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Dehn Functions and the Space of Marked Groups

Mohammad Shahryari*
Department of Mathematics
College of Sciences
Sultan Qaboos University
Muscat, Oman

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Abstract. In the space of marked group, we suppose that a sequence (G_i, X_i) converges to (G, X), where G is finitely presented. We obtain an inequality which connects Dehn functions of G_is and G.

Keywords: space of marked groups, Gromov-Grigorchuk metric, finitely presented groups, Dehn functions.

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In the space of marked groups, consider the situation a sequence (G_i, X_i) in which converges to a finitely presented marked group (G, X). What can we say about the relation between corresponding Dehn functions of the groups G_i and G? Suppose $\Gamma_i = \langle X_i | R_i \rangle$ is an arbitrary presentation for G_i and $\Gamma = \langle X | R \rangle$ is an arbitrary finite presentation for G. Let $L = \max_{r \in R} \|r\|$. We will prove that for any $n \geqslant 0$,

$$\limsup_i \frac{\delta_i(n)}{\delta_i(L)} \leqslant \delta(n).$$

Here of course, δ_i is the Dehn function of G_i corresponding to Γ_i. Also δ is the Dehn function of G corresponding to a finite presentation Γ. As a result, it shows that if the set $\{\delta_i(L) : \ i \geqslant 1\}$ is finite, then so is the set $\{\delta_i(n) : \ i \geqslant 1\}$, for all $n \geqslant 0$.

1. Basic notions

The idea of Gromov-Grigorchuk metric on the space of finitely generated groups is proposed by M. Gromov in his celebrated solution to the Milnor's conjecture on the groups with polynomial growth (see [5]). For a detailed discussion of this metric, the reader can consult [2]. Here, we give some necessary basic definitions. A marked group (G, X) consists of a group G and an m-tuple of its elements $X = (x_1, \dots, x_m)$ such that G is generated by X. Two marked groups (G, X) and (G', X') are *the same*, if there exists an isomorphism $G \to G'$ sending any x_i to x'_i. The set of all such marked groups is denoted by $\mathcal{G}_m$. This set can be identified by the set of all normal subgroup of the free group $\mathbb{F} = \mathbb{F}_m$. Since the later is a closed subset of the compact topological space $2^{\mathbb{F}}$ (with the product topology), so it is also a compact space. This space is in fact metrizable: let B_λ be the closed ball of radius λ in $\mathbb{F}$ (having the identity as the center) with respect to its word metric. For any two normal subgroups N and N', we say that they are in distance at most $e^{-\lambda}$, if $B_\lambda \cap N = B_\lambda \cap N'$. So, if Λ is the largest of such numbers, then we can define

$$d(N, N') = e^{-\Lambda}.$$

*m.ghalehlar@squ.edu.om

This induces a corresponding metric on $\mathcal{G}_m$. To see what is this metric exactly, let (G, X) be a marked group. For any non-negative integer λ, consider the set of *relations* of G with length at most λ, i.e.

$$\mathrm{Rel}_\lambda(G, X) = \{w \in \mathbb{F} : \|w\| \leqslant \lambda, w_G = 1\}.$$

Then $d((G, X), (G', X')) = e^{-\Lambda}$, where Λ is the largest number such that $\mathrm{Rel}_\Lambda(G, X) = \mathrm{Rel}_\Lambda(G', X')$. Note that we identify here X and X' using the correspondence $x_i \to x_i'$. This metric on $\mathcal{G}_m$ is the so called Gromov-Grigorchuk metric.

Many topological properties of the space $\mathcal{G}_m$ is discussed in [2]. In this note, we need just one elementary fact: any finitely presented marked group (G, X) in $\mathcal{G}_m$ has a neighborhood, every element in which is a quotient of G.

We also need to review some basic notions concerning *Dehn* and *isoperimtry* functions. Let $\langle X|R\rangle$ be a presentation for a finitely generated group G (X is finite). Let $w \in \mathbb{F} = F(X)$ be a word such that $w_G = 1$. Clearly in this case w belongs to $\langle R^{\mathbb{F}}\rangle$, the normal closure of R in $\mathbb{F}$. Hence, we have

$$w = \prod_{i=1}^{k} u_i r_i^{\pm 1} u_i^{-1},$$

for some elements $u_1, \ldots, u_k \in \mathbb{F}$ and $r_1, \ldots, r_k \in R$. The smallest possible k is called the *area* of w and it is denoted by $\mathrm{Area}_R(w)$. A function $f : \mathbb{N} \to \mathbb{R}$ is an *isoperimetric* function for the given presentation, if for all $w \in \mathbb{F}$, with $w_G = 1$, we have

$$\mathrm{Area}_R(w) \leqslant f(\|w\|).$$

The corresponding Dehn function is the smallest isoperimetric function, i.e.

$$\delta(n) = \max\{\mathrm{Area}_R(w) : w \in \mathbb{F}, w_G = 1, \|w\| \leqslant n\}.$$

This function measures the complexity of the word problem in the case of finitely presented group G: the word problem for the presentation $\langle X|R\rangle$ is solvable, if and only if, the corresponding Dehn function is recursive. In fact the recursive Dehn functions measures the time complexity of fastest non-deterministic Turing machine solving word problem of G (see [3] and [7]). Also in the case of finitely presented groups, the *type* of Dehn function is a quasi-isometry invariant of groups. Although, in this note, we use the exact values of Dehn function, we give the definition of *type*. Let $f, g : \mathbb{N} \to \mathbb{N}$ be two arbitrary functions. We say that f is *dominated* by g, if there exists a positive number C, such that for all n,

$$f(n) \leqslant Cg(Cn + C) + Cn + C.$$

We denote the domination by $f \preceq g$. These two functions are said to be equivalent, if $f \preceq g$ and $g \preceq f$. The type of a Dehn function is its equivalence class with respect to this relation. Two Dehn functions of a fixed group with respect to different finite presentations have the same type. There are many classes of finitely presented groups having Dehn functions of type n^α for a dense set of exponents $\alpha \geqslant 2$ (see [1]). Hyperbolic groups are the only groups having linear type Dehn functions. Olshanskii proved that there is no group with Dehn function of type n^α with $1 < \alpha < 2$ (see [6]). For a study of Dehn functions of non-finitely presented groups, the reader can consult [4].

2. Main results

We work within the space of marked groups $\mathcal{G}_m$.

Theorem 1. *Let (G_i, X_i) be a sequence converging to (G, X), where G is finitely presented. Then for any n, we have*

$$\limsup_i \frac{\delta_i(n)}{\delta_i(L)} \leqslant \delta(n),$$

where δ_i is any Dehn function of G_i and δ is the Dehn function of G corresponding to any finite presentation $\Gamma = \langle X|R\rangle$ and $L = \max_{r\in R}\|r\|$.

Proof. As G is finitely presented, we may assume that all G_i is a quotient of G. We also identify X_i by X using the obvious correspondence. Let $\mathbb{F} = F(X)$ be the free group on X and assume that $w \in \mathbb{F}$. Suppose that $w_G = 1$ and $l = \mathrm{Area}_R(w)$. Then we have

$$w = \prod_{j=1}^{l} a_j r_j^{\pm 1} a_j^{-1},$$

where $a_1, \ldots, a_l \in \mathbb{F}$ and $r_1, \ldots, r_l \in R$. We know that $(r_j)_{G_i} = 1$, for all i and j, hence

$$r_j = \prod_{t_j=1}^{l_{ij}} u_{it_j} r_{it_j}^{\pm 1} u_{it_j}^{-1},$$

where $l_{ij} = \mathrm{Area}_{R_i}(r_j)$, $r_{i1}, \ldots, r_{il_{ij}} \in R_i$ and $u_{i1}, \ldots, u_{il_{ij}} \in \mathbb{F}$. Therefore, we have

$$w = \prod_{j=1}^{l} a_j (\prod_{t_j=1}^{l_{ij}} u_{it_j} r_{it_j}^{\pm 1} u_{it_j}^{-1})^{\pm 1} a_j^{-1} = \prod_{j=1}^{l}\prod_{t_j=1}^{l_{ij}} a_j u_{it_j} r_{it_j}^{\mp 1} u_{it_j}^{-1} a_j^{-1}.$$

This shows that

$$\mathrm{Area}_{R_i}(w) \leqslant \sum_{j=1}^{l} l_{ij} = \sum_{j=1}^{l} \mathrm{Area}_{R_i}(r_j).$$

Suppose $K_i = \max_{r\in R}(\mathrm{Area}_{R_i}(r))$. Then, we have

$$(*) \quad \mathrm{Area}_{R_i}(w) \leqslant K_i \cdot \mathrm{Area}_R(w).$$

Now, let $n \geqslant 1$. There exists an integer i_0 such that for any $i \geqslant i_0$, we have

$$d((G_i, X_i), (G, X)) \leqslant e^{-n}.$$

This shows that $\mathrm{Rel}_n(G_i, X_i) = \mathrm{Rel}_n(G, X)$, for $i \geqslant i_0$. In other words

$$\{w \in \mathbb{F} : \|w\| \leqslant n, w_{G_i} = 1\} = \{w \in \mathbb{F} : \|w\| \leqslant n, w_G = 1\}.$$

By $(*)$ and by the definition of Dehn function, we conclude $\delta_i(n) \leqslant K_i \cdot \delta(n)$. Hence, for $i \geqslant i_0$, we have

$$\frac{\delta_i(n)}{K_i} \leqslant \delta(n),$$

and therefore

$$\sup_{i\geqslant i_0} \frac{\delta_i(n)}{K_i} \leqslant \delta(n).$$

For any j, define

$$a_j(n) = \sup_{i\geqslant j} \frac{\delta_i(n)}{K_i} \leqslant \delta(n),$$

which a decreasing sequence in j. Since $a_{i_0}(n) \leqslant \delta(n)$, so $\lim_j a_j(n) \leqslant \delta(n)$. This shows that

$$\limsup_i \frac{\delta_i(n)}{K_i} \leqslant \delta(n).$$

Now, note that

$$K_i = \max_{r\in R} \mathrm{Area}_{R_i}(r) \leqslant \max_{r\in R, \|r\|=\|w\|} \mathrm{Area}_{R_i}(w) \leqslant \delta_i(L).$$

This completes the proof. □

As a result, we see that if the set $\{\delta_i(L) : i \geqslant 1\}$ is finite, then so is the set $\{\delta_i(n) : i \geqslant 1\}$, for all $n \geqslant 0$. This is because, if we put $M = \max_i \delta_i(L)$, then

$$\limsup_i \delta_i(n) \leqslant M \cdot \delta(n).$$

Now, if the second set is infinite, then the sequence $(\delta_i(n))_i$ has a divergent subsequence, which is a contradiction.

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Функции Дена и пространство отмеченных групп

Мохаммед Шахриари
Колледж Науки
Университет Султана Кабуса
Маскат, Оман

Аннотация. Предположим, что в пространстве отмеченной группы последовательность (G_i, X_i) сходится к (G, X), где G конечно представлена. Получаем неравенство, связывающее функции Дена G_is и G.

Ключевые слова: пространство отмеченных групп, метрика Громова–Григорчука, конечно определенные группы, функции Дена.