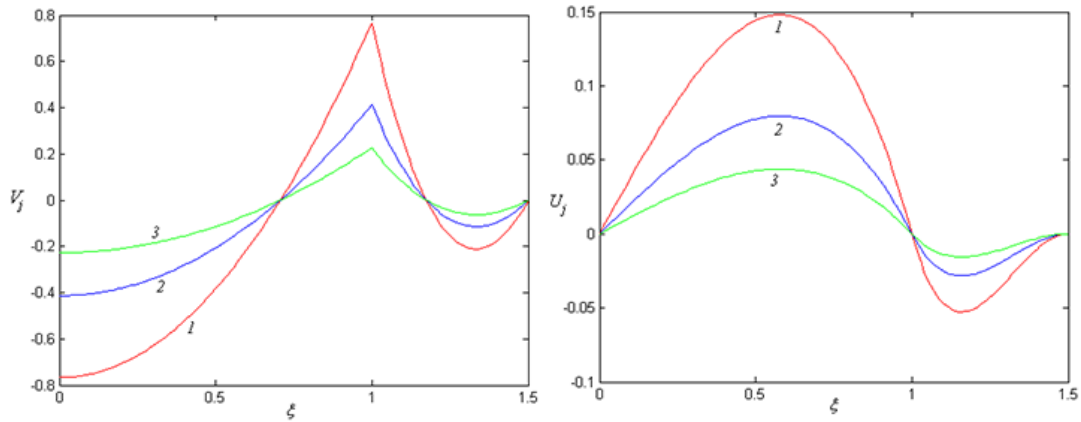
Fig. 3. The relationship between functions $V_j(\xi,\tau)$, $U_j(\xi,\tau)$ and parameter E: $1 - E = 0.05$, $2 - E = 0.2$, $3 - E = 0.7$

movement. Thus, return flow zones appear in liquid layers near the interface. This happens not only due to the gradient of surface tension but also due to the non-stationary pressure drop in the layers that occurs during heating.

Let us consider the obtained numerical results for other functions. Due to the formation of heat function $A_j(\xi,\tau)$ increases in both layers. As for "concentration", function $H_j(\xi,\tau)$ decreases (see Fig. 4). One should take into account the Soret number Sr_j. This dimensionless parameter has a great impact on the concentration distribution in mixtures. Depending on the thermal diffusion coefficient α_j, the Soret number can be either positive or negative. If the Soret number for both mixtures is negative then the directions of the temperature gradient and the diffusion mass flow are the same. As a result, light components move to the more heated area, and the heavy ones stay in areas with reduced temperature. This corresponds to the phenomenon of normal thermal diffusion. For positive Soret numbers, the direction of movement of components changes to the opposite. At the same time, the corresponding effect of thermal diffusion is abnormal. The results of numerical calculation allow one to conclude that abnormal thermal diffusion takes place in this model.
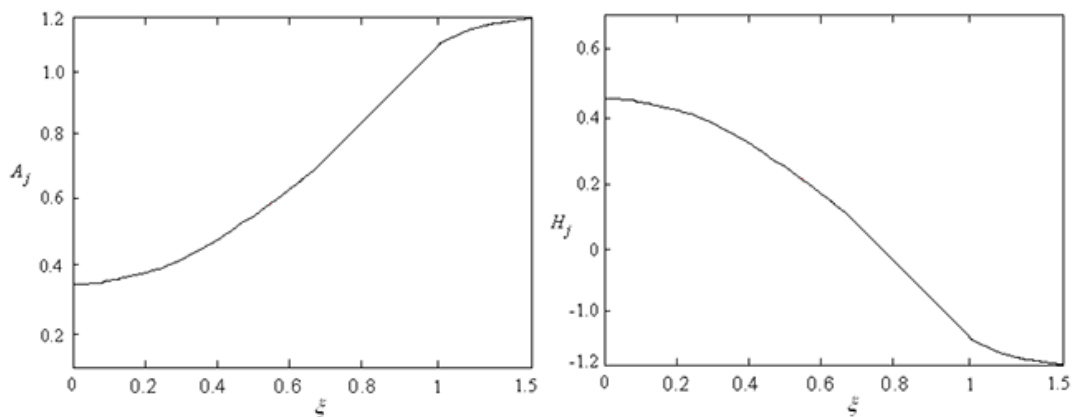

Fig. 4. Plots of functions $A_j(\xi,\tau)$, $H_j(\xi,\tau)$ at fixed τ

Conclusion

A study of the unsteady two-layer flow of binary mixtures in a cylindrical capillary was carried out with consideration for changes in internal energy on the inter-facial surface. The resulting conjugate initial-boundary value problem is non-linear and inverse with respect to pressure gradients along the axis of the cylindrical capillary. To solve the problem the tau method was used. Shifted Jacobi polynomials were taken as basis functions. As a result, the system of integro-differential equations was reduced to a closed system of ordinary differential equations of the first order. To solve the system of equations the Runge-Kutta method of the fourth order was used. It was shown that with increasing time the solution of the non-stationary problem tends to the stationary mode. As a result of calculations for the model problem it was found that when energy parameter increases the characteristic convection velocity changes and intensity decreases. The increase of the energy parameter also promotes laminarization of the flow near the inter-facial surface.

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Нестационарное течение двух бинарных смесей в цилиндрическом капилляре с учетом изменений внутренней энергии поверхности

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Аннотация. Исследуется задача о двумерном нестационарном течении двух несмешивающихся несжимаемых бинарных смесей в цилиндрическом капилляре в отсутствие массовых сил. Смеси контактируют через общую поверхность раздела, на которой учитывается энергетическое условие. Температура и концентрация в них распределены по квадратичному закону, что хорошо согласует-ся с полем скоростей типа Хименца. Возникающая сопряженная начально-краевая задача является нелинейной и обратной относительно градиентов давлений вдоль оси цилиндрической трубки. Для ее решения применяется тау-метод. Показано, что с ростом времени численное решение нестаци-онарной задачи выходит на стационарный режим. Установлено, что влияние приращений внут-ренней энергии межфазной поверхности существенно сказывается на динамике течения смесей в слоях.

Ключевые слова: нестационарное решение, бинарная смесь, поверхность раздела, энергетическое условие, внутренняя энергия, обратная задача, градиент давления, тау-метод, термодиффузия.