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Second Hankel Determinant for Bi-univalent Functions Associated with q-differential Operator

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Abstract. The objective of this paper is to obtain an upper bound to the second Hankel determinant denoted by $H_2(2)$ for the class $S_q^*(\alpha)$ of bi-univalent functions using q-differential operator.

Keywords: Hankel determinant, bi-univalent functions, q-differential operator, Fekete-Szegö functional.

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Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=n}^{\infty} a_k z^k \quad (a_k \geqslant 0;\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and univalent in the open unit disk given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The Koebe one-quarter theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ \ (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geqslant \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all functions f which are bi-univalent in $\mathbb{U}$ and are given by the Taylor–Maclaurin series expansion (1). The behavior of the coefficients is unpredictable when the biunivalency condition is imposed on the function $f \in \mathcal{A}$. A systematic study of the class Σ of bi-univalent function in $\mathbb{U}$, which is introduced in 1967 by Lewin [12]. For a brief history and interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties of the bi-univalent function class Σ, one can refer to the work of

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Srivastava *et al.* [21] and references therein. Ever since then, several authors investigated various subclasses of the class Σ of bi-univalent functions. For some more recent works see [22–27]. The class of bi-starlike functions is introduce by Brannan and Taha [2] (see also [14]). For $0 \leqslant \alpha < 1$, a function $f \in \mathcal{A}$ is in the class $S_q^*(\alpha)$ of bi-starlike function of order α if both f and f^{-1} are starlike in $\mathbb{U}$ and obtained estimates on the initial coefficients conjectured that $|a_2| \leqslant \sqrt{2}$. It may be noted that for $\alpha = 0$, $q \longrightarrow 1^-$, $S_q^*(\alpha) = S^*$, the familiar subclass of starlike functions in $\mathbb{U}$.

For the univalent function in the class $\mathcal{A}$, it is well known that the n^{th} coefficient a_n is bounded by n. The bounds for the coefficients gives information about the geometric properties of these functions. For example growth and distortion properties of normalized univalent function are obtained by using the bounds of its second coefficient a_2. In 1966, Pommerenke [15] define the Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}. \tag{2}$$

A good amount of literature is available about the importance of Hankel determinant. It plays an important role in the study of singularities as well as in the study of power series with integral coefficients ([3,4]). In 1916, Bieberbach proved that if $f \in S$, then $|a_2^2 - a_3| \leqslant 1$. In 1933, Fekete and Szegö [5] proved that

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} 4\mu - 3 & \text{if } \ \mu \geqslant 1, \\ 1 + 2exp[-2\mu/(1-\mu)] & \text{if } \ 0 \leqslant \mu < 1, \\ 3 - 4\mu & \text{if } \ \mu \leqslant 0. \end{cases} \tag{3}$$

The Hankel functional $H_2(1) = \left|a_3 - a_2^2\right|$ and $H_2(2) = \left|a_2 a_4 - a_3^2\right|$ is also known as Fekete–Szegö functional and second Hankel determinant respectively. The Hankel functional has many applications in functional theory. For example $|a_3 - a_2^2|$ is equal to $S_f(z)/6$, where $S_f(z)$ is the schwarzian derivative of the locally univalent function defined $S_f(z) = (f''(z)/f'(z))' - 1/2(f''(z)/f(z))^2$ (See [19]). In 1969, Keough and Merkers [11] solved Fekete–Szegö problem for the classes of starlike and convex functions. Lee *et al.* [13] established the sharp bounds to $|H_2(2)|$ by generalizing several classes defined by subordination. Janteng *et al.* [9] (see also [1,18]) provided a brief survey on Hankel determinants and obtained bounds for $|H_2(2)|$ for the classes of starlike and convex functions.

The theory of q-calculus in recent years has attracted the attention of researchers. The q-analogy of the ordinary derivative was initiated at the beginning of century by Jackson [8]. Ismail *et al.* [7] first introduce and explore class of generalized complex functions via q-calculus on the open unit disk $\mathbb{U}$. Recently many newsworthy results related to subclass of analytic functions and q-operators are meticulously studied by various authors (see [10,17,20]). For $0 < q < 1$, the q-derivative of a function f given by (1) is defined as

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & \text{for } \ z \neq 0, \\ f'(0) & \text{for } \ z = 0. \end{cases} \tag{4}$$

We note that $\lim\limits_{q\to 1^-} D_q f(z) = f'(z)$. From (4), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{5}$$

where as $q \to 1^-$

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^k \longrightarrow k. \tag{6}$$

In this connection, our aim is to study upper bounds for functional $|a_2a_4 - a_3^2|$ for functions belonging to the class $f \in S_q^*(\alpha)$, which is defined as follows.

Definition 0.1. *A function $f(z)$ given by (1) is said to be in the class $f \in S_q^*(\alpha)$, $0 < q < 1$, $0 \leqslant \alpha < 1$ if the following conditions are satisfied:*

$$f \in \Sigma, \qquad \frac{z(D_q f(z))}{f(z)} > \beta \qquad (0 \leqslant \beta < 1; z \in \mathbb{U})$$

$$\text{and} \qquad \frac{z(D_q g(w))}{g(w)} > \beta \qquad (0 \leqslant \beta < 1; z \in \mathbb{U}), \tag{7}$$

where the function g is the extension of f^{-1} to $\mathbb{U}$.

In order to derive our main results, we have to recall here the following lemma.

Lemma 0.1 ([16]). *If $h \in \mathcal{H}$, then $|B_k| \leqslant 2$, for each $k \geqslant 1$ and the inequality is sharp for the function $\dfrac{1+z}{1-z}$.*

Lemma 0.2 ([6]). *If $p \in \mathcal{P}$, $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$ then $2c_2 = c_1^2 + x(4 - c_1^2)$, $4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$, for some x, z with $|x| \leqslant 1$ and $|z| \leqslant 1$.*

Another result that will required is the maximum value of a quadratic expression. Stranded computation shows

$$\max_{(0\leqslant t\leqslant 4)} \left(Pt^2 + Qt + R\right) = \begin{cases} (4PR - Q^2)/4P & \text{if } Q > 0,\ P \leqslant -Q/8, \\ R & \text{if } Q \leqslant 0,\ P \leqslant -Q/4, \\ 16P + 4Q + R & \text{if } Q \geqslant 0,\ P \geqslant -Q/8 \ \text{ or } \ Q \leqslant 0,\ P \geqslant -Q/4. \end{cases} \tag{8}$$

1. Main results

In this section, we investigate second Hankel determinant $|H_2(2)|$ for functions belonging to the class $S_q^*(\alpha)$ using q-differential operator. For convenience, in the sequel we use the abbreviation $q_2 = [2]_q - 1,\ q_3 = [3]_q - 1,\ q_4 = [4]_q - 1$.

Theorem 1.1. *Let $0 \leqslant \alpha < 1$, $0 < q < 1$. If function $f \in \mathcal{A}$ given by (1) belongs to the class $\mathcal{S}_q^*(\alpha)$ then*
i. For $Q > 0$, $P \leqslant -Q/8$

$$\left|a_2a_4 - a_3^2\right| \leqslant T\left(R - \frac{Q^2}{4P}\right). \tag{9}$$

ii. For $Q \leqslant 0$, $P \leqslant -Q/4$

$$\left|a_2a_4 - a_3^2\right| \leqslant TR. \tag{10}$$

iii. For $Q \geqslant 0$, $P \geqslant -Q/4$

$$\left|a_2a_4 - a_3^2\right| \leqslant T\left(16P + 4Q + R\right), \tag{11}$$

where

$$\begin{aligned} &P = 4\beta^2 L + \beta M + N, \quad Q = U - 4\beta V, \\ &R = 64q_2^4q_4, \quad L = (q_4 - q_3)q_3^2, \quad M = q_2^2q_4 + 8q_3^2 - 8q_3q_4, \\ &N = 4(q_4 - q_3) - q_2^2q_3q_4 + 4q_2^4q_4, \quad U = 4q_2^2q_3q_4 + 12q_2^3q_3^2 - 32q_2^4q_4, \quad V = q_2^2q_3q_4 \quad \text{and} \\ &T = \frac{(1-\beta)^2}{4q_2^4q_3^2q_4}. \end{aligned} \tag{12}$$

Proof. If $f \in \mathcal{S}_q^*(\alpha)$ and $g \in f^{-1}$. Then

$$\frac{z(D_qf(z))}{f(z)} = \beta + (1-\beta)p(z)$$

and

$$\frac{w(D_qg(w))}{g(w)} = \beta + (1-\beta)q(w). \tag{13}$$

We obtain

$$\frac{z(D_qf(z))}{f(z)} = 1 + q_2a_2z + \left[q_3a_3 - q_2a_2^2\right]z^2 + \left[q_4a_4 - (q_2+q_3)a_3a_2 + q_2a_2^3\right]z^3 + \cdots. \tag{14}$$

Also

$$\begin{aligned} \frac{w(D_qg(w))}{g(w)} = 1 - q_2a_2z + \left[q_3\left(2a_2^2 - a_3\right) - q_2a_2^2\right]w^2 + \\ + \left[(q_2+q_3)a_2\left(2a_2^2 - a_3\right) - q_4\left(5a_2^3 - 5a_2a_3 + a_4\right) - q_2a_2^3\right]w^3 + \cdots. \end{aligned} \tag{15}$$

From (13), (14) and (15), it is easily seen that

$$a_2 = \frac{(1-\beta)c_1}{q_2}, \tag{16}$$

$$a_3 = \frac{(1-\beta)^2c_1^2}{q_2^2} + \frac{(1-\beta)(c_2 - d_2)}{2q_3} \tag{17}$$

and

$$a_4 = \frac{q_3(1-\beta)^3c_1^3}{q_2^3q_4} + \frac{5(1-\beta)^2c_1(c_2-d_2)}{4q_2q_3} + \frac{(1-\beta)(c_3-d_3)}{2q_4}. \tag{18}$$

Upon simplification, we easily establish

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg|\frac{(q_3-q_4)(1-\beta)^4}{q_2^4q_4}c_1^4 + \frac{(1-\beta)^3}{4q_2^2q_3}c_1^2(c_2-d_2) + \\ + \frac{(1-\beta)^2}{2q_2q_4}c_1(c_3-d_3) - \frac{(1-\beta)^2}{4q_3^2}(c_2-d_2)^2\Bigg|. \end{aligned} \tag{19}$$

According to Lemmas 1 and 2, we write

$$c_2 - d_2 = \frac{4 - c_1^2}{2}(x - y) \tag{20}$$

and

$$4c_3 - 4d_3 = \frac{c_1^3}{2} + \frac{c_2\left(4 - c_1^2\right)}{2}(x + y) - \frac{c_1\left(4 - c_1^2\right)}{2}(x^2 + y^2) + \\ + \frac{\left(4 - c_1^2\right)}{2}\left(\left(1 - |x|^2\right)z - \left(1 - |y|^2\right)w\right), \tag{21}$$

for some x, y, z and w with $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$ and $|w| \leqslant 1$. Substituting values of c_2, c_3, d_2 and d_3 from (20), (21) on the right side of (19), we have

$$\left|a_2 a_4 - a_3^2\right| \leqslant \mathrm{M}_1 + \mathrm{M}_2(\varrho_1 + \varrho_2) + \mathrm{M}_3(\varrho_1^2 + \varrho_2^2) + \mathrm{M}_4(\varrho_1 + \varrho_2) := F(\varrho_1, \varrho_2), \tag{22}$$

where

$$\mathrm{M}_1 = \frac{(q_4 - q_3)(1 - \beta)^4}{q_2^4 q_4} c_1^4 + \frac{(1 - \beta)^2}{4q_2 q_4} c_1^4 + \frac{(1 - \beta)^2}{2q_2 q_4} c_1\left(4 - c_1^2\right), \tag{23}$$

$$\mathrm{M}_2 = \left[\frac{(1 - \beta)^3}{8q_2^2 q_3} c_1^2\left(4 - c_1^2\right) + \frac{(1 - \beta)^2}{4q_2 q_4} c_1^2\left(4 - c_1^2\right)\right](|x| + |y|), \tag{24}$$

$$\mathrm{M}_3 = \left[\frac{(1 - \beta)^2}{8q_2 q_4} c_1^2\left(4 - c_1^2\right) - \frac{(1 - \beta)^2}{4q_2 q_4} c_1\left(4 - c_1^2\right)\right]\left(|x|^2 + |y|^2\right), \tag{25}$$

$$\mathrm{M}_4 = \frac{(1 - \beta)^2}{8q_3^2}\left(4 - c_1^2\right)^2](|x| + |y|)^2. \tag{26}$$

Applying Lemma 1, without loss of generality assume $c_1 \equiv c \in [0, 2]$ for $\varrho_1 = |x| \leqslant 1$ and $\varrho_2 = |y| \leqslant 1$ and using triangle inequality, we have

$$\mathrm{M}_1 = \frac{(1 - \beta)^2}{4q_2^4 q_4}\left[4(q_4 - q_3)(1 - \beta)^2 - 2c^3 + 8c + q_2^3\right]c^4 \geqslant 0, \tag{27}$$

$$\mathrm{M}_2 = \frac{(1 - \beta)^2}{8q_2^2 q_3 q_4}\left[(1 - \beta)q_4 + 2q_2 q_3\right]c^2\left(4 - c^2\right) \geqslant 0, \tag{28}$$

$$\mathrm{M}_3 = \frac{(1 - \beta)^2}{8q_2 q_4}\left(4 - c^2\right)c(c - 2) \leqslant 0, \tag{29}$$

$$\mathrm{M}_4 = \frac{(1 - \beta)^2}{4q_3^2}\left(4 - c^2\right)^2 \geqslant 0. \tag{30}$$

To maximize the function $F(\varrho_1, \varrho_2)$ on the closed region $\mathfrak{S} = \{(\varrho_1, \varrho_2) : 0 \leqslant \varrho_1 \leqslant 1, 0 \leqslant \varrho_2 \leqslant 1\}$. Differentiating $F(\varrho_1, \varrho_2)$ partially with respect to ϱ_1 and ϱ_2, we get

$$F_{\varrho_1\varrho_1} \cdot F_{\varrho_2\varrho_2} - \left(F_{\varrho_1\varrho_2}\right)^2 < 0. \tag{31}$$

This shows that the function $F(\varrho_1, \varrho_2)$ cannot have local maximum in the interior of the region $\mathfrak{S}$. Now we investigate the maximum of $F(\varrho_1, \varrho_2)$ on the boundary of the region $\mathfrak{S}$. For $\varrho_1 = 0$ and $0 \leqslant \varrho_2 \leqslant 1$ (similarly $\varrho_2 = 0$ and $0 \leqslant \varrho_1 \leqslant 1$), we obtain

$$F(0, \varrho_2) = \Omega(\varrho_2) = (\mathrm{M}_3 + \mathrm{M}_4)\varrho_2^2 + \mathrm{M}_2\varrho_2 + \mathrm{M}_1. \tag{32}$$

i. $M_3+M_4 \geqslant 0$: In this case for $0 \leqslant \varrho_2 \leqslant 1$ and any fixed c with $0 \leqslant c \leqslant 2$, it is clear that $\Omega'(\varrho_2) = 2(M_3+M_4)\varrho_2 + M_2 > 0$, that is $\Omega(\varrho_2)$ is an increasing function hence for fixed $c \in [0,2)$, the maximum of $\Omega(\varrho_2)$ occurs at $\varrho_2 = 1$ and maximum of $\varrho_2 = M_1+M_2+M_3+M_4$.

ii. $M_3+M_4 < 0$: Since $M_2 + 2(M_3+M_4) \geqslant 0$ for $0 < \varrho_2 < 1$ and for any fixed c with $0 \leqslant c < 2$, it is clear that $M_2 + 2(M_3+M_4) < 2(M_3+M_4)\varrho_2 + M_2 < M_2$ and so $\Omega'(\varrho_2) > 0$. Hence for fixed c with $0 \leqslant c < 2$, the maximum $\Omega'(\varrho_2)$ occurs at $\varrho_2 = 1$. Also for $c = 2$ we obtain

$$F(\varrho_1,\varrho_2) = \frac{4(1-\beta)^2(q_4-q_3)}{q_2^4 q_4}\left[(1-\beta)^2 + \frac{q_2^3}{(q_4-q_3)}\right]. \tag{33}$$

For $\varrho_1 = 1$ and $0 \leqslant \varrho_2 < 1$ (similarly $\varrho_2 = 1$ and $0 \leqslant \varrho_1 \leqslant 1$), we obtain

$$F(1,\varrho_2) = \mho(\varrho_2) = (M_3+M_4)\varrho_2^2 + (M_2+2M_4)\varrho_2 + M_1+M_2+M_3+M_4. \tag{34}$$

Thus from above cases of M_3+M_4 we get that

$$\max \mho(\varrho_2) = \mho(1) = M_1 + 2M_2 + 2M_3 + 4M_4. \tag{35}$$

Since $\Omega(1) \leqslant \mho(1)$ for $c \in [0,2]$, we obtain $\max F(\varrho_1,\varrho_2) = F(1,1)$ on the boundary of the square $\mathfrak{S}$. Thus , the maximum of F occurs at $\varrho_1 = 1$ and $\varrho_2 = 1$ in the closed square $\mathfrak{S}$.

Let $\Bbbk : [0,2] \longrightarrow \mathbb{R}$ defined by

$$\Bbbk(c) = \max(\varrho_1,\varrho_2) = F(1,1) = M_1 + 2M_2 + 2M_3 + 4M_4. \tag{36}$$

Substituting the values of M_1, M_2, M_3, M_4 in the function $\Bbbk$ defined by (36), we get

$$\begin{aligned}\Bbbk(c) =& \frac{(1-\beta)^2}{4q_2^4 q_3^2 q_4}\Big(\left|4(q_4-q_3)(1-\beta)^2 q_3^2 - (1-\beta)q_2^2 q_3 q_4 + 4q_2^4 q_4\right| c^4 + \\ &+ \left|4(1-\beta)q_2^2 q_3 q_4 + 12 q_2^3 q_3^2 - 32 q_2^4 q_4\right| c^2 + \left|64 q_2^4 q_4\right|\Big)\end{aligned} \tag{37}$$

which is quadratic in c^2. Using the standard computation, we get

$$\left|a_2 a_4 - a_3^2\right| \leqslant T \begin{cases} (4PR - Q^2)/4P & \text{if } Q > 0,\ P \leqslant -Q/8, \\ R & \text{if } Q \leqslant 0,\ P \leqslant -Q/4, \\ 16P + 4Q + R & \text{if } Q \geqslant 0,\ P \geqslant -Q/8 \text{ or } Q \leqslant 0,\ P \geqslant -Q/4 \end{cases} \tag{38}$$

where P, Q, R and T are given by (12).
This completes the proof. □

Theorem 1.2. *Let $0 < q < 1,\ 0 \leqslant \alpha < 1$ and $f \in S_q^*(\alpha)$. Then for complex μ*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2}{q_2^2}. \tag{39}$$

Proof. Letting $c := c_1 > 0$. Then for complex μ, using (16) and (17), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{(1-\beta)^2 c^2}{q_2^2} + \frac{(1-\beta)(c_2-d_2)}{2q_3} - \mu\frac{(1-\beta)^2 c^2}{q_2^2} = \\ &= \frac{(2-\mu)(1-\beta)^2 c^2 q_3 + (1-\beta)(c_2 - d2)}{2q_2^2 q_3}. \end{aligned} \tag{40}$$

By (16), we obtain

$$a_3 - \mu a_2^2 = \frac{2(2-\mu)(1-\beta)^2 c^2 q_3 + (1-\beta)(4-c^2)(x-y)}{4q_2^2 q_3}, \tag{41}$$

where x and y satisfying $|x| \leqslant 1$, $|y| \leqslant 1$

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2 c^2}{4q_2^2}, \tag{42}$$

using $c \leqslant 2$, we get

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{(2-\mu)(1-\beta)^2}{q_2^2}. \tag{43}$$

This completes the proof. □

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Второй определитель Ганкеля для биунивалентных функций, ассоциированных с q-дифференциальным оператором

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Аннотация. Целью данной статьи является получение верхней оценки второго определителя Ганкеля, обозначаемого $H_2(2)$, для класса $S_q^*(\alpha)$ биунивалентных функций используя q-дифференциальный оператор.

Ключевые слова: определитель Ганкеля, биоднолистные функции, q-дифференциальный оператор, функционал Фекете-Сегаö.