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Study of a Deformation Localization Direction in Slow Motion of a Granular Medium

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Abstract. This paper is devoted to the study of the direction of the deformation localization lines in a slow gravity flow of a granular medium in convergent channels with various geometric characteristics. Variational principles of the theory of limiting equilibrium, established within the framework of a special mathematical model of a material that resist tension and compression differently, are used. Assuming a linear deformation localization zone we obtain safety factors and carry out their comparative analysis.

Keywords: variational inequality, materials with different strengths, deformation localization.

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Introduction

The theory of materials with different strengths is one of the most interesting and actively developing branch of mechanics. The field of application of this theory is the problems of mechanics of geomaterials. Such materials have significantly different tensile and compressive strength properties. The range of problems related to the mechanics of geomaterials is diverse. In engineering practice, the analysis of the behavior of geomaterials is important in connection with the issues of mechanical treatment of soils, as well as in relation to the issues of mining, construction of engineering structures etc.

The study of the process of localization of deformations in samples made of a material with different strengths is of constant interest. The importance of solving of such problems is dictated by the fact that in practice in narrow zones of localization of tensile deformations where malleability of the material is significantly higher than in the rest of the sample micro-destructions occur. Therefore, when analyzing the structural design for strength, such zones must be determined. At the same time, the possibilities of constructing exact solutions in such problems are limited, thus the development of computational methods is very relevant.

In the branch of geomechanics related to the study of the behavior of granular media, there is an important problem of analyzing movement of granular media in converging channels. Problems of this kind arise when emptying granular media or geomaterials from storage chambers and bunkers, as well as in many mining technologies. The approximate (engineering) solution of the problem and the results of field experiments are presented in works [1,2]. In the work [3] the problem of a flat slow gravity flow of a granular medium in a converging channel was considered.

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For granular sample the safety factor was computed and formulas were obtained for calculating the inclination angle of a narrow linear zone of deformation localization of simple shear deformation with dilatancy. A numerical experiment was also carried out using the finite element method that showed results close to the solution.

The purpose of this work is to construct an approximate solution to the problem of slow gravity flow of a granular medium in converging channels with various geometric structures. During the transition from the static stress-strain state to the movement of a granular medium, the deformation is localized along some surfaces, followed by the movement of the formed blocks. Under assumption of a linear deformation localization zone, it is necessary to calculate the safety factors for various channel samples and conduct the comparative analysis. The solution of the problem will be based on a model that takes into account different strengths of the material [4].

1. Mathematical model

For the description of the stress-strain state of a granular medium as a different strengths material having different tensile and compressive strength limits, we will use a model of a medium with plastic bonds. This model has been developed by V. P. Myasnikov and V. M. Sadovskii in the work [4]. Under compressive or tensile strain lower than the adhesion coefficient (the limit bond strength) such a medium does not deform. As the limit bond strength is approached, the deformation develops according to the theory of linear strain hardening. The rheological scheme of the model is given on Fig. 1 [5]. According to this scheme we have the following

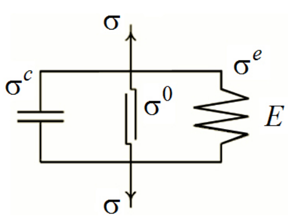


Fig. 1. The rheological scheme

additive representation $\sigma_{ij} = \sigma^c_{ij} + \sigma^0_{ij} + \sigma^e_{ij}$, where σ_{ij} is the total strain tensor, σ^c_{ij} is the rigid contact component, σ^0_{ij} is the cohesion tensor, $\sigma^e_{ij} = E_{ijkl}\varepsilon_{kl}$ is the elastic tensor, $\varepsilon = (\varepsilon_{ij})$ is the deformation tensor, E_{ijkl} is the symmetric positively defined elastic modulus tensor (we assume summing in repeating indices). The tensor σ^c_{ij} satisfies the variational inequality

$$\sigma^c_{ij} \cdot (\tilde{\varepsilon}_{ij} - \varepsilon_{ij}) \leqslant 0, \quad \varepsilon, \tilde{\varepsilon} \in C, \tag{1}$$

where C is the cone of admissible deformations of the form $C = \{\varepsilon \mid \kappa\gamma(\varepsilon) \leqslant \theta(\varepsilon)\}$, κ is the dilatancy parameter, $\gamma(\varepsilon)$ is the intensity of shear, $\theta(\varepsilon)$ is the volume deformation [5].

In this notation, the inequality (1) takes the form

$$\left(E_{ijkl}\varepsilon_{kl} - \sigma_{ij} + \sigma^0_{ij}\right) \cdot (\tilde{\varepsilon}_{ij} - \varepsilon_{ij}) \geqslant 0, \quad \varepsilon, \tilde{\varepsilon} \in C.$$

By definition of a projection, this means that

$$\varepsilon_{ij} = \pi_{ij}\left[E^{-1}_{ijkl}\left(\sigma_{ij} - \sigma^0_{ij}\right)\right],$$

here E^{-1}_{ijkl} are the components of the inverse tensor, π_{ij} the components of the projection of C with respect to the norm $|\varepsilon| = \sqrt{\varepsilon_{ij} E_{ijkl} \varepsilon_{kl}}$.

Consider [3] an element of a construction from a material with different strengths filling a planar domain Ω with the boundary $\partial\Omega = \Gamma$ that consists of two non-intersecting parts Γ_u and Γ_σ. On the first part displacements are absent and on the second part the distributed load p is given. There hold equilibrium equations in variational form and boundary conditions

$$\int\limits_{\Omega} \left(\frac{\partial \sigma_{ij}}{\partial x_j} + f_i \right) (\tilde{u}_i - u_i)\, d\Omega = 0, \tag{2}$$

$$u_i = \tilde{u}_i = 0 \ \text{ on } \ \Gamma_u, \quad \sigma_{ij} \cdot n_j = p_i \ \text{ on } \ \Gamma_\sigma. \tag{3}$$

The problem (2)–(3) reduces to the problem of finding the minimum $\min\limits_{\tilde{u}_i \in U_c} J(\tilde{u}) = J(u)$, where

$$J(u) = \iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma^0_{11} + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma^0_{12} + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma^0_{22} - (f_1 \tilde{u}_1 + f_2 \tilde{u}_2) \right) dx_1 dx_2 - \\ - \int\limits_{\Gamma_\sigma} (p_1 \tilde{u}_1 + p_2 \tilde{u}_2)\, d\Gamma,$$

$$U_C = \left\{ u_i \in H^1(\Omega) \mid u_i|_{\Gamma_u} = 0, \ \varepsilon(u) \in C \right\},$$

for the components of the deformation tensor we have kinematic equations

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

A load (f, p) is called safe if $u \equiv 0$.

Let $p_i = 0$, $f_i = m \cdot f^0_i$, where m is the loading parameter. A load is safe for m varying from zero to the limit value (safety factor)

$$m^* = \min_{\substack{(\tilde{u}_1, \tilde{u}_2) \in U_c \\ (\tilde{u}_1, \tilde{u}_2) \neq 0}} \frac{\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma^0_{11} + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma^0_{12} + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma^0_{22} \right) dx_1 dx_2}{\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2)\, dx_1 dx_2}. \tag{4}$$

This statement is a formulation of a kinematic theorem on limiting equilibrium from plasticity theory [6].

2. Linear deformation localization zone

In the paper [3] we considered a problem of planar gravity flow of a granular medium in a convergent asymmetric channel with sides inclined at angles α and β with the base a, assuming $\alpha > \beta$, $\alpha \in \left(0; \frac{\pi}{2}\right)$ (Fig. 2). The convergent channel fills a planar domain Ω with the boundary $\partial\Omega = \Gamma = \Gamma_u \bigcup \Gamma_\sigma$. On the boundary Γ_u displacements are absent. The vector p of the distributed load on Γ_σ is equal to zero.

The condition $(\tilde{u}_1, \tilde{u}_2) \in U_C$ takes the form

$$\gamma_0 \leqslant \nu \varepsilon_0, \tag{5}$$

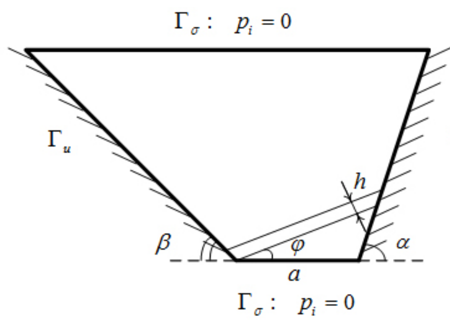


Fig. 2. Direction of a narrow linear localization zone

where $\nu = \sqrt{1/\kappa^2 - 4/3}$, $0 < \kappa < \sqrt{3}/2$.

By (4) the safety factor is m_1:

$$m_1 = \frac{2\tau_s}{\kappa\rho g a}\frac{1}{\sin\alpha}\frac{1}{(\nu\sin\alpha - \cos\alpha)}, \tag{6}$$

here τ_s is the yield point.

In paper [3] we obtained that deformation for a simple shear with dilatancy is localized in a narrow linear zone of thickness h inclined at an angle φ:

$$\varphi = \alpha - \arcsin\frac{1}{\sqrt{\nu^2+1}} \qquad \text{or} \qquad \varphi = \alpha - \operatorname{arctg}\frac{1}{\nu}. \tag{7}$$

In this case for the angle φ we compute

$$\sin\varphi = \frac{\nu\sin\alpha - \cos\alpha}{\sqrt{\nu^2+1}}, \quad \cos\varphi = \frac{\nu\sin\alpha - \cos\alpha}{\sqrt{\nu^2+1}}. \tag{8}$$

In this paper we shall consider two problems of planar gravity flow of a granular medium in a convergent asymmetric channel. The geometry of a channel in each case will differ from the one on Fig. 2. Under assumption of linearity of the deformation localization zone we compute the safety factors m^* for such channels and compare them. The boundary conditions for the domain Ω are analogous to the conditions of the problem (Fig. 2) considered in [3].

Consider a planar deformed state of a homogeneous sample (Fig. 3). Its geometry differs from that of the one considered in [3] (Fig. 2): the base of the channel is inclined at an angle φ (7).

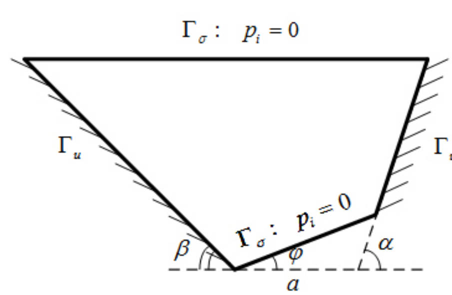
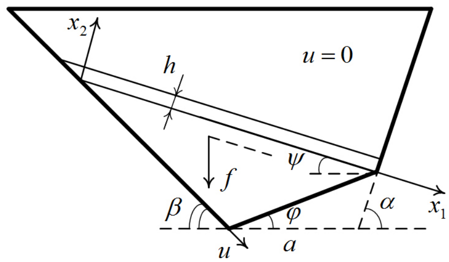


Fig. 3. Cross-section of the sample ($\alpha > \beta$)

Problem 1. Compute the safety factor m^* for the sample (Fig. 3) according to (4).



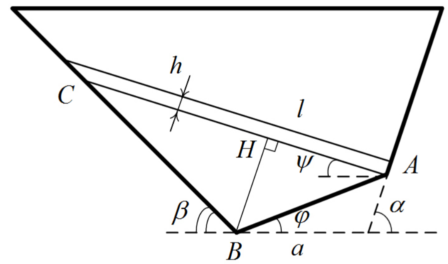


Fig. 4. Geometric constructions

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ be the admissible displacement field describing the deformation localization of simple shear with dilatancy in a narrow linear zone of thickness h inclined at an angle ψ (Fig. 4).

Let us compute some angles. Consider

$$\begin{aligned} \triangle ABH: &\quad \angle BAH = \varphi + \psi, \quad \angle ABH = \frac{\pi}{2} - (\varphi + \psi), \\ \triangle CBH: &\quad \angle CBH = \pi - (\beta + \varphi + \angle ABH) = \frac{\pi}{2} - (\beta - \psi), \\ &\quad \angle BCH = \frac{\pi}{2} - \angle CBH = \beta - \psi. \end{aligned}$$

In the Cartesian coordinates related to the narrow linear zone

$$\begin{cases} \tilde{u}_1 = u_0 \cos(\beta - \psi), \\ \tilde{u}_2 = -u_0 \sin(\beta - \psi), \end{cases} \qquad \begin{cases} \gamma_0 = \dfrac{u_0}{h} \cos(\beta - \psi), \\ \varepsilon_0 = \dfrac{u_0}{h} \sin(\beta - \psi). \end{cases} \tag{9}$$

Then we get

$$\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1} \sigma^0_{11} + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma^0_{12} + \frac{\partial \tilde{u}_2}{\partial x_2} \sigma^0_{22} \right) dx_1 dx_2 = \varepsilon_0 \, \sigma^0 \cdot S_{\square}, \tag{10}$$

where

$$\sigma^0 = \frac{\tau_s}{\kappa}, \qquad S_{\square} = h\, l.$$

The 'separating' triangle domain ABC moves as a solid body, therefore

$$\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2)\, dx_1 dx_2 = f^0 \cdot S_{\triangle}, \tag{11}$$

where

$$f^0 = \rho\, g\, u_0 \sin\beta, \qquad S_{\triangle} = \frac{1}{2} H\, l, \qquad H = L_1 \sin(\varphi + \psi),$$

$$L_1 = AB = a\,(\cos\varphi + \sin\varphi \cdot \mathrm{ctg}\,(\alpha - \varphi)) = a\,(\cos\varphi + \nu \sin\varphi),$$

substituting values from (8) we get

$$L_1 = a\sqrt{1 + \nu^2}\, \sin\varphi. \tag{12}$$

Having in mind that (10) and (11), the safety factor m^* is equal to

$$m^* = \frac{\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c\\(\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\frac{u_0}{h}\,\sin(\beta-\psi)\,h\,l}{u_0\,\sin\beta\cdot\frac{1}{2}\,H\,l} = \frac{2\,\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1,\tilde{u}_2)\in U_c\\(\tilde{u}_1,\tilde{u}_2)\neq 0}} \frac{\sin(\beta-\psi)}{\sin\beta\cdot L_1\sin(\varphi+\psi)}.$$

Taking into account (5) and (9), we obtain the relation

$$\cos(\beta-\psi)\leqslant\nu\sin(\beta-\psi),$$

$$\frac{\varepsilon_0}{\gamma_0}=\mathrm{tg}(\beta-\psi)\geqslant\frac{1}{\nu}\quad\text{ or }\quad\sin(\beta-\psi)\geqslant\frac{1}{\sqrt{\nu^2+1}}.$$

Then

$$\psi\leqslant\beta-\mathrm{arctg}\,\frac{1}{\nu}\quad\text{ or }\quad\psi\leqslant\beta-\arcsin\frac{1}{\sqrt{\nu^2+1}}.$$

In this case

$$m^*=\frac{2\,\tau_s}{\kappa\rho g L_1\sin\beta}\min_{\psi}\frac{\sin(\beta-\psi)}{\sin(\varphi+\psi)}. \tag{13}$$

Let us find the minimum of the expression from (13) with respect to ψ

$$\begin{aligned}\frac{d}{d\psi}\left(\frac{\sin(\beta-\psi)}{\sin(\varphi+\psi)}\right)&=\frac{-\cos(\beta-\psi)\sin(\varphi+\psi)-\sin(\beta-\psi)\cos(\varphi+\psi)}{\sin^2(\varphi+\psi)}=\\&=-\frac{\sin(\varphi+\psi)\cos(\beta-\psi)+\cos(\varphi+\psi)\sin(\beta-\psi)}{\sin^2(\varphi+\psi)}=\\&=-\frac{\sin(\varphi+\psi+\beta-\psi)}{\sin^2(\varphi+\psi)}=-\frac{\sin(\varphi+\beta)}{\sin^2(\varphi+\psi)}<0\qquad\forall\psi,\end{aligned}$$

since $\sin(\varphi+\beta)>0$ for $\beta,\ \varphi\in\left(0;\ \dfrac{\pi}{2}\right)$.

Thus, the minimum of the function (13) in ψ is attained at

$$\psi=\beta-\mathrm{arctg}\,\frac{1}{\nu}\quad\text{ or }\quad\psi=\beta-\arcsin\frac{1}{\sqrt{\nu^2+1}}. \tag{14}$$

Then formula (13) for the safety factor m^* assumes the form

$$m_2=\frac{2\,\tau_s}{\kappa\rho g L_1\sin\beta}\frac{1}{\sqrt{\nu^2+1}\sin(\varphi+\psi)}. \tag{15}$$

From (14) we obtain

$$\sin\psi=\frac{\nu\sin\beta-\cos\beta}{\sqrt{\nu^2+1}},\qquad\cos\psi=\frac{\nu\cos\beta+\sin\beta}{\sqrt{\nu^2+1}}. \tag{16}$$

Using geometric constructions (Fig. 4) we find the value of $l=L_2$, which is needed further below.

We have $L_2=AC=l_1+l_2$. Consider $\triangle ABC=\triangle ABH+\triangle CBH$. Then

$$\triangle ABH\ \Rightarrow\ \mathrm{tg}(\varphi+\psi)=\frac{H}{l_1},\quad H=L_1\sin(\varphi+\psi),$$

$$\triangle CBH\ \Rightarrow\ \mathrm{tg}(\beta-\psi)=\frac{H}{l_2}=\frac{1}{\nu}.$$

We get

$$L_2 = l_1 + l_2 = \frac{H}{\operatorname{tg}(\varphi+\psi)} + H\cdot\nu = L_1\cdot\sin(\varphi+\psi)\cdot\left(\frac{\cos(\varphi+\psi)}{\sin(\varphi+\psi)} + \nu\right)$$

or

$$L_2 = L_1\left(\cos(\varphi+\psi) + \nu\sin(\varphi+\psi)\right), \tag{17}$$

here L_1 is given by (12).

Consider now a planar deformed state of a homogeneous sample (Fig. 5) assuming that $\alpha \in \left(0; \frac{\pi}{2}\right)$ and $\alpha > \beta$. The geometry in Problem 2 differs from the geometry in Problem 1: the base of the channel is inclined at an angle ψ. The values of the angles φ and ψ are computed by formulas (7) and (14), respectively, $\varphi < \alpha$, $\psi < \beta$. The boundary conditions are the same as in Problem 1.

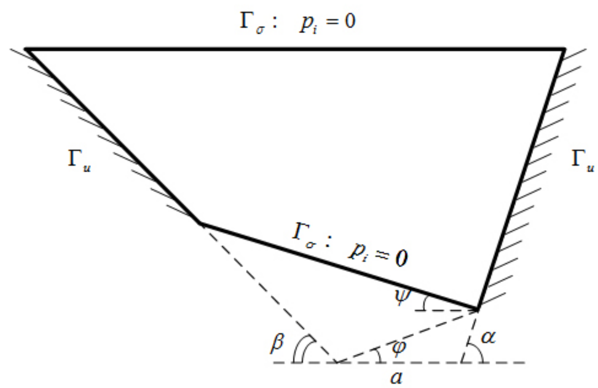


Fig. 5. Cross-section of the sample ($\alpha > \beta$)

Problem 2. Compute the safety factor m^* for the sample (Fig. 5) according to (4).

Let $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ be the admissible displacement field describing the deformation localization of simple shear with dilatancy in a narrow linear zone of thickness h inclined at an angle ϕ (Fig. 6).

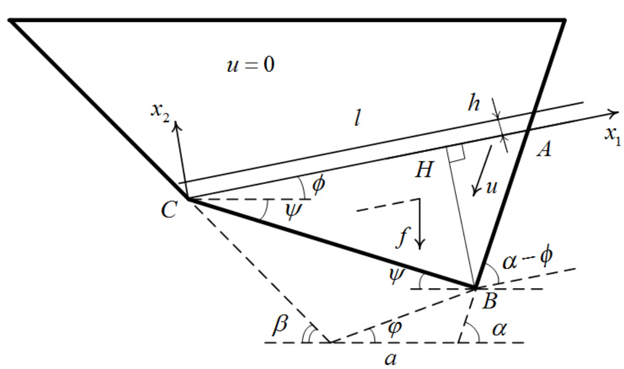


Fig. 6. Geometric constructions

In the Cartesian coordinates related to this zone

$$\begin{cases} \tilde{u}_1 = -u_0\cos(\alpha-\phi), \\ \tilde{u}_2 = -u_0\sin(\alpha-\phi), \end{cases} \qquad \begin{cases} \gamma_0 = \dfrac{u_0}{h}\cos(\alpha-\phi), \\ \varepsilon_0 = \dfrac{u_0}{h}\sin(\alpha-\phi). \end{cases} \tag{18}$$

Then we get

$$\iint\limits_{\Omega} \left(\frac{\partial \tilde{u}_1}{\partial x_1}\, \sigma_{11}^0 + \left(\frac{\partial \tilde{u}_1}{\partial x_2} + \frac{\partial \tilde{u}_2}{\partial x_1} \right) \sigma_{12}^0 + \frac{\partial \tilde{u}_2}{\partial x_2}\, \sigma_{22}^0 \right) dx_1 dx_2 = \varepsilon_0\, \sigma^0 \cdot S_{\square}, \tag{19}$$

where

$$\sigma^0 = \frac{\tau_s}{\kappa}, \quad S_{\square} = h\, l.$$

The 'separating' triangle domain ABC moves as a solid body, hence,

$$\iint\limits_{\Omega} (f_1 \tilde{u}_1 + f_2 \tilde{u}_2)\, dx_1 dx_2 = f^0 \cdot S_{\triangle}, \tag{20}$$

where

$$f^0 = \rho\, g\, u_0\, \sin\alpha, \qquad S_{\triangle} = \frac{1}{2}\, H\, l, \qquad H = L_2 \sin(\psi + \phi),$$

here L_2 is computed by formula (17).

With (19) and (20), the safety factor m^* is

$$m^* = \frac{\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1, \tilde{u}_2) \in U_c \\ (\tilde{u}_1, \tilde{u}_2) \neq 0}} \frac{\frac{u_0}{h}\, \sin(\alpha - \phi)\, h\, l}{u_0\, \sin\alpha \cdot \frac{1}{2}\, H\, l} = \frac{2\,\tau_s}{\kappa\rho g} \min_{\substack{(\tilde{u}_1, \tilde{u}_2) \in U_c \\ (\tilde{u}_1, \tilde{u}_2) \neq 0}} \frac{\sin(\alpha - \phi)}{\sin\alpha \cdot L_2 \cdot \sin(\psi + \phi)}.$$

Taking into account (5) and (18), we get the relation

$$\cos(\alpha - \phi) \leqslant \nu \sin(\alpha - \phi),$$

$$\frac{\varepsilon_0}{\gamma_0} = \operatorname{tg}(\alpha - \phi) \geqslant \frac{1}{\nu} \quad \text{or} \quad \sin(\alpha - \phi) \geqslant \frac{1}{\sqrt{\nu^2 + 1}}.$$

Then

$$\phi \leqslant \alpha - \operatorname{arctg} \frac{1}{\nu} \quad \text{or} \quad \phi \leqslant \alpha - \arcsin \frac{1}{\sqrt{\nu^2 + 1}}.$$

In this case

$$m^* = \frac{2\,\tau_s}{\kappa\rho g L_2 \sin\alpha} \min_{\phi} \frac{\sin(\alpha - \phi)}{\sin(\psi + \phi)}. \tag{21}$$

Let us find the minimum of the expression from (21) with respect to ϕ

$$\frac{d}{d\phi} \left(\frac{\sin(\alpha - \phi)}{\sin(\psi + \phi)} \right) = -\frac{\cos(\alpha - \phi)\sin(\psi + \varphi) + \sin(\alpha - \phi)\cos(\psi + \phi)}{\sin^2(\psi + \phi)} =$$

$$= -\frac{\sin(\alpha - \phi + \psi + \phi)}{\sin^2(\psi + \phi)} = -\frac{\sin(\alpha + \psi)}{\sin^2(\psi + \phi)} < 0 \quad \forall \phi,$$

since $\sin(\alpha + \psi) > 0$ for $\alpha, \psi \in \left(0; \dfrac{\pi}{2}\right)$. Hence, the minimum is attained at

$$\operatorname{tg}(\alpha - \phi) = \frac{1}{\nu} \quad \text{or} \quad \sin(\alpha - \phi) = \frac{1}{\sqrt{\nu^2 + 1}},$$

which means that

$$\phi = \alpha - \operatorname{arctg} \frac{1}{\nu} \quad \text{or} \quad \phi = \alpha - \arcsin \frac{1}{\sqrt{\nu^2 + 1}}. \tag{22}$$

Comparing the expressions (7) and (22) we deduce that the deformation localization zone is inclined at the angle $\phi = \varphi$. Thus, the safety factor (21) takes the form

$$m_3 = \frac{2\,\tau_s}{\kappa\rho g L_2 \sin\alpha} \frac{1}{\sqrt{\nu^2 + 1}\, \sin(\varphi + \psi)} \tag{23}$$

for the values of L_2 from (17).

3. Safety factors comparison

Let us compare the safety factors m_1, m_2, and m_3 of the form (6), (15), and (23), respectively. For that we consider the quotients m_2/m_1 and m_3/m_2.

By conditions, we know the angles

$$\alpha > \beta, \quad \alpha, \beta \in \left(0; \frac{\pi}{2}\right), \quad \varphi < \alpha, \quad \psi < \beta.$$

Let us carry out auxiliary computations. Using (8) and (16), we find

$$\sin(\varphi + \psi) = \sin\varphi\cos\psi + \cos\varphi\sin\psi =$$
$$= \frac{(\nu\sin\alpha - \cos\alpha)(\nu\cos\beta + \sin\beta) + (\nu\cos\alpha + \sin\alpha)(\nu\sin\beta - \cos\beta)}{\nu^2 + 1}.$$

Modify the numerator of this expression

$$\begin{aligned}
&\nu^2\sin\alpha\cos\beta + \nu\sin\alpha\sin\beta - \nu\cos\alpha\cos\beta - \cos\alpha\sin\beta + \\
&+ \nu^2\cos\alpha\sin\beta - \nu\cos\alpha\cos\beta + \nu\sin\alpha\sin\beta - \sin\alpha\cos\beta = \\
&= (\nu^2 - 1)(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 2\nu(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = \\
&= (\nu^2 - 1)\sin(\alpha + \beta) - 2\nu\cos(\alpha + \beta).
\end{aligned}$$

Then

$$\sin(\varphi + \psi) = \frac{(\nu^2 - 1)\sin(\alpha + \beta) - 2\nu\cos(\alpha + \beta)}{\nu^2 + 1}. \tag{24}$$

Analogously,

$$\cos(\varphi + \psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi =$$
$$= \frac{(\nu\sin\alpha + \cos\alpha)(\nu\cos\beta + \sin\beta) - (\nu\cos\alpha - \sin\alpha)(\nu\sin\beta - \cos\beta)}{\nu^2 + 1}.$$

After rearranging the numerator

$$\begin{aligned}
&\nu^2\sin\alpha\cos\beta + \nu\sin\alpha\sin\beta + \nu\cos\alpha\cos\beta + \cos\alpha\sin\beta - \\
&- \nu^2\cos\alpha\sin\beta + \nu\cos\alpha\cos\beta + \nu\sin\alpha\sin\beta - \sin\alpha\cos\beta = \\
&= (\nu^2 - 1)(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\nu(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \\
&= (\nu^2 - 1)\cos(\alpha + \beta) + 2\nu\sin(\alpha + \beta),
\end{aligned}$$

we obtain

$$\cos(\varphi + \psi) = \frac{(\nu^2 - 1)\cos(\alpha + \beta) + 2\nu\sin(\alpha + \beta)}{\nu^2 + 1}. \tag{25}$$

Problem 3. Find the condition for ν for which

$$\frac{m_2}{m_1} < 1.$$

Taking into account (24), we consider the quotient

$$\frac{m_2}{m_1} = \frac{\dfrac{2\,\tau_s}{\kappa\rho g L_1\sin\beta}\,\dfrac{1}{\sqrt{\nu^2+1}\,\sin(\varphi+\psi)}}{\dfrac{2\,\tau_s}{\kappa\rho g a}\,\dfrac{1}{\sin\alpha\,(\nu\sin\alpha - \cos\alpha)}} = \frac{a\sin\alpha\,(\nu\sin\alpha - \cos\alpha)}{\sqrt{\nu^2+1}\,L_1\sin\beta\sin(\varphi+\psi)} =$$

$$= \frac{\sin\alpha}{\sin\beta}\cdot\frac{1}{(\nu^2+1)}\cdot\frac{\nu\sin\alpha - \cos\alpha}{\sin\alpha\sin(\varphi+\psi)} = \frac{1}{\sin\beta}\cdot\frac{\nu\sin\alpha - \cos\alpha}{(\nu^2-1)\sin(\alpha+\beta) - 2\nu\cos(\alpha+\beta)}.$$

Then

$$\frac{\nu \sin\alpha - \cos\alpha}{\sin\beta} < (\nu^2 - 1)\sin(\alpha+\beta) - 2\nu\cos(\alpha+\beta)$$

i.e., we get a quadratic inequality in ν:

$$\nu^2 \sin(\alpha+\beta) - \nu\left(2\cos(\alpha+\beta) + \frac{\sin\alpha}{\sin\beta}\right) - \sin(\alpha+\beta) + \frac{\cos\alpha}{\sin\beta} > 0.$$

Solving it, we find

$$\begin{aligned} D &= \left(2\cos(\alpha+\beta) + \frac{\sin\alpha}{\sin\beta}\right)^2 - 4\sin(\alpha+\beta)\left(-\sin(\alpha+\beta) + \frac{\cos\alpha}{\sin\beta}\right) = \\ &= 4\cos^2(\alpha+\beta) + 4\cos(\alpha+\beta)\cdot\frac{\sin\alpha}{\sin\beta} + \frac{\sin^2\alpha}{\sin^2\beta} + 4\sin^2(\alpha+\beta) - 4\sin(\alpha+\beta)\cdot\frac{\cos\alpha}{\sin\beta} = \\ &= 4 + \frac{\sin^2\alpha}{\sin^2\beta} - \frac{4}{\sin\beta}\left(\sin(\alpha+\beta)\cos\alpha - \cos(\alpha+\beta)\sin\alpha\right) = \\ &= 4 + \frac{\sin^2\alpha}{\sin^2\beta} - \frac{4}{\sin\beta}\sin(\alpha+\beta-\alpha) = 4 + \frac{\sin^2\alpha}{\sin^2\beta} - 4 = \frac{\sin^2\alpha}{\sin^2\beta}, \end{aligned}$$

and

$$\nu_{1,2} = \frac{2\cos(\alpha+\beta) + \dfrac{\sin\alpha}{\sin\beta} \pm \dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)}.$$

Let $\nu_1 < \nu_2$, namely,

$$\nu_1 = \frac{2\cos(\alpha+\beta) + \dfrac{\sin\alpha}{\sin\beta} - \dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)} = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \operatorname{ctg}(\alpha+\beta), \tag{26}$$

$$\nu_2 = \frac{2\cos(\alpha+\beta) + \dfrac{\sin\alpha}{\sin\beta} + \dfrac{\sin\alpha}{\sin\beta}}{2\sin(\alpha+\beta)} = \frac{\cos(\alpha+\beta) + \dfrac{\sin\alpha}{\sin\beta}}{\sin(\alpha+\beta)}. \tag{27}$$

Thus, the inequality $\dfrac{m_2}{m_1} < 1$ holds for $\nu < \nu_1$ and $\nu > \nu_2$. If $\nu_1 < \nu < \nu_2$ then $m_2 > m_1$ and the second fragment does not move.

Problem 4. Find the condition for ν for which

$$\frac{m_3}{m_2} < 1.$$

Consider the relation

$$\frac{m_3}{m_2} = \frac{\dfrac{2\,\tau_s}{\kappa\rho g L_2 \sin\alpha}\dfrac{1}{\sqrt{\nu^2+1}\sin(\varphi+\psi)}}{\dfrac{2\,\tau_s}{\kappa\rho g L_1 \sin\beta}\dfrac{1}{\sqrt{\nu^2+1}\sin(\varphi+\psi)}} = \frac{L_1}{L_2}\frac{\sin\beta}{\sin\alpha} = \frac{1}{\cos(\varphi+\psi) + \nu\sin(\varphi+\psi)}\frac{\sin\beta}{\sin\alpha}.$$

Using (24) and (25), we obtain the value of the expression

$$\begin{aligned}
&\cos(\varphi+\psi)+\nu\sin(\varphi+\psi)=\\
&\qquad=\frac{\left(\nu^2-1\right)\cos(\alpha+\beta)+2\nu\sin(\alpha+\beta)}{\nu^2+1}+\nu\,\frac{\left(\nu^2-1\right)\sin(\alpha+\beta)-2\nu\cos(\alpha+\beta)}{\nu^2+1}=\\
&\qquad=\frac{\left(\nu^2-1-2\nu^2\right)\cos(\alpha+\beta)+\left(2\nu+\nu^3-\nu\right)\sin(\alpha+\beta)}{\nu^2+1}=\\
&\qquad=\frac{-\left(\nu^2+1\right)\cos(\alpha+\beta)+\nu\left(\nu^2+1\right)\sin(\alpha+\beta)}{\nu^2+1}=-\cos(\alpha+\beta)+\nu\sin(\alpha+\beta).
\end{aligned}$$

Then

$$\frac{\sin\beta}{\sin\alpha}<\nu\sin(\alpha+\beta)-\cos(\alpha+\beta)$$

or

$$\nu>\nu_0=\frac{\cos(\alpha+\beta)+\dfrac{\sin\beta}{\sin\alpha}}{\sin(\alpha+\beta)}. \tag{28}$$

Thus, the inequality $\dfrac{m_3}{m_2}<1$ holds for $\nu>\nu_0$, if $\nu<\nu_0$ then $m_3>m_2$, and the third fragment does not move.

Let us compare the obtained values $\nu_1<\nu_2$ and ν_0 from (26), (27), and (28).

We have $\sin\alpha>\sin\beta$, or $\dfrac{\sin\alpha}{\sin\beta}>1$, since $\alpha>\beta$ and $\alpha,\ \beta\in\left(0;\dfrac{\pi}{2}\right)$. Therefore $\nu_1<\nu_0<\nu_2$.

Consequently, for $\nu>\nu_2$ of the form (27) the inequalities $\dfrac{m_2}{m_1}<1$ and $\dfrac{m_3}{m_2}<1$ hold simultaneously.

Thus, depending on the value of the coefficient ν that characterize the dilatancy of the medium the deformation zones are localized differently.

Conclusion

In this paper we use the model of a granular medium with different strengths by V. P. Myasnikov and V. M. Sadovskii to study a slow gravity flow of a granular medium in convergent channels. Assuming a linear deformation localization zone we obtain an approximate value of the safety factor and formulas for the slope of a narrow linear zone of the deformation localization for a simple shear with dilatancy. A comparative analysis of the obtained factors is carried out.

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Исследование направления локализации деформации при медленном движении сыпучей среды

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Аннотация. В статье исследуется направление линий локализации деформации при медленном движении сыпучей среды под действием собственного веса в сходящемся канале с разной геометрической структурой. Используются вариационные принципы теории предельного равновесия, установленные на основе специальной математической модели материала, по-разному сопротивляющегося растяжению и сжатию. В рамках предположения о линейной зоне локализации деформации вычислены коэффициенты безопасности и проведен их сравнительный анализ.

Ключевые слова: вариационное неравенство, разнопрочная среда, локализация деформации.