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Estimates for Mittag-Leffler Functions with Smooth Phase Depending on Two Variables

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Abstract. In this paper we consider the problem on estimates for Mittag-Leffler functions with the smooth phase functions of two variables having singularities of type D_∞, $D_4^\pm$ and A_r. The generalisation is that we replace the exponential function with the Mittag–Leffler-type function, to study oscillatory type integrals. We extend results of paper [1] and [2] to two-dimensional integrals with phase having some simple singularities.

Keywords: Mittag–Leffler functions, phase function, amplitude.

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Introduction

The Mittag-Leffler function $E_\alpha(z)$ is named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846–1927) who defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \ Re(\alpha) > 0, \tag{1}$$

and studied its properties in 1902–1905 in five subsequent notes [3–6] in connection with his summation method for divergent series.

A classic generalization of the Mittag–Leffler function, namely the two-parametric Mittag–Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \ Re(\alpha) > 0, \tag{2}$$

which was deeply investigated independently by Humbert and Agarval in 1953 [6–8] and by Dzherbashyan in 1954 [9–11], see also [12] and the references therein.

In this paper we also consider a special case the generalized Mittag–Leffler function defined as in (2) by

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta \in \mathbb{R}.$$

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Obviously,

$$E_{1,1}(x) = e^x. \tag{3}$$

We consider the following integral with phase f and amplitude ψ

$$I_{\alpha,\beta} = \int_a^b E_{\alpha,\beta}(i\lambda f(x))\psi(x)dx, \tag{4}$$

where $0 < \alpha \leqslant 1$, $\beta > 0$ and $\lambda > 0$.

If $\alpha = \beta = 1$ in the integral (4) the integral $I_{1,1}$ is called an oscillatory integral. In harmonic analysis estimates, the most important estimates for oscillatory integrals is van der Corput lemma [13]. Estimates for oscillatory integrals with polynomial phase can be bound, for instance, in papers [14–19]. In the this paper we replace the exponential function with the Mittag–Leffler-type function and study oscillatory type integrals (4). In the papers [1] and [2] analogues of the van der Corput lemmas involving Mittag–Leffler functions for one dimensional integrals have been considered. We extend results of [1] and [2] for two-dimensional integrals with phase having some simple singularities.

The main result of the paper is the following

Theorem 1. *Let $-\infty < a < b < \infty$. Assume that the phase function is a homogenous polynomial of third degree in two variables and let $\psi \in L^p[a,b]^2$, $1 < p \leqslant \infty$. Then for any $\alpha \in (0,1)$, $\beta, \lambda \in (0,+\infty)$*

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{2}-\frac{1}{2p}}}, \tag{5}$$

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda (x_1^2 x_2 \pm x_2^3))\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}, \tag{6}$$

$$\left| \int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^3)\psi(x)dx \right| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}, \tag{7}$$

where the constant C depends only on p.

1. Some auxiliary statements

First we give auxiliary statements. Let us consider a homogeneous polynomial of third degree in two variables.

Proposition 1 ([20])**.** *A homogeneous polynomial of third degree in two variables may be reduced by a R-linear transformation to one of the forms: 1) $x_1^2 x_2$, 2) $x_1^2 x_2 \pm x_2^3$, 3) x_1^3, 4) 0.*

Definition 1. *Given $\mu \in (1,\infty]$, a critical point equivalent to the critical point of the function $x_1^2 x_2 \pm x_2^{\mu-1}$ is said to be a critical point of type $D_\mu^\pm$, where $x_2^{\mu-1} \equiv 0$ for $\mu = \infty$.*

Definition 2. *A critical point equivalent to the critical point of the function x_1^{r+1}, $r \geqslant 1$ is said to be a critical point of type A_r.*

Proposition 2 ([21])**.** *If $0 < \alpha < 2$, β is an arbitrary real number and μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, then there is $C > 0$ such that*

$$|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \ z \in \mathbb{C}, \ \mu \leqslant |\arg(z)| \leqslant \pi. \tag{8}$$

Proposition 3 ([1])**.** *Let $\alpha, \beta > 0$ and $f : [a,b] \to \mathbb{C}$. Then for all $\lambda \in \mathbb{C}$*

$$E_{\alpha,\beta}(i\lambda f(x)) = E_{2\alpha,\beta}(-\lambda^2 f^2(x)) + i\lambda f(x) E_{2\alpha,\beta+\alpha}(-\lambda^2 f^2(x)). \tag{9}$$

2. Proof of the main results

Proof of Theorem 1. Since for small λ the integral (4) is clearly bounded, we prove Theorem 1 only for $\lambda \geqslant 1$. Without loss of generality, we can consider the integral on $[0,1]^2$, otherwise we reduce to this case using a linear transformation. Since we are given a homogeneous polynomial of third degree in two variables, by Proposition 1 we can represent it as one of the following: 1) $x_1^2x_2$; 2) $x_1^2x_2 \pm x_2^3$; 3) x_1^3 4) 0. If the phase function is $f(x) \equiv 0$ it is clear that integral will be identically zero. So we will consider the other three cases separately.

Using the inequalities (8) and (9) we obtain:

$$|E_{\alpha,\beta}(i\lambda f(x))| \leqslant |E_{2\alpha,\beta}(-\lambda^2 f^2(x))| + \lambda|f(x)|\,|E_{2\alpha,\beta+\alpha}(-\lambda^2 f^2(x))| \leqslant$$

$$\leqslant \frac{C}{1+\lambda^2 f^2(x)} + \frac{C\lambda|f(x)|}{1+\lambda^2 f^2(x)} \leqslant \frac{C(1+\lambda|f(x)|)}{1+\lambda^2 f^2(x)} \leqslant \frac{C}{1+\lambda|f(x)|}. \tag{10}$$

Case I. First we assume that the phase function has a critical point of type D_∞ so that $f(x) = x_1^2x_2$.

We consider the integral (4) of the form:

$$I_{\alpha,\beta} = \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda x_1^2x_2)\psi(x)dx. \tag{11}$$

We use the inequality (10) in the integral (11) and we obtain:

$$|I_{\alpha,\beta}| = \left|\int_{[0,1]^2} E_{\alpha,\beta}(i\lambda x_1^2x_2)\psi(x)dx\right| \leqslant \int_{[0,1]^2} \left|E_{\alpha,\beta}(i\lambda x_1^2x_2)\right| |\psi(x)|\,dx \leqslant$$
$$\leqslant C\int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_2}{1+\lambda x_1^2x_2}. \tag{12}$$

Let q be such that $\frac{1}{p}+\frac{1}{q}=1$. Assume first that $p \neq \infty$, so that $q>1$. Then using the Hölder inequality in the inner integral we get

$$J_{in1} := \int_0^1 \frac{|\psi(x)|dx_2}{1+\lambda x_1^2x_2} \leqslant \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\int_0^1 \frac{dx_2}{|1+\lambda x_1^2x_2|^q}\right)^{\frac{1}{q}} =$$
$$= \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^{\frac{1}{q}}.$$

Thus,

$$|I_{\alpha,\beta}| \leqslant C\int_0^1 \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\right)^{\frac{1}{q}} dx_1.$$

Then using again the Hölder inequality in this integral we obtain

$$|I_{\alpha,\beta}| \leqslant C\left(\int_0^1\int_0^1 |\psi(x)|^p dx_2 dx_1\right)^{\frac{1}{p}} \left(\int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\,dx_1\right)^{\frac{1}{q}} \leqslant$$
$$\leqslant C\|\psi\|_{L^p} \left(\int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\,dx_1\right)^{\frac{1}{q}}.$$

Let

$$K := \int_0^1 \frac{1-(1+\lambda x_1^2)^{1-q}}{(q-1)\lambda x_1^2}\, dx_1.$$

Since $(1+\lambda x_1^2)^{1-q} = 1 + O(\lambda x_1^2)$ near $x_1 = 0$ and $q > 1$, the integral K is convergent. To estimate K, first we use the change of variables $t = \sqrt{\lambda} x_1$ to get

$$\begin{aligned} K =& \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{1-(1+t^2)^{1-q}}{t^2} dt = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt = \\ =& \frac{1}{(q-1)\sqrt{\lambda}} \int_0^1 \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt + \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt =: K_1 + K_2. \end{aligned}$$

Since $q-1 \leqslant [q]$, where $[q] \geqslant 1$ is the integer part of $q > 1$, by the Newton binomial formula

$$(1+t^2)^{q-1} \leqslant (1+t^2)^{[q]} = 1 + [q]t^2 + \frac{[q]([q]-1)}{2} t^4 + \ldots + t^{2[q]},$$

and hence

$$K_1 = \frac{1}{(q-1)\sqrt{\lambda}} \int_0^1 \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt \leqslant \frac{C_q}{\sqrt{\lambda}},$$

where

$$C_q := \frac{1}{q-1} \int_0^1 \frac{[q] + \frac{[q]([q]-1)}{2} t^2 + \ldots + t^{2[q]-2}}{(1+t^2)^{q-1}} dt\,.$$

Moreover, since $\dfrac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} < \dfrac{1}{t^2}$,

$$\begin{aligned} K_2 =& \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{(1+t^2)^{q-1}-1}{t^2(1+t^2)^{q-1}} dt < \frac{1}{(q-1)\sqrt{\lambda}} \int_1^{\sqrt{\lambda}} \frac{1}{t^2} dt = \\ =& \frac{1}{(q-1)\sqrt{\lambda}} \Big(1 - \frac{1}{\sqrt{\lambda}}\Big) < \frac{1}{(q-1)\sqrt{\lambda}}\,. \end{aligned}$$

Hence,

$$K \leqslant \frac{C_q'}{\sqrt{\lambda}}, \qquad C_q' := C_q + \frac{1}{q-1}\,,$$

and

$$|I_{\alpha,\beta}| \leqslant \frac{C_q'' \|\psi\|_{L^p}}{\lambda^{\frac{1}{2q}}},$$

where C_q'' is some coefficient depending only on q, and hence only on p.

Now we consider the case $q = 1$. Notice that the coefficient $C_q'' \to +\infty$ as $q \to 1$ and therefore we cannot directly conclude the required estimate from the one for $q > 1$. As $q = 1$, we have $p = \infty$ and $\psi \in \mathbb{L}^\infty$. In view of (12), first we estimate the inner integral as

$$|J_{in1}| = \int_0^1 \frac{|\psi(x)| dx_2}{1+\lambda x_1^2 x_2} \leqslant \sup_{x_2 \in [0,1]} |\psi(x)| \int_0^1 \frac{dx_2}{1+\lambda x_1^2 x_2} \leqslant$$

$$\leqslant \frac{\sup\limits_{x_2\in[0,1]} |\psi(x)|}{\lambda x_1^2} \ln(1+\lambda x_1^2 x_2)\Bigg|_0^1 = \frac{\sup\limits_{x_2\in[0,1]} |\psi(x)| \ln(1+\lambda x_1^2)}{\lambda x_1^2}.$$

Thus

$$|I_{\alpha,\beta}| \leqslant \int_0^1 \frac{\sup\limits_{x_2\in[0,1]} |\psi(x)| \ln(1+\lambda x_1^2)}{\lambda x_1^2} dx_1 \leqslant C\|\psi\|_{L^\infty} \int_0^1 \frac{\ln(1+\lambda x_1^2)}{\lambda x_1^2} dx_1.$$

We use the change of variables $\lambda x_1^2 = y$ in the last integral and get

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}} \int_0^\lambda \frac{\ln(1+y)}{y^{\frac{3}{2}}} dy \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}} \int_0^\infty \frac{\ln(1+y)}{y^{\frac{3}{2}}} dy. \tag{13}$$

Note that the last integral converges. Now, using integration by parts we obtain

$$\int_0^\infty \frac{\ln(1+y)}{y^{\frac{3}{2}}} dy = -\lim_{N_1\to 0, N_2\to\infty} \frac{2\ln(1+y)}{y^{\frac{1}{2}}}\Big|_{N_1}^{N_2} + \int_0^\infty \frac{2dy}{(1+y)y^{\frac{1}{2}}} =$$

$$= \int_0^\infty \frac{4dy^{\frac{1}{2}}}{1+y} = 4\arctan y|_0^\infty = 2\pi.$$

Thus from (13) we get

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^\infty}}{\lambda^{\frac{1}{2}}}.$$

Case II. Assume that the phase function has a critical point of type $D_4^\pm$ so that $f(x) = x_1^2 x_2 \pm x_2^3$. We estimate the integral (4) when the phase function has a critical point of type D_4^+ and the case D_4^- can be done similarly.

We consider the integral

$$I_{\alpha,\beta} = \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3))\psi(x)dx. \tag{14}$$

Using the inequality (10) for the integral (14) we get

$$|I_{\alpha,\beta}| = \left| \int_{[0,1]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3))\psi(x)dx \right| \leqslant \int_{[0,1]^2} \left|E_{\alpha,\beta}(i\lambda(x_1^2 x_2 + x_2^3))\right| |\psi(x)|\, dx \leqslant$$

$$\leqslant \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda(x_1^2 x_2 + x_2^3)} = \int_0^1 dx_1 \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda x_2^3 + \lambda x_1^2 x_2}.$$

We use the Hölder inequality for the last inner integral and obtain

$$|J_{in2}| := \int_0^1 \frac{|\psi(x)|dx_1}{|1+\lambda x_2^3 + \lambda x_2 x_1^2|} \leqslant \left(\int_0^1 |\psi(x)|^p dx_2\right)^{\frac{1}{p}} \left(\int_0^1 \frac{dx_1}{|1+\lambda x_2^3 + \lambda x_2 x_1^2|^q}\right)^{\frac{1}{q}}.$$

Then using again the Hölder inequality for this integral we establish

$$|I_{\alpha,\beta}| \leqslant \left(\int_0^1\int_0^1 |\psi(x)|^p dx_2 dx_1\right)^{\frac{1}{p}} \left(\int_0^1\int_0^1 \frac{dx_1}{|1+\lambda x_2^3 + \lambda x_2 x_1^2|^q} dx_2\right)^{\frac{1}{q}}.$$

Changing the variables $x_1 = \left(\frac{1+\lambda x_2^3}{\lambda x_2}\right)^{\frac{1}{2}} t$ we get

$$|I_{\alpha,\beta}| \leqslant \|\psi\|_{L^p} \left(\int_0^1\int_0^1 \frac{dx_1 dx_2}{|1+\lambda x_2^3 + \lambda x_2 x_1^2|^q}\right)^{\frac{1}{q}} =$$

$$= \|\psi\|_{L^p} \left(\int_0^1 \frac{(1+\lambda x_2^3)^{\frac{1}{2}-q}}{(\lambda x_2)^{\frac{1}{2}}} dx_2 \int_0^A \frac{dt}{(1+t^2)^q}\right)^{\frac{1}{q}},$$

where $A = \left(\dfrac{\lambda x_2}{1+\lambda x_2^3}\right)^{\frac{1}{2}}$ and $\int\limits_0^A \dfrac{dt}{(1+t^2)^q} < C$ as $A \to \infty$. Thus,

$$|I_{\alpha,\beta}| \leqslant C\|\psi\|_{L^p}\left(\int_0^1 \frac{(1+\lambda x_2^3)^{\frac{1}{2}-q}}{(\lambda x_2)^{\frac{1}{2}}} dx_2\right)^{\frac{1}{q}}.$$

Replacing x_2 by $\lambda^{-\frac{1}{3}}\tau$ and using $\dfrac{1}{q} = 1 - \dfrac{1}{p}$ we get

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}\left(\int_0^{\lambda^{\frac{1}{3}}} \frac{d\tau}{\tau^{\frac{1}{2}}(\tau^3+1)^{q-\frac{1}{2}}}\right)^{\frac{1}{q}} \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}\left(\int_0^{\infty} \frac{d\tau}{\tau^{\frac{1}{2}}(\tau^3+1)^{q-\frac{1}{2}}}\right)^{\frac{1}{q}}.$$

Since the last integral is covergent,

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}.$$

Case III. Assume that the phase function has a critical point of type A_2 so that $f(x) = x_1^3$. We estimate the integral (4) with the phase function $f(x) = x_1^3$

$$|I_{\alpha,\beta}| \leqslant \int_0^1\int_0^1 |E_{\alpha,\beta}(i\lambda x_1^3)||\psi(x)|dx_1dx_2.$$

First, we use the inequality (10) for the last inner integral to obtain

$$|J_{in3}| := \int_0^1 \frac{|\psi(x)|dx_1}{1+\lambda x_1^3}.$$

Then we use the Hölder inequality for the last integral $I_{\alpha,\beta}$ twice and we get:

$$|I_{\alpha,\beta}| \leqslant \left(\int_0^1\int_0^1 |\psi(x)|^p dx_1dx_2\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \frac{dx_1}{|1+\lambda x_1^3|^q}dx_2\right)^{\frac{1}{q}}.$$

Replacing $\lambda^{-\frac{1}{3}}x_1$ by t in the above inequality, we obtain

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3q}}}\left(\int_0^{\lambda^{\frac{1}{3}}} \frac{dt}{|1+t^3|^q}\right)^{\frac{1}{q}} \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3q}}}\left(\int_0^{\infty} \frac{dt}{|1+t^3|^q}\right)^{\frac{1}{q}}.$$

Since $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and the last integral converges,

$$|I_{\alpha,\beta}| \leqslant \frac{C\|\psi\|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}.$$

The proof is complete.

Remark. If $\alpha = \beta = 1$ in the integral (4), it is called an oscillatory integral and the theorem holds for it.

Declaration of competing interest

This work does not have any conflicts of interest.

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Оценка для функции Миттаг-Леффлера с гладкой фазой, зависящей от двух переменных

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Аннотация. В статье рассматривается задача об оценках функций Миттаг-Леффлера с гладкими фазовыми функциями двух переменных, имеющими особенности типа D_∞, $D_4^\pm$ и A_r. Мы обобщаем результаты статей [1] и [2] на двумерные интегралы с фазой, имеющей некоторые простые особенности. Обобщение состоит в том, что мы заменяем экспоненциальную функцию функцией типа Миттаг–Леффлера для изучения типа осцилляторного интеграла.

Ключевые слова: функция Миттаг–Леффлера, фаза функция, амплитуда.