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Numerical-and-analytic Method for Solving Cauchy Problem of One-dimensional Gas Dynamics

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Abstract. The equations describing the flow of a one-dimensional barotropic perfect gas are considered in this paper. Conservation laws of a special kind were found for these equations. The Cauchy problem is reduced to several quadratures along a curve on which boundary conditions are set using these conservation laws.

Keywords: one-dimensional barotropic perfect gas, Cauchy problem, conservation laws.

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Introduction

Equations of gas dynamics due to their relative simplicity and importance are among the most studied equations of continuum mechanics. Almost all new mathematical methods and techniques are used to solve the equations. These are hodograph method (S. A. Chaplygin), methods of the theory of complex variable functions (N.Ė. Zhukovskiy, S. A. Chaplygin), method of differential constraints (N. N. Yanenko), group analysis (L. V. Ovsyannikov) etc.

From the point of view of advanced symmetries and conservation laws one-dimensional equations of isentropic gas dynamics were studied [1]. A review of methods of studying gas dynamics equations was presented [1]. The conservation laws for polytropic gas equations were found [2]. These laws were used to solve Cauchy problem, and the problem was reduced to a boundary value problem for the Euler-Darboux equations. In this work the special types of conservation laws were found. They allowed us to solve the boundary value problem for the equations of isentropic gas dynamics. We have the same situation as for the equations of perfect plasticity. For these equations all the conservation laws were built [3] but they did not allow us to solve the boundary value problems. Only when the special type of conservation laws was constructed the Cauchy and Riemann problems were solved [2, 4].

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1. Problem formulation

Let us consider the equations of barotropic perfect gas flow [5]

$$u_t + uu_x = -p_x/\rho, \quad \rho_t + (\rho u)_x = 0, \tag{1}$$

where u, p, ρ are velocity, pressure and density, respectively. If $\dfrac{dp}{d\rho} = a^2$ then a is the local velocity of gas. System of equations (1) is transformed into the following system of equations [5]

$$F_1 = r_t + (u+a)r_x = 0, \quad F_2 = s_t + (u-a)s_x = 0, \quad u = \frac{r-s}{2}, \tag{2}$$

where r, s are Riemann invariants.
It is known that system of equations (2) is of hyperbolic type, and it has two sets of characteristics

$$\frac{dx}{dt} = u + a, \quad \frac{dx}{dt} = u - a. \tag{3}$$

The following conditions are satisfied along these characteristics

$$r = r_0 - const, \quad s = s_0 - const. \tag{4}$$

Purpose of Work: To construct a special type of conservation laws for system of equations (2), and to use them to solve the boundary value problem

$$r|_L = F(\tau), \quad s|_L = G(\tau), \tag{5}$$

where the smooth curve is represented parametrically $t = t(\tau)$, $x = x(\tau)$.

2. Conservation laws for system of equations (2)

Definition 1. *By the conservation law of system of equations (2) is meant the relation of the form*

$$A_t(r,s) + B_x(r,s) = \omega_1 F_1 + \omega_2 F_2, \tag{6}$$

where ω_i, $i = 1, 2$ are functions that are never both identically zero together. Values A, B are conserved currents.

From (2) we obtain

$$A_r = \omega_1, \quad A_s = \omega_2, \quad B_r = \omega_1(u+a), \quad B_s = \omega_2(u-a). \tag{7}$$

From (7) we obtain the system of two equations

$$B_r = A_r(u+a), \quad B_s = A_s(u-a). \tag{8}$$

Let us convert system (8) with the use of the following change of variables

$$\begin{aligned} A &= \phi(r,s)\exp\frac{u}{2a} + \psi(r,s)\exp(-\frac{u}{2a}), \\ B &= (u-a)\phi(r,s)\exp\frac{u}{2a} + (u+a)\psi(r,s)\exp(-\frac{u}{2a}). \end{aligned} \tag{9}$$

Then system (8) can be rewritten in the form

$$
\begin{aligned}
&a\phi_r(r,s)\exp\frac{u}{2a} - 1/4\psi(r,s)\exp(-\frac{u}{2a}) = 0,\\
&a\psi_r(r,s)\exp(-\frac{u}{2a}) - 1/4\psi(r,s)\exp\frac{u}{2a} = 0.
\end{aligned} \tag{10}
$$

Next system of equations (10) is reduced to the equation

$$16a^2\phi_{rs} - 8a\phi_r - \phi = 0. \tag{11}$$

Let us assume that

$$\phi = v(r,s)\exp\frac{s}{2a}. \tag{12}$$

Then it follows from (11) that

$$16a^2 v_{rs} - v = 0. \tag{13}$$

From (6) the conservation law follows in the form (see Fig. 1.)

$$\oint_C -Adx + Bdt = 0. \tag{14}$$

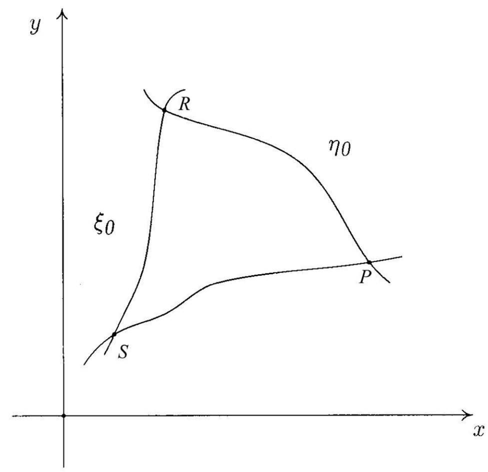

Fig. 1. The Cauchy problem

The contour C consists of three parts: Contour L, characteristics $KN(r = r_0)$ and characteristics $MN(s = s_0)$. Let us divide the integral into three parts

$$\oint_C -Adx + Bdt = \int_L -Adx + Bdt + \int_{MN} -Adx + Ddt + \int_{NK} -Adx + Bdt = 0. \tag{15}$$

Let us calculate the integral $NK(r = r_0)$. We have

$$
\begin{aligned}
\int_{NK} -Adx + Bdt &= \int_{NK} (-A(u+a) + B)dt = \\
&= t(-A(u+a) + B)|_{t_N}^{t_K} - \int_{NK} td(-A(u+a) + B).
\end{aligned} \tag{16}
$$

From this the first condition on the conserved current components is obtained:

$$-A(u+a) + B|_{r=r_0} = 1. \tag{17}$$

In a similar way the integral over MN is calculated:

$$\int_{MN} -Adx + Bdt = \int_{MN} (-A(u-a) + B)dt = \\ = t(-A(u-a)+B)|_{t_M}^{t_N} - \int_{MN} td(-A(u-a)+B). \tag{18}$$

The second condition on the conserved current components is

$$-A(u-a) + B|_{s=s_0} = -1. \tag{19}$$

Changing variables in (17) and (19) according to (10) and (12), we obtain the following boundary value problem

$$16a^2 v_{rs} - v = 0, \tag{20}$$

$$v|_{r=r_0} = -\frac{1}{2a}\exp\left(\frac{r_0+s}{2a}\right), \quad v_r|_{s=s_0} = -\frac{1}{8a^2}\exp\left(-\frac{r+s_0}{2a}\right). \tag{21}$$

The solution of problem (19), (20) is written as

$$v = -\frac{1}{2a} I_0(\sqrt{(r-r_0)(s-s_0)/(2a)})\exp\left(\frac{r_0+s_0}{2a}\right) - \\ -\frac{1}{4a^2}\int_{s_0}^{s} I_0(\sqrt{\frac{(r-r_0)(\tau-s_0)}{2a}})\exp\left(\frac{r_0+\tau}{2a}\right)d\tau - \\ -\frac{1}{8a^2}\int_{r_0}^{r} I_0(\sqrt{\frac{(\tau-r_0)(s-s_0)}{2a}})\exp\left(\frac{\tau+s_0}{2a}\right)d\tau. \tag{22}$$

In order to calculate the second coordinate of point N consider again relation (14).
Let us divide the integral into three parts

$$\oint_C -Adx + Bdt = \int_L -Adx + Bdt + \int_{MN} -Adx + Ddt + \int_{NK} -Adx + Bdt = 0. \tag{23}$$

Let us calculate the integral $NK(r = r_0)$. We have

$$\int_{NK} -Adx + Bdt = \int_{NK}\Big(-A + \frac{B}{u+a}\Big)dx = \\ = x\Big(-A+\frac{B}{u+a}\Big)\Big|_{x_N}^{x_K} - \int_{NK} x\,d\Big(-A+\frac{B}{u+a}\Big). \tag{24}$$

From this the first condition on the conserved current components is obtained

$$-A + \frac{B}{u+a}|_{r=r_0} = 1. \tag{25}$$

In a similar way the integral over MN is calculated

$$\int_{MN} -Adx + Bdt = \int_{MN}\Big(-A + \frac{B}{u-a}\Big)dx = \\ = x\Big(-A+\frac{B}{u-a}\Big)\Big|_{x_M}^{x_N} - \int_{MN} x\,d\Big(-A+\frac{B}{u-a}\Big). \tag{26}$$

The second condition on the conserved current components is

$$-A + \frac{B}{u-a}\Big|_{s=s_0} = -1. \qquad (27)$$

Changing variables in (17) and (19) according to (10) and (12), we obtain the following boundary value problem

$$16a^2 v_{rs} - v = 0, \qquad (28)$$

$$v|_{r=r_0} = -\frac{1}{2a}\exp(-\frac{r_0+s}{2a}), \quad v_r|_{s=s_0} = -\frac{1}{8a^2}\exp(-\frac{r+s_0}{2a}). \qquad (29)$$

The solution of problem (27), (28) is written as

$$\begin{aligned} v = &-2aI_0\Big(\sqrt{(r-r_0)(s-s_0)/(2a)}\Big)\frac{\exp\frac{r_0+s_0}{2a}}{\frac{r_0+s_0}{2}+a} - \\ &-2a\int_{s_0}^{s} I_0\Big(\sqrt{\frac{(r-r_0)(\tau-s_0)}{2a}}\Big)\frac{d}{d\tau}\left(\frac{\exp\frac{r_0+\tau}{2a}}{\frac{r_0+\tau}{2}+a}\right)d\tau + \\ &+8a^2\int_{r_0}^{r} I_0\Big(\sqrt{\frac{(\tau-r_0)(s-s_0)}{2a}}\Big)\frac{d}{d\tau}\left(\frac{\exp\frac{\tau+s_0}{2a}}{\frac{\tau+s_0}{2}-a}\right)d\tau. \end{aligned} \qquad (30)$$

Taking into account (17), (19) and also (22), we obtain from (16) and (18) that

$$2t_N = t_k + t_M + \int_L -Adx + Bdt. \qquad (31)$$

Taking into account (25), (27) and using (30), we obtain from (24) and (26) that

$$2x_N = x_k + x_M + \int_L -Adx + Bdt. \qquad (32)$$

Relations (31) and (32) allow us to find the coordinates of the cross points of the characteristics. This in its turn allows us to construct the characteristics of the Cauchy problem with numerical-and-analytic method.

Conclusion

An infinite series of conservation laws for the equations describing one-dimensional flow of isentropic gas is found. Relations that allow one to find the coordinates of the cross points of the characteristics are obtained on the basis of these laws. The problem is reduced to the calculation of several quadratures.

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Численно-аналитический метод решения задачи Коши одномерной газовой динамики

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Аннотация. В работе рассмотрены уравнения, описывающие течения одномерного баротропного совершенного газа. Для этих уравнений найдены законы сохранения специального вида. С помощью этих законов сохранения задача Коши сведена к нескольким квадратурам по кривой, на которой заданы граничные условия.

Ключевые слова: одномерный баротропный совершенный газ, задача Коши, законы сохранения.