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Central Limit Theorem for Weakly Dependent Random Variables with Values in $D\,[0,1]$

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Abstract. The main goal of this article is to prove the central limit theorem for sequences of random variables with values in the space $D\,[0,1]$. We assume that the sequence satisfies the mixing conditions. In the paper the central limit theorems for sequences with strong mixing and ρ_m-mixing are proved.

Keywords: central limit theorem, mixing sequence, $D\,[0,1]$ space.

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1. Preliminaries

A central limit theorem for Banach space valued dependent random variables have been studied by many authors (see [6, 11, 15–17] and references therein). It is known that validity of the central limit theorem depends on the geometric structure of Banach space. One of the most difficult space in this sense is $D\,[0,1]$ (the space of all real-valued functions that are right continuous and have left limits, which is endowed with the Skorohod topology) space. In this paper we will prove the central limit theorem for mixing random variables with values in $D\,[0,1]$.

Let $\{X_n(t),\, t \in [0,1]\,,\, n \geqslant 1\}$ be a sequence of random variables with values in $D\,[0,1]$. We say that $\{X_n(t),\, t \in [0,1]\,,\, n \geqslant 1\}$ satisfies a central limit theorem if the distribution of $\frac{1}{\sqrt{n}}\,(X_1(t) + \ldots + X_n(t))$ converges weakly to a Gaussian distribution in $D\,[0,1]$.

The central limit theorem in $D\,[0,1]$ is very important from applications point of view. It immediately implies asymptotic normality of empirical and weighted empirical processes. The central limit theorem for the sequence of independent identically distributed (i.i.d) random variables with values in $D\,[0,1]$ were studied by many authors (see [1–3,8,12]) and references therein).

The first central limit theorem was proved by Hahn [8]. Later the central limit theorem in $D\,[0,1]$ was proved by D. Juknevičienė (1985), V. Paulauskas and Ch. Stive (1990), P.H. Bezandry

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and X. Fernique (1992), M. Bloznelis and V. Paulauskas (1993), X. Fernique (1994). The result of M.G. Hahn [8] can be formulated as follows.

Theorem 1.1 ([8])**.** *Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad EX_1^2(t) < \infty \quad \text{for all} \quad t \in [0,1]. \tag{1}$$

Assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha > 0.5$, $\beta > 1$ such that for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$ the following two conditions hold:

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{\alpha},$$

$$E\left(X_1(u) - X_1(t)\right)^2 \left(X_1(t) - X_1(s)\right)^2 \leqslant \left(F(u) - F(s)\right)^{\beta}. \tag{2}$$

Then $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies the central limit theorem in $D[0,1]$ and the limiting Gaussian process is sample continuous.

As it was already noticed in [2], the condition (2) is connected with the fourth moments of the process $X_1(t)$. This conditions does not allow us to apply Theorem 1.1 to a wide class of weighted empirical processes. In [2] and [3] authors obtained the following results (where $a \wedge b$ denotes $\min(a,b)$):

Theorem 1.2 ([2])**.** *Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ satisfying the condition (1) and assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha, \beta > 0$ such that for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$ the following two conditions hold:*

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{1/2} \log^{-4,5-\alpha}\left(1 + \left(G(u) - G(t)\right)^{-1}\right), \tag{3}$$

$$\begin{aligned} &E\left(|X_1(t) - X_1(s)| \wedge 1\right)^2 \left(X_1(u) - X_1(t)\right)^2 \leqslant \\ &\leqslant \left(F(u) - F(s)\right) \log^{-5-\beta}\left(1 + \left(F(u) - F(s)\right)^{-1}\right). \end{aligned} \tag{4}$$

Then $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies the central limit theorem in $D[0,1]$ and the limiting Gaussian process is sample continuous.

Theorem 1.3 ([2])**.** *The statement of Theorem 1.2 remains true if conditions (3) and (4) are replaced by*

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{1/2} \log^{-2,5-\alpha}\left(1 + \left(G(u) - G(t)\right)^{-1}\right), \tag{5}$$

$$E\left(X_1(t) - X_1(s)\right)^2 \left(X_1(u) - X_1(t)\right)^2 \leqslant \left(F(u) - F(s)\right) \log^{-5-\beta}\left(1 + \left(F(u) - F(s)\right)^{-1}\right). \tag{6}$$

Theorem 1.4 ([3])**.** *Assume $p, q \geqslant 2$. Let f, g be nonnegative functions on $[0, +\infty)$ which are nondecreasing near 0 and let F, G be increasing continuous functions on $[0,1]$. Let $X(t)$ be a random process with mean 0, finite second moment, and sample path in D satisfying*

$$E\left(|X(s) - X(t)| \wedge |X(t) - X(u)|\right)^p \leqslant f\left(F(u) - F(s)\right),$$

$$E\left|X(s) - X(t)\right|^q \leqslant g\left(G(t) - G(s)\right),$$

for $0 \leqslant s \leqslant t \leqslant u \leqslant 1$, $u - s$ small and

$$\int_0 f^{1/p}(u) \cdot u^{-1-1/p} du < \infty, \qquad \int_0 g^{1/q}(u) \cdot u^{-1-1/(2q)} du < \infty.$$

Then $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies the central limit theorem in $D[0,1]$ and the limiting Gaussian process is sample continuous.

2. Main results

The main goal of this article is to prove the central limit theorem for mixing sequences of random variables with values in space $D[0,1]$.

Below, we give the definitions of mixing coefficients for a sequence of random variables with values in a separable Banach space **B**. In Definition 2 it is assumed that **B** is an infinite-dimensional space.

Definition 1. *For a sequence $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ the coefficients of ρ, α-mixing are defined by the following equalities.*

$$\rho(n) = \sup \left\{ \frac{|E(\xi - E\xi)(\eta - E\eta)|}{E^{\frac{1}{2}}(\xi - E\xi)^2 E^{\frac{1}{2}}(\eta - E\eta)^2} : \xi \in L_2(F_1^k),\ \eta \in L_2(F_{n+k}^{\infty}),\ k \in N \right\}.$$

$$\alpha(n) = \sup \left\{ |P(AB) - P(A)P(B)| : A \in F_1^k,\ B \in F_{k+n}^{\infty},\ k \in N \right\}.$$

where F_a^b is the σ-algebra generated by random processes $X_a(t), \ldots, X_b(t)$ and $L_2(F_a^b)$ is the space of all square integrable random variables measurable with respect to F_a^b.

Definition 2. *For the sequence $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ the coefficients of $\rho_m(n)$-mixing and $\alpha_m(n)$-mixing are defined by the following equalities*

$$\rho_m(n) = \sup_{\boldsymbol{R}^m} \sup \left\{ \frac{|E(\xi - E\xi)(\eta - E\eta)|}{E^{\frac{1}{2}}(\xi - E\xi)^2 E^{\frac{1}{2}}(\eta - E\eta)^2} : \xi \in L_2(F_1^k(\boldsymbol{R}^m)),\ \eta \in L_2(F_{n+k}^{\infty}(\boldsymbol{R}^m)),\ k \in N \right\}.$$

$$\alpha_m(n) = \sup_{\boldsymbol{R}^m} \sup \left\{ |\, P(AB) - P(A)P(B)| : A \in F_1^k(\boldsymbol{R}^m),\ B \in F_{k+n}^{\infty}(\boldsymbol{R}^m),\ k \in N \right\}.$$

where $F_a^b(\boldsymbol{R}^m)$ is the σ-algebra generated by random processes $\prod_m X_a(t), \ldots, \prod_m X_b(t)$ and $\prod_m$ is a projection operator $\boldsymbol{B}$ in m-dimensional subspace $\boldsymbol{R}^m$ i.e. $\prod_m : \boldsymbol{B} \to \boldsymbol{R}^m$. A sequence is called ρ-mixing (or ρ_m-, $\alpha-$, α_m- mixing) if

$$\rho(k) \to 0 \quad as \quad k \to \infty, \tag{7}$$

$$\rho_m(k) \to 0 \quad as \quad k \to \infty \quad and \quad m = 1, 2, \ldots, \tag{8}$$

$$\alpha(k) \to 0 \quad as \quad k \to \infty, \tag{9}$$

$$\alpha_m(k) \to 0 \quad as \quad k \to \infty \quad and \quad m = 1, 2, \ldots \tag{10}$$

respectively.

As the example given in Zhurbenko [13] shows, in general (8) does not imply (7), though (7) always implies (8), the same is true with (9) and (10). In (8) and (10) it is actually required that all finite-dimensional projections of the sequence $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfy the mixing condition and these conditions are weaker than the conditions (7) and (9).

Set $S_n(t) = \frac{1}{\sqrt{n}}(X_1(t) + \cdots + X_n(t))$ and in what follows $\Rightarrow$ denotes weak convergence.

Now we formulate our theorems.

Theorem 2.1. *Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a strictly stationary sequence of ρ_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E|X_1(t)|^2 < \infty \quad for\ all \quad t \in [0,1].$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and $\varepsilon > 0$ the following hold:

$$E\left(X_1(t) - X_1(s)\right)^2 \leqslant (F(t) - F(s)) \log^{-(3+\varepsilon)}\left(1 + (F(t) - F(s))^{-1}\right), \tag{11}$$

$$\lim_{n\to\infty} E\left(X_1 + \cdots + X_n\right)^2 = \infty \quad \textit{for all} \quad t \in [0,1],$$

$$\sum_{k=1}^{n} \rho_m\left(2^k\right) < \infty, \quad m = 1, 2, \ldots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ satisfies the central limit theorem i.e.

$$S_n(t) \Rightarrow N(t) \quad \textit{as} \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F\left(t_1, t_2\right) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorem 2.2. *Let $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ be a strictly stationary sequence of ρ_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E\left|X_1(t)\right|^{2+\varepsilon} < \infty, \quad \textit{for all} \quad t \in [0,1] \quad \textit{and some} \quad \varepsilon > 0.$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and the following hold:

$$E\left|X_1(s) - X_1(t)\right|^{2+\varepsilon} \leqslant (F(s) - F(t)) \log^{-(3+2\varepsilon)}\left(1 + (F(s) - F(t))^{-1}\right), \tag{12}$$

$$\sum_{k=1}^{n} \rho_m^{\frac{2}{2+\varepsilon}}(2^k) < \infty, \quad m = 1, 2, \ldots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ satisfies the central limit theorem i.e.

$$S_n(t) \Rightarrow N(t) \quad \textit{as} \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F\left(t_1, t_2\right) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorem 2.3. *Let $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ be a strictly stationary sequence of α_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E\left|X_1(t)\right|^{2+\delta} < \infty, \quad \textit{for all} \quad t \in [0,1] \quad \textit{and some} \quad \delta > 0.$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and $\varepsilon > 0$ the following hold:

$$\left(E\left|X_1(t) - X_1(s)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}} \leqslant \left(F\left(t\right) - F\left(s\right)\right) \log^{-(3+2\varepsilon)}\left(1 + \left(F\left(t\right) - F\left(s\right)\right)^{-1}\right), \tag{13}$$

$$\sum_{k=1}^{n} \alpha_m^{\frac{\delta}{2+\delta}}\left(k\right) < \infty, \quad m = 1, 2, \ldots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ *satisfies the central limit theorem i.e.*

$$S_n(t) \Rightarrow N(t) \quad as \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F(t_1, t_2) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorems 2.1–2.2 improve the results of [11].

3. Preliminary results

The proofs of the theorems are based on the following lemmas.

Lemma 1 ([2]). *Let* $X_1(t), X_2(t), \ldots, X_n(t), \ldots$ *be random variables with values in* $D[0,1]$. *Assume that there exist a nondecreasing continuous function* F *on* $[0,1]$ *and positive numbers* $\gamma_1,\ c_1,\ \varepsilon_1$ *such that for all* $\lambda > 0$ *and* $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.

$$P\left(|X_n(t) - X_n(s)| \wedge |X_n(u) - X_n(t)| \geqslant \lambda\right) \leqslant c_1 \lambda^{-2\gamma_1} g_{2\gamma_1+1+\varepsilon_1}\left(F(u) - F(s)\right),$$

where $g_p(u) = u\,|\log u|^{-p},\ p > 0.$ *If for all* $t_1, \ldots, t_k \in [0,1],\ \ k = 1, 2, \ldots$

$$(X_n(t_1), \ldots, X_n(t_k)) \Rightarrow (X(t_1), \ldots, X(t_k)) \quad as \quad n \to \infty$$

and

$$P(X(1) = \lim_{t\to 1} X(t)) = 1.$$

Then $X_n \Rightarrow X$ *as* $n \to \infty$.

Lemma 2 ([9]). *Let* $\{X_i,\ i \geqslant 1\}$ *be a strictly stationary sequence of real valued random variables with* ρ*-mixing and*

$$EX_1 = 0, \quad EX_1^2 < \infty,$$
$$\lim_{n\to\infty} E\left(X_1 + \cdots + X_n\right)^2 = \infty,$$
$$\sum_{k=1}^{n} \rho(2^k) < \infty.$$

Then

$$\frac{1}{\sqrt{n}}\left(X_1 + \cdots + X_n\right) \Rightarrow N\left(0, \sigma^2\right) \quad as \ \ n \to \infty,$$

where $N\left(0,\sigma^2\right)$ *Gaussion random variable with zero-mean and variance*

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} E\left(X_1 + \cdots + X_n\right)^2 > 0.$$

Lemma 3 ([14]). *Let* $\{X_i,\ i \geqslant 1\}$ *be a sequence of real-valued random variables with* ρ*-mixing and for some* $q \geqslant 2$

$$EX_1 = 0, \quad E|X_1|^q < \infty,$$
$$\sum_{k=1}^{n} \rho^{\frac{2}{q}}(2^k) < \infty.$$

Then there exists a constant K *such that the following inequality holds:*

$$E|X_1 + \cdots + X_n|^q \leqslant K\left(n^{\frac{q}{2}} \max_{1\leqslant i\leqslant n}\left(E|X_i|^2\right)^{\frac{q}{2}} + n \max_{1\leqslant i\leqslant n} E|X_i|^q\right).$$

Lemma 4 ([13]). *Let $\{X_i,\, i \geqslant 1\}$ be a stationary sequence of random variables with α-mixing and*

$$EX_1 = 0, \quad E\,|X_1|^{2+\delta} < \infty,$$

$$\sum_{k=1}^{\infty} \alpha^{\frac{\delta}{2+\delta}}(k) < \infty,$$

for some $\delta > 0$. Then

$$\sigma^2 = EX_1^2 + 2\sum_{j=2}^{\infty} E(X_1 X_j) < \infty \quad when \quad \sigma^2 > 0,$$

$$\frac{1}{\sigma\sqrt{n}}\,(X_1 + \cdots + X_n) \Rightarrow N\,(0,1) \quad as \quad n \to \infty.$$

Lemma 5 ([4]). *Let $\{X_i,\, i \geqslant 1\}$ be a strictly stationary sequence of random variables with α-mixing and*

$$EX_1 = 0, \quad E\,|X_1|^{2+\delta} < \infty,$$

$$\sum_{k=1}^{\infty} n^{\frac{t}{2}-1} \alpha^{\frac{2+\delta-t}{2+\delta}}(k) < \infty,$$

for some $0 < \delta \leqslant \infty$ and $2 \leqslant t < 2+\delta$. Then

$$E\left|\sum_{k=1}^{n}(X_k - \mu)\right|^t \leqslant C n^{\frac{t}{2}} \left(E\,|X_1|^{2+\delta}\right)^{\frac{t}{2+\delta}}.$$

4. Proof of Theorems

Proof of Theorem 2.1. We will use the method developed in the papers [2,8] and [12]. It follows from Lemma 1 that it suffices to prove

$$P\left(|S_n(t) - S_n(s)| \wedge |S_n(u) - S_n(t)| \geqslant \lambda\right) \leqslant c_1 \lambda^{-(2+\varepsilon)} g_{3+\varepsilon}\left(F(u) - F(s)\right),$$

where $\lambda \in (0,1]$, $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.

It is easy to see that the following inequality holds for $\lambda \in (0,1]$.

$$P\left(|S_n(t) - S_n(s)| \wedge |S_n(u) - S_n(t)| \geqslant \lambda\right) \leqslant P\left(|S_n(t) - S_n(s)|\,|S_n(u) - S_n(t)| \geqslant \lambda^2\right).$$

We have

$$J = |S_n(t) - S_n(s)|\,|S_n(u) - S_n(t)| =$$

$$= \left(\left|n^{-\frac{1}{2}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|\right)\left(\left|n^{-\frac{1}{2}}\sum_{k=1}^{n}(X_k(u) - X_k(t))\right|\right) \leqslant$$

$$\leqslant \frac{1}{2}\left(n^{-\frac{1}{2}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right)^2 + \frac{1}{2}\left(n^{-\frac{1}{2}}\sum_{k=1}^{n}(X_k(u) - X_k(t))\right)^2 = J_1 + J_2.$$

In what follows we denote by C the constants (possibly depending on different parameters) which can be different even in the same chain of inequalities.

We have
$$P\left(J \geqslant \lambda^2\right) \leqslant P\left(J_1 \geqslant \frac{1}{2}\lambda^2\right) + P\left(J_2 \geqslant \frac{1}{2}\lambda^2\right).$$
We evaluate each of the summands individually. Using the Markov inequality and Lemma 3, we obtain
$$\begin{aligned} P\left(J_1 \geqslant \lambda^2\right) &= P\left(\left(n^{-\frac{1}{2}}\sum_{k=1}^{n}\left(X_k\left(t\right) - X_k\left(s\right)\right)\right)^2 \geqslant \lambda^2\right) \leqslant \\ &\leqslant \lambda^{-2}E\left(n^{-\frac{1}{2}}\sum_{k=1}^{n}\left(X_k\left(t\right) - X_k\left(s\right)\right)\right)^2 \leqslant \lambda^{-2}CE\left(X_1\left(t\right) - X_1\left(s\right)\right)^2, \end{aligned} \tag{14}$$
$$P\left(J_2 \geqslant \lambda^2\right) \leqslant \lambda^{-2}CE\left(X_1\left(u\right) - X_1\left(t\right)\right)^2. \tag{15}$$
From (14) and (15) we get
$$P\left(J \geqslant \lambda^2\right) \leqslant \lambda^{-2}CE\left(X_1\left(t\right) - X_1\left(s\right)\right)^2 + \lambda^{-2}CE\left(X_1\left(u\right) - X_1\left(t\right)\right)^2.$$
From the conditions of Theorem 2.1
$$\begin{aligned} &P\left(\left|S_n\left(t\right) - S_n\left(s\right)\right|\left|S_n\left(u\right) - S_n\left(t\right)\right| \geqslant \lambda^2\right) \leqslant \\ &\leqslant \lambda^{-2}CE\left(X_1\left(t\right) - X_1\left(s\right)\right)^2 + \lambda^{-2}CE\left(X_1\left(u\right) - X_1\left(t\right)\right)^2 \leqslant \\ &\leqslant \lambda^{-2}C\left(F(t) - F(s)\right)\log^{-(3+\varepsilon)}\left(1 + \left(F(t) - F(s)\right)^{-1}\right) + \\ &+\lambda^{-2}C\left(F(u) - F(t)\right)\log^{-(3+\varepsilon)}\left(1 + \left(F(u) - F(t)\right)^{-1}\right) \leqslant \\ &\leqslant 2\lambda^{-2}C\left(F(u) - F(s)\right)\log^{-(3+\varepsilon)}\left(1 + \left(F(u) - F(s)\right)^{-1}\right) \leqslant 2C\lambda^{-2}g_{3+\varepsilon}\left(F(u) - F(s)\right). \end{aligned}$$
Above we used the inequality
$$log^{-1}\left(1 + \left(F(u) - F(s)\right)^{-1}\right) \leqslant 2\left|\log\left(F(u) - F(s)\right)\right|^{-1} \tag{16}$$
for
$$F(u) - F(s) \leqslant 0.25.$$

Now, to complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n\left(t\right)$. The convergence of finite-dimensional distributions follows from Lemma 2 and the Cramer-Wold device [5]. Thus, Theorems 2.1 is proved. □

Proof of Theorem 2.2.

We will prove Theorem 2.2 by the same method as Theorem 2.1. It follows from Lemma 1 that it suffices to prove
$$P\left(\left|S_n\left(t\right) - S_n\left(s\right)\right| \wedge \left|S_n\left(u\right) - S_n\left(t\right)\right| \geqslant \lambda\right) \leqslant c_1\lambda^{-(2+\varepsilon)}g_{3+2\varepsilon}\left(F\left(u\right) - F\left(s\right)\right),$$
where $\lambda \in (0,1]$, $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.

It is easy to see that the following inequality holds for $\lambda \in (0,1]$.
$$P\Big(\left|S_n\left(t\right) - S_n\left(s\right)\right| \wedge \left|S_n\left(u\right) - S_n\left(t\right)\right| \geqslant \lambda\Big) \leqslant P\Big(\left|S_n\left(t\right) - S_n\left(s\right)\right|^{\frac{2+\varepsilon}{2}}\left|S_n\left(u\right) - S_n\left(t\right)\right|^{\frac{2+\varepsilon}{2}} \geqslant \lambda^{2+\varepsilon}\Big).$$

We have

$$I = |S_n(t) - S_n(s)|^{\frac{2+\varepsilon}{2}} |S_n(u) - S_n(t)|^{\frac{2+\varepsilon}{2}} =$$

$$= \left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|^{\frac{2+\varepsilon}{2}} \left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(u) - X_k(t))\right|^{\frac{2+\varepsilon}{2}} \leqslant$$

$$\leqslant \frac{1}{2}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|^{2+\varepsilon} + \frac{1}{2}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(u) - X_k(t))\right|^{2+\varepsilon} = I_1 + I_2.$$

We have

$$P\left(I \geqslant \lambda^{2+\varepsilon}\right) \leqslant P\left(I_1 \geqslant \frac{1}{2}\lambda^{2+\varepsilon}\right) + P\left(I_2 \geqslant \frac{1}{2}\lambda^{2+\varepsilon}\right).$$

Using the Markov inequality and Lemma 3, we obtain

$$P\left(I_1 \geqslant \lambda^{2+\varepsilon}\right) = P\left(\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|^{2+\varepsilon} \geqslant \lambda^{2+\varepsilon}\right) \leqslant$$

$$\leqslant \lambda^{-(2+\varepsilon)} E\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|^{2+\varepsilon} \leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)} n^{-(2+\varepsilon)/2} n^{(2+\varepsilon)/2} \left(E|X_1(t) - X_1(s)|^2\right)^{(2+\varepsilon)/2} +$$

$$+ C\lambda^{-(2+\varepsilon)} n^{-(2+\varepsilon)/2} n E|X_1(t) - X_1(s)|^{2+\varepsilon} \leqslant$$

$$\leqslant \lambda^{-(2+\varepsilon)} C\left(E|X_1(t) - X_1(s)|^2\right)^{(2+\varepsilon)/2} + \lambda^{-(2+\varepsilon)} C n^{-\varepsilon/2} E|X_1(t) - X_1(s)|^{2+\varepsilon} \leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)} E|X_1(t) - X_1(s)|^{2+\varepsilon}.$$

$$P\left(I_2 \geqslant \lambda^{2+\varepsilon}\right) \leqslant 2C\lambda^{-(2+\varepsilon)} E|X_1(u) - X_1(t)|^{2+\varepsilon}.$$

From the conditions of Theorem 2.2 and using (16) we have

$$P\left(|S_n(t) - S_n(s)|^{\frac{2+\varepsilon}{2}} |S_n(u) - S_n(t)|^{\frac{2+\varepsilon}{2}} \geqslant \lambda^{2+\varepsilon}\right) \leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)} (F(t) - F(s)) \log^{-(3+2\varepsilon)}\left(1 + (F(t) - F(s))^{-1}\right) +$$

$$+ C\lambda^{-(2+\varepsilon)} (F(u) - F(t)) \log^{-(3+2\varepsilon)}\left(1 + (F(u) - F(t))^{-1}\right) \leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)} (F(u) - F(s)) \log^{-(3+2\varepsilon)}\left(1 + (F(u) - F(s))^{-1}\right) \leqslant 2C\lambda^{-(2+\varepsilon)} g_{3+2\varepsilon}(F(u) - F(s)).$$

To complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n(t)$. The convergence of finite-dimensional distributions follows from Lemma 2 and the Cramer-Wold device [5]. Thus, Theorems 2.2 is proved. □

Proof of Theorem 2.3.

To prove Theorem 2.3, we estimate I as in the proof of Theorem 2.2 by I_1 and I_2. Using the Markov inequality and Lemma 5, we have (where $\varepsilon + \varepsilon_1 = \delta,\ \varepsilon_1 > 0$)

$$P\left(I_1 \geqslant \lambda^{2+\varepsilon}\right) = P\left(\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t) - X_k(s))\right|^{2+\varepsilon} \geqslant \lambda^{2+\varepsilon}\right) \leqslant$$

$$\leqslant C\alpha\left(k\right)\lambda^{-(2+\varepsilon)}\frac{1}{n^{\frac{2+\varepsilon}{2}}}E\left|\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right|^{2+\varepsilon}\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(t)-X_1(s)\right|^{2+\varepsilon+\varepsilon_1}\right)^{\frac{2+\varepsilon}{2+\varepsilon+\varepsilon_1}}\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(t)-X_1(s)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}.$$

$$P\left(I_2\geqslant\lambda^{2+\varepsilon}\right)=C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(u)-X_1(t)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}.$$

$$P\left(I\geqslant\lambda^2\right)\leqslant C\lambda^{-(2+\varepsilon)}\left(\left(E\left|X_1(t)-X_1(s)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}+\left(E\left|X_1(u)-X_1(t)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}\right).$$

From the conditions of Theorem 2.3 and using (16) we have

$$P\left(\left|S_n\left(t\right)-S_n\left(s\right)\right|^{\frac{2+\varepsilon}{2}}\left|S_n\left(u\right)-S_n\left(t\right)\right|^{\frac{2+\varepsilon}{2}}\geqslant\lambda^{2+\varepsilon}\right)\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(F(t)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(t)-F(s)\right)^{-1}\right)+$$

$$+C\lambda^{-(2+\varepsilon)}\left(F(u)-F(t)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(t)\right)^{-1}\right)\leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)}\left(F(u)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(s)\right)^{-1}\right)\leqslant 2C\lambda^{-(2+\varepsilon)}g_{3+2\varepsilon}\left(F(u)-F(s)\right).$$

Again as in the proof of previous theorems, to complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n\left(t\right)$. The convergence of finite-dimensional distributions follows from Lemma 4 and the Cramer–Wold device [5]. Thus, Theorems 2.3 is proved. □

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Центральная предельная теорема для слабо зависимых случайных величин со значениями в $D[0,1]$

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Аннотация. Основной целью настоящей статьи является доказательство центральной предельной теоремы для последовательностей случайных величин со значениями в пространстве $D[0,1]$. Мы предполагаем, что последовательность удовлетворяет условиям перемешивания. В статье доказаны центральные предельные теоремы для последовательностей с сильным перемешиванием и ρ_m-перемешиванием.

Ключевые слова: центральная предельная теорема, последовательность с перемешиванием, пространство $D[0,1]$.