DOI: 10.17516/1997-1397-2022-15-3-398-408
УДК 515

Coincidence Point Results and its Applications in Partially Ordered Metric Spaces

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Received 12.09.2021, received in revised form 29.01.2022, accepted 16.03.2022

Abstract. The purpose of this paper is to establish some common fixed point theorems for f-nondecreasing self-mapping satisfying a certain rational type contraction condition in the frame of a metric spaces endowed with partial order. Also, some consequences of the results in terms of an integral type contractions are presented in the space. Further, the monotone iterative technique has been used to find a unique solution of an integral equation.

Keywords: ordered metric space, rational contraction, compatible mappings, coincidence point, common fixed point.

Citation: N.Seshagiri Rao, K.Kalyani, Tekle Gemechu, Coincidence Point Results and its Applications in Partially Ordered Metric Spaces, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 398–408.
DOI: 10.17516/1997-1397-2022-15-3-398-408.

Introduction

Ever since in fixed point theory and approximation theory, the classical Banach contraction principle [1] plays a vital role to acquire the unique solution of many known results. It is very important and popular tool in various disciplines of mathematics to solve the existing problems in nonlinear analysis. Later, a lot of variety of generalizations of this Banach contraction principle [1] have been taken place in a metrical fixed point theory by improving the underlying contraction condition, some of which are in [2–11]. Thereafter, vigorous research work has been noticed by weakening its hypotheses in various spaces with topological properties such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces etc. Prominent works on the existence and uniqueness of a fixed point in partially ordered

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metric spaces with different contractive conditions have been acquired by several researchers, the readers may refer to [12–21] and the references therein, which generate natural interest to establish usable fixed point theorems.

The concept of coupled fixed point for a certain mapping in ordered metric space was first introduced by Bhaskar and Lakshmikantham [22] and then applied their results to a periodic boundary value problem to obtain the unique solution. While, the theory of coupled coincidence point and common fixed point results was first initiated by Lakshmikantham and Ćirić [23] which generalized and extended the results of [22] by considering the monotone property of a mapping in ordered metric spaces. Some generalized results on fixed point, coupled fixed point and common fixed point under various contractive conditions in different spaces can be found from [24–37]. Recently, Seshagiri Rao et al. [38–42] and Kalyani et al. [43] have investigated some coupled fixed point theorems for the self mappings satisfying generalized rational contractions in partially ordered metric spaces.

The aim of this paper is to present some common fixed point results for a pair of self-mappings satisfying a generalized rational contraction condition in the context of complete partially ordered metric space. These results generalized and extended the results of Harjani et al. [15] and Chandok [30] in the literature. Some consequences of the main result in terms of integral contractions are presented. A numerical example has been provided to support the result obtained. Moreover, an application of the result has been given by taking the integral equation using the monotone iterative technique.

1. Mathematical preliminaries

Definition 1 ([38]). *The triple $(X, d, \preceq)$ is called a partially ordered metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2 ([38]). *If (X, d) is a complete metric space, then the triple $(X, d, \preceq)$ is called complete partially ordered metric space.*

Definition 3 ([38]). *Let $(X, \preceq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be strictly increasing (or strictly decreasing), if $f(x) \prec f(y)$ (or $f(x) \succ f(y)$), for all $x, y \in X$ with $x \prec y$.*

Definition 4 ([42]). *Let $f, T : A \to A$ be two mappings, where $A \neq \emptyset$ subset of X. Then*

(a) f and T are commutative, if $fTx = Tfx$ for all $x \in A$.

(b) f and T are compatible, if for very sequence $\{x_n\}$ with $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0$.

(c) f and T are said to be weakly compatible if they commute only at their coincidence points (i.e., if $fx = Tx$ then $fTx = Tfx$).

(d) T is called monotone f-nondecreasing, if

$$fx \preceq fy \Rightarrow Tx \preceq Ty \ for\, all\ x, y \in X.$$

(e) A is a well ordered set, if every two elements of it are comparable.

(f) a point $x \in A$ is a common fixed (or coincidence) point of f and T, if $fx = Tx = x$ $(or fx = Tx)$.

2. Main results

We begin this section with the following coincidence point theorem.

Theorem 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following contraction condition*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta \left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy), \tag{1}$$

for all $x, y \in X$ for which the distinct fx and fy are comparable and for some $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leqslant \alpha + 2\beta + \gamma < 1$. If there exists certain $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Let $x_0 \in X$ such that $fx_0 \preceq Tx_0$. Since, $T(X) \subseteq f(X)$ then we can choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$, then there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. Similarly by continuing the same procedure, we construct a sequence $\{x_n\} \subseteq X$ such that $fx_{n+1} = Tx_n$ for all $n \geqslant 0$.

Again from the hypothesis, we have $fx_0 \preceq Tx_0 = fx_1$. Since T is monotone f-nondecreasing then we obtain that $Tx_0 \preceq Tx_1$. As by the similar argument, we get $Tx_1 \preceq Tx_2$, since $fx_1 \preceq fx_2$. Continuing the process, we acquire that

$$Tx_0 \preceq Tx_1 \preceq \cdots \preceq Tx_n \preceq Tx_{n+1} \preceq \dots .$$

Case 1. Suppose that $d(Tx_n, Tx_{n+1}) = 0$ for some $n \in \mathbb{N}$, then we have $Tx_{n+1} = Tx_n$. Therefore, $Tx_{n+1} = Tx_n = fx_{n+1}$. Hence, x_{n+1} is a coincidence point of T and f in X and we have the result.

Case 2. Suppose $d(Tx_n, Tx_{n+1}) > 0$ for all $n \geqslant 0$, then from (1), we have

$$d(Tx_{n+1}, Tx_n) \leqslant \alpha \frac{d(fx_{n+1}, Tx_{n+1})\ d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} + \beta \left[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})\right] + \\ + \gamma d(fx_{n+1}, fx_n),$$

which intern implies that

$$d(Tx_{n+1}, Tx_n) \leqslant \alpha d(Tx_n, Tx_{n+1}) + \beta \left[d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})\right] + \gamma d(Tx_n, Tx_{n-1}).$$

Finally, we arrive at

$$d(Tx_{n+1}, Tx_n) \leqslant \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(Tx_n, Tx_{n-1}).$$

Inductively, we get

$$d(Tx_{n+1}, Tx_n) \leqslant \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right)^n d(Tx_1, Tx_0).$$

Let $k = \dfrac{\beta + \gamma}{1 - \alpha - \beta} < 1$, then from the triangular inequality of a metric d for $m \geqslant n$, we have

$$d(Tx_m, Tx_n) \leqslant d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \cdots + d(Tx_{n+1}, Tx_n) \leqslant \\ \leqslant \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) d(Tx_1, Tx_0) \leqslant \frac{k^n}{1 - k} d(Tx_1, Tx_0),$$

as $m, n \to +\infty$, $d(Tx_m, Tx_n) \to 0$, which shows that the sequence $\{Tx_n\}$ is a Cauchy sequence in X. Therefore from the completeness of X, there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to +\infty$. Further the continuity of T implies that

$$\lim_{n\to+\infty} T(Tx_n) = T\left(\lim_{n\to+\infty} Tx_n\right) = T\mu.$$

Since $fx_{n+1} = Tx_n$ and then $fx_{n+1} \to \mu$ as $n \to +\infty$. Furthermore, from the compatibility of the mappings T and f, we have

$$\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0.$$

By the triangular inequality, we have

$$d(T\mu, f\mu) = d(T\mu, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, f\mu),$$

on taking $n \to +\infty$ and from the fact that T and f are continuous, we obtain that $d(T\mu, f\mu) = 0$. Thus, $T\mu = f\mu$. Hence, μ is a coincidence point of T and f in X. □

We have the following consequences from Theorem 1.

Corollary 1. *Suppose $(X, d, \preceq)$ be a complete partially ordered metric space. Let the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Ty) + d(fy, Tx)\right],$$

for all $x, y \in X$ for which the distinct fx and fy are comparable and where $\alpha, \beta \in [0, 1)$ such that $0 \leqslant \alpha + 2\beta < 1$. If $fx_0 \preceq Tx_0$ for some $x_0 \in X$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. The required proof can be obtained by setting $\gamma = 0$ in Theorem 1. □

Corollary 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \beta\left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy),$$

for all $x, y \in X$ for which fx, fy are comparable and $\beta, \gamma \in [0, 1)$ such that $0 \leqslant 2\beta + \gamma < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Set $\alpha = 0$ in Theorem 1. □

We extract the continuity criteria of T in Theorem 1 is still valid by assuming the following hypotheses in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy), \tag{2}$$

for all $x, y \in X$ for which $fx \neq fy$ are comparable and where $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leqslant \alpha + 2\beta + \gamma < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and $\{x_n\}$ is a nondecreasing *sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X.*

Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Suppose $f(X)$ is a complete subset of X. From Theorem 1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence, $\{fx_n\}$ is also a Cauchy sequence in $(f(X), d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$. Since $f(X)$ is complete, then there exists $fu \in f(X)$ such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu.$$

Notice that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing. Then from the hypothesis, we have $Tx_n \preceq fu$ and $fx_n \preceq fu$ for all $n \in \mathbb{N}$. But T is monotone f-nondecreasing, then we get $Tx_n \preceq T\mu$ for all n. Letting $n \to +\infty$, we obtain that $fu \preceq Tu$.

Assume that $fu \prec Tu$. Define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 1 yields that $\{fu_n\}$ is a nondecreasing sequence and $\lim\limits_{n\to+\infty} fu_n = \lim\limits_{n\to+\infty} Tu_n = fv$ for some $v \in X$. Now, from the hypothesis, we have $\sup fu_n \preceq fv$ and $\sup Tu_n \preceq fv$, for all $n \in \mathbb{N}$. Notice that

$$fx_n \preceq fu \preceq fu_1 \preceq fu_2 \preceq \cdots \preceq fu_n \preceq \cdots \preceq fv.$$

Case 1. If $fx_{n_0} = fu_{n_0}$ for some $n_0 \geqslant 1$ then we have

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu.$$

Thus, u is a coincidence point of T and f in X.

Case 2. If $fx_{n_0} \neq fu_{n_0}$ for all n, then from (2), we have

$$\begin{aligned} d(fx_{n+1}, fu_{n+1}) &= d(Tx_n, Tu_n) \leqslant \\ &\leqslant \alpha \frac{d(fx_n, Tx_n)\ d(fu_n, Tu_n)}{d(fx_n, fu_n)} + \beta\left[d(fx_n, Tu_n) + d(fu_n, Tx_n)\right] + \gamma d(fx_n, fu_n). \end{aligned}$$

On taking limit as $n \to +\infty$ in the above inequality, we obtain that

$$d(fu, fv) \leqslant (2\beta + \gamma) d(fu, fv) < d(fu, fv), \text{ since } 2\beta + \gamma < 1.$$

Therefore, we have

$$fu = fv = fu_1 = Tu.$$

Hence, we conclude that u is a coincidence point of T and f in X.

Assume that T and f are weakly compatible. Let w be the coincidence point of T and f, then we have

$$Tw = Tfz = fTz = fw, \text{ since } w = Tz = fz \text{ for some } z \in X.$$

From (2), we have

$$\begin{aligned} d(Tz, Tw) &\leqslant \alpha \frac{d(fz, Tz)\ d(fw, Tw)}{d(fz, fw)} + \beta\left[d(fz, Tw) + d(fw, Tz)\right] + \gamma d(fz, fw) \leqslant \\ &\leqslant (2\beta + \gamma) d(Tz, Tw), \end{aligned}$$

as $2\beta + \gamma < 1$, we obtain that $d(Tz, Tw) = 0$. Thus, $Tz = Tw = fw = w$. Hence, w is a common fixed point of T and f in X.

Suppose that the set of common fixed points of T and f is well ordered. It is enough to prove that the common fixed point of T and f is unique. Assume in contrary that, $u \neq v$ be two common fixed points of T and f. Then from (2), we have

$$d(u,v) \leqslant \alpha \frac{d(fu,Tu)\ d(fv,Tv)}{d(fu,fv)} + \beta\left[d(fu,Tv)+d(fv,Tu)\right] + \gamma d(fu,fv) \leqslant$$
$$\leqslant (2\beta+\gamma)\ d(u,v) < d(u,v), \text{ since } 2\beta+\gamma<1,$$

which is a contradiction. Hence, $u = v$. Conversely, suppose T and f have only one common fixed point then the set of common fixed points of T and f being a singleton is well ordered. This completes the proof. □

We have the following results as a consequence of Theorem 2.

Corollary 3. *Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose the mappings $T,f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies the following contraction conditions for all $x,y \in X$ for which $fx \neq fy$ are comparable*

$$(i) \qquad d(Tx,Ty) \leqslant \alpha \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \beta\left[d(fx,Ty)+d(fy,Tx)\right],$$

for some $\alpha,\beta \in [0,1)$ with $0 \leqslant \alpha+2\beta<1$,

$$(ii) \qquad d(Tx,Ty) \leqslant \beta\left[d(fx,Ty)+d(fy,Tx)\right] + \gamma d(fx,fy),$$

where $\beta,\gamma \in [0,1)$ such that $0 \leqslant 2\beta+\gamma<1$.

If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Moreover, if T and f are weakly compatible, then T and f have a common fixed point in X.

Furthermore, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Setting $\gamma = 0$ and $\alpha = 0$ in the Theorem 2, we obtain the required proof. □

Remarks

(1) If $\beta = 0$ in Theorems 1 & 2, we obtain Theorems 2.1 & 2.3 of Chandok [30].

(2) If $f = I$ and $\beta = 0$ in Theorems 1 & 2, then we get Theorems 2.1 & 2.3 of Harjani et al. [15].

Now, we have the following consequence of Theorem 1 involving the integral type contraction.

Let Φ denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(i) each φ is Lebesgue integrable function on every compact subset of $[0,+\infty)$ and

(ii) $\int_0^{\epsilon} \varphi(t)dt > 0$, for any $\epsilon > 0$.

Corollary 4. *Let $(X,d,\preceq)$ be a complete partially ordered metric space. Suppose that the mappings $T,f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfies*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \beta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \gamma \int_0^{d(fx,fy)} \varphi(t)dt, \qquad (3)$$

for all $x, y \in X$ for which the distinct fx and fy are comparable, $\varphi(t) \in \Phi$ and there exist $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leqslant \alpha + 2\beta + \gamma + < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

We obtain some consequences of Corollary 4 by taking $\gamma = 0$ and $\alpha = 0$.

Corollary 5. *If $\beta = 0$ in Corollary 4, we obtain the Corollary 2.5 of Chandok [30].*

Example 1. *Define a metric $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|$, where $X = [0, 1]$ with usual order $\leqslant$. Let us define two self mappings T and f on X by $Tx = \dfrac{x}{2}$ and $fx = \dfrac{x}{1+x}$, then T and f have a coincidence point in X.*

Proof. By definition of a metric d, it is clear that (X, d) is a complete metric space. Obviously, $(X, d, \leqslant)$ is complete partially ordered metric space with usual order. Let $x_0 = 0 \in X$, then $f(x_0) \leqslant T(x_0)$. By definitions; T, f are continuous, T is monotone f-nondecreasing and $T(X) \subseteq f(X)$.

Now for any $x, y \in X$ with $x < y$, we have

$$\begin{aligned} d(Tx, Ty) = \Big|\frac{x}{2} - \frac{y}{2}\Big| = \frac{1}{2}|x - y| < \\ &< \frac{\alpha}{4}xy(1-y) + \frac{\beta}{2}\left[\frac{|x(2-y) - y|}{(1+x)} + \frac{|y(2-x) - x|}{(1+y)}\right] + \gamma\frac{|x-y|}{(1+x)(1+y)} < \\ &< \alpha\frac{\left|\frac{x}{1+x} - \frac{x}{2}\right|\,\left|\frac{y}{1+y} - \frac{y}{2}\right|}{\frac{x}{1+x} - \frac{y}{1+y}} + \beta\left[\Big|\frac{x}{1+x} - \frac{y}{2}\Big| + \Big|\frac{y}{1+y} - \frac{x}{2}\Big|\right] + \gamma\frac{|x-y|}{(1+x)(1+y)} < \\ &< \alpha\frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy), \end{aligned}$$

holds the contraction condition in Theorem 1 for some α, β, γ in $[0, 1)$ such that $0 \leqslant \alpha + 2\beta + \gamma < 1$. Therefore T and f have a coincidence point $0 \in X$. □

Similarly the following is one more example of main Theorem 1.

Example 2. *A distance function $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|$, where $X = [0, 1]$ with usual order $\leqslant$. Define the two self mappings T and f on X by $Tx = x^3$ and $fx = x^4$, then T and f have two coincidence points 0, 1 in X with $x_0 = \dfrac{1}{3}$.*

3. Application

In this section, we discuss a unique solution of the integral equation by the method of upper and lower solutions. The monotone iterative technique is the one to find the minimal and a maximal solution between the lower and upper solutions which validate the maximal principle.

Let $\Lambda \in \mathbb{R}^n$ be a bounded and open set and $\mathcal{H} = \mathscr{L}(\Lambda)^2$ be a Hilbert space with usual inner product and norm then a linear operation $\mathscr{L} : D(\mathscr{L}) \subset \mathcal{H} \to \mathcal{H}$ is said to be a valid maximum principle if there exists some $\lambda \in \mathbb{R}$ such that

$$\begin{gathered} \mathscr{L}u + \lambda u \geqslant 0 \text{ on } \Lambda \text{ implies that } u \geqslant 0 \text{ on } \Lambda, u \in D(\mathscr{L}), \\ \text{where } u \geqslant 0 \text{ on } \Lambda \text{ if } u(x) \geqslant 0 \ \text{ for a.e.} x \in \mathbb{R}. \end{gathered} \tag{4}$$

Now consider the first order periodic boundary value problem of an integral equation given in [18].

$$u(x) = \int_0^M f(x, u(x), [\mathscr{K}u](x)])dx \quad \text{for a.e.} x \in \Lambda = (0, M), \ u(0) = u(M).$$

or

$$u'(x) = f(x, u(x), [\mathscr{K}u](x)) \ \text{ for a.e.} x \in (0, M), \ u(0) = u(M). \tag{5}$$

where f is Caratheodary function, $\mathscr{K}$ is an integral operator

$$[\mathscr{K}u](x) = \int_0^M k(x, y)u(y)dy \tag{6}$$

with kernal $\mathscr{K} \in \mathscr{L}^2(\Lambda \times \Lambda)$. It is clear that for any $u \in \mathcal{H} = \mathscr{L}^2(\Lambda \times \Lambda)$ then $\mathscr{K}u \in \mathcal{H}$.

For a solution of (5), first we study the linear problem for $\lambda \neq 0$

$$u' + \lambda u + \delta \mathscr{K}u = \sigma, \ u(0) = u(M). \tag{7}$$

Its known that u is a solution of (7) if and only if

$$u(x) = \int_0^M g(x, y)[\sigma(y) - \delta \mathscr{K}u(y)]dy = w(x) + \int_0^M \mathscr{R}(x, y)u(y)dy \tag{8}$$

where

$$w(x) = \int_0^M g(x, y)\sigma(y)dy,$$

and

$$\mathscr{R}(x, y) = -\delta \int_0^M g(x, z)\mathscr{K}(z, y)dz.$$

Let us define the linear operator $\mathscr{L} : D(\mathscr{L}) \subset \mathcal{H} \to \mathcal{H}$, where $D(\mathscr{L}) = \{u \in \mathcal{H}^1(\Lambda) : u(0) = = u(M)\}$ as

$$[\mathscr{L}u](x) = u'(x) + \delta[\mathscr{K}u](x). \tag{9}$$

Similarly, let $\mathscr{N} : D(\mathscr{N}) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, where

$$[\mathscr{N}u](x) = f(x, u(x), [\mathscr{K}u](x)) + \delta[\mathscr{K}u](x). \tag{10}$$

Hence, (5) is equivalent to $\mathscr{L}u = \mathscr{N}u$, $u \in D(\mathscr{L}) \cap D(\mathscr{N})$ and $D(\mathscr{L}) \subset \mathscr{L}^\infty(\Lambda) \subset D(\mathscr{N})$, where $\mathscr{K} \in \mathscr{L}^\infty(\Lambda \times \Lambda)$. Suppose that $\lambda \neq 0$ and follow the conditions of [44], we have

$$||\mathscr{K}||_2 < \frac{(2|\lambda|)^{\frac{1}{2}}|1 - e^{-\lambda M}|}{|\delta|(M(1 - e^{-2\lambda M}))} = d_1 \tag{11}$$

From Lemma 5.1 of [18] followed by above condition, the equation (7) has a unique solution $u \in \mathcal{H}$ for each $\sigma \in \mathcal{H}$ and $G : (\mathscr{L} + \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous, where

$$u(x) = [G\sigma](x) = \int_0^M g(x, y)\sigma(y)dy.$$

Hence from Theorem 2.2 in [44] shows that the maximum principal (4) is valid for $\lambda > 0$ whenever $\mathscr{K} \in \mathscr{L}^\infty(\Lambda \times \Lambda)$ and

$$||\mathscr{K}||_\infty < \frac{\lambda^2}{|\delta|(e^{\lambda M} + \lambda M - 1)} = d_2 \tag{12}$$

From (11) and (12), we get $D(\mathscr{L}) \subset D(\mathscr{N})$. The functions $\alpha, \beta \in D(\mathscr{L})$ are said to be lower and upper solution of (5) if $\alpha'(x) \leqslant f(x, u(x), [\mathscr{K}u](x)) \leqslant \beta'(x)$ for a.e. $x \in \mathbb{R}$.

Now suppose there exist the constants m, δ, λ with $0 < m \leqslant \lambda$ such that (12) is satisfied and the following inequality holds.

$$\begin{aligned} f(x, u(x), [\mathscr{K}u](x)) - f(x, v(x), [\mathscr{K}v](x)) \geqslant \\ \geqslant -m(u(x) - v(x)) - \delta([\mathscr{K}u](x)) - [\mathscr{K}v](x)), \end{aligned} \tag{13}$$

whenever $x \in \Lambda$, $\alpha(x) \leqslant v(x) \leqslant u(x) \leqslant \beta(x)$. Then applying Theorem 3.1 of [18] it is possible to approximate the external solutions of (5) by monotone iterates between the lower solution α and the upper solution β.

The authors do thankful to the editor and anonymous referees for their valuable suggestions and comments which improved the contents of the paper.

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Результаты точки совпадения и их приложения в частично упорядоченных метрических пространствах

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Аннотация. Целью данной работы является установление некоторых общих теорем о неподвижной точке для f-неубывающего отображения в себя, удовлетворяющего некоторому условию сжатия рационального типа в репере метрических пространств, наделенных частичным порядком. Также в пространстве представлены некоторые следствия результатов в терминах сжатий интегрального типа. Кроме того, метод монотонной итерации был использован для нахождения единственного решения интегрального уравнения.

Ключевые слова: упорядоченное метрическое пространство, рациональное сжатие, согласованные отображения, точка совпадения, общая неподвижная точка.