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Variational Formulas of the Monodromy Group for a Third-Order Equation on a Compact Riemann Surface

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Abstract. In the present article, we deduce explicit variational formulas for a solution vector and the elements of its monodromy group for a third-order ordinary differential equation on a compact Riemann surface of genus $g \geq 2$ in the spaces of quadratic and cubic holomorphic differentials.

Keywords: Riemann surface, third-order equation on a Riemann surface, variational formula, holomorphic differential.

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Introduction

In the present article, we deduce explicit variational formulas for a solution vector and for the elements of its monodromy group to a third-order ordinary differential equation on a compact Riemann surface of genus $g \geq 2$ with respect to a variation in the spaces of quadratic and cubic holomorphic differentials. These theorems are a continuation of results by D. Hejhal, V. V. Chueshev, and M. I. Tulina.

In [1–3], D. Hejhal began to study the dependence of a solution vector and the generators of the monodromy group of the equation on small variations in the space of holomorphic differentials.

Variational formulas found applications in the theory of Teichmüller spaces in connection with the uniformization of compact Riemann surfaces (see [3–4]).

The coefficients of a third-order differential equation on a compact Riemann surface must be the quadratic and cubic differentials at the corresponding derivatives (see [5]).

In the previous papers [4, 6, 7], a compact method was proposed for deducing the variational formulas for the vector solution and the elements of its monodromy group with the use of matrix-vector notation.

In the present article, we obtain formulas for the first variation with respect to a basis in spaces of holomorphic cubic differentials for a solution vector and the monodromy group on a compact Riemann surface for a third-order linear ordinary differential equation with any holomorphic coefficients. Moreover, we find explicit variational formulas for a variation in spaces

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of holomorphic quadratic differentials for a solution vector as well as the formula for the first variation of the solution vector for a variation with respect to a basis of quadratic holomorphic differentials on a compact Riemannian surface of genus $g \geq 2$.

1. Preliminaries

Let F be a compact Riemann surface of genus $g \geq 2$, D be an open disk on the plane \mathbb{C} . Denote by Γ a Fuchsian group of the first kind uniformizing F in the disk D , i.e., F is conformally equivalent to D/Γ .

Consider an linear ordinary differential equation

$$\frac{d^n v}{dt^n} + q_2(t) \frac{d^{n-2} v}{dt^{n-2}} + q_3(t) \frac{d^{n-3} v}{dt^{n-3}} + \cdots + q_n(t)v = 0, \quad t \in D, \quad (1)$$

where $q_j(t)$ is a meromorphic function on D , $j = 2, \dots, n$. Equation (1) has Fuchsian type on F if it has only regular Fuchsian points and is preserved after the change of variables

$$\omega = v(s)L'(t)^{\frac{n-1}{2}}, \quad (t, v) \rightarrow (s, \omega), \quad s = L(t), \quad L \in \Gamma. \quad (2)$$

A solution vector is a column-vector consisting of a basis in the space of holomorphic solutions to an equation with holomorphic coefficients. Holomorphic differentials of order q have the form $\Phi(z)dz^q$ and are invariant under a change of coordinates on the surface, i.e.,

$$\Phi(Lz)L'(z)^q = \Phi(z), \quad z \in D, \quad L \in \Gamma.$$

Denote by $\Omega^q(F)$ the vector space of holomorphic q -differentials on D/Γ , where $q \in \mathbf{N}$ (see [5]).

Lemma 1 ([2,3]). *Suppose that a column vector $U(t)$ consists of n linearly independent solutions to equation (1) on $F = D/\Gamma$. Then the equality*

$$U(Lt) = \chi(L)U(t)\xi_L(t)^{n-1}, \quad L \in \Gamma, \quad \xi_L(t) = \sqrt{L'(t)}, \quad (3)$$

uniquely determines the monodromy homomorphism $\chi : \Gamma \rightarrow GL(n, \mathbb{C})$ defined by the mapping $L \rightarrow \chi(L)$, $L \in \Gamma$.

The monodromy group of equation (1) is the image $\chi(\Gamma)$ of the group Γ . This is a matrix group describing the multivaluedness of a solution vector.

Note that for $n = 2$ the variation is possible only with respect to one coefficient of the equation

$$u^{(2)}(z) + (Q_0(z) - \mu r(z))u(z) = 0.$$

For $n = 3$, for the equation

$$u^{(3)}(z) + (Q_0(z) - \lambda q(z))u^{(1)}(z) + (R_0(z) - \mu r(z))u(z) = 0 \quad (4)$$

we have already three substantially different variations: (1) with respect to r , i.e., with respect to μ , in the space of cubic differentials; (2) with respect to q , i.e., with respect to λ , in the space of quadratic differentials; (3) with respect to r and q , i.e., with respect to λ and μ .

Let $U(z) = (u(z), v(z), w(z))^T$ be the solution vector to the Cauchy problem at a point $z_0 \in D$,

$$\begin{pmatrix} u(z_0) \\ v(z_0) \\ w(z_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u'(z_0) \\ v'(z_0) \\ w'(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u''(z_0) \\ v''(z_0) \\ w''(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (5)$$

for the unperturbed equation, i.e., for $\lambda = 0$ and $\mu = 0$.

Put

$$W(x) = \begin{pmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{pmatrix}, \quad W(z_0) = E,$$

$$W_1(x) = \begin{vmatrix} 0 & v & w \\ 0 & v' & w' \\ f & v'' & w'' \end{vmatrix} = f(-1)^4 \begin{vmatrix} v & w \\ v' & w' \end{vmatrix} = [f = ru] = ru \begin{vmatrix} v & w \\ v' & w' \end{vmatrix},$$

$$W_2(x) = \begin{vmatrix} u & 0 & w \\ u' & 0 & w' \\ u'' & f & w'' \end{vmatrix} = f(-1)^5 \begin{vmatrix} u & w \\ u' & w' \end{vmatrix} = [f = rv] = rv \begin{vmatrix} u & w \\ u' & w' \end{vmatrix},$$

$$W_3(x) = \begin{vmatrix} u & v & 0 \\ u' & v' & 0 \\ u'' & v'' & f \end{vmatrix} = f(-1)^6 \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = [f = rw] = rw \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}.$$

Then

$$V(z) = \begin{pmatrix} W_1(z) & 0 & 0 \\ 0 & W_2(z) & 0 \\ 0 & 0 & W_3(z) \end{pmatrix},$$

is a solution to the Lagrange adjoint unperturbed third-order equation on D/Γ . It is known from [3] that it satisfies the equality

$$V(Lz) = \xi_L(z)^2 V(z) \chi(L)^{-1}, \quad L \in \Gamma, \quad \xi_L(z) = \sqrt{L'(z)}, \quad z \in D.$$

2. Expansion of the solution vector in a series under variation in the space of quadratic differentials

Consider the perturbed vector equation

$$U^{(3)}(z) + (Q_0(z) - \lambda q(z))U^{(1)}(z) + R_0(z)U(z) = 0, \quad (6)$$

where $\lambda \in \mathbb{C}$, $|\lambda| < \varepsilon$, ε is a sufficiently small number, and $q(z)dz^2$ is a nonzero holomorphic differential on D/Γ .

Denote by

$$U(z; \lambda; 0) = \begin{pmatrix} u(z; \lambda; 0) & 0 & 0 \\ 0 & v(z; \lambda; 0) & 0 \\ 0 & 0 & \omega(z; \lambda; 0) \end{pmatrix} = \begin{pmatrix} u(z; \lambda; 0) \\ v(z; \lambda; 0) \\ \omega(z; \lambda; 0) \end{pmatrix}$$

the solution vector to the Cauchy problem (5) at a point z_0 for the perturbed equation (6). By Poincaré's small parameter method and the Cauchy–Kovalevskaya theorem, expand the solution vector in the Taylor series

$$U(z; \lambda; 0) = U(z) + \lambda U_{10}(z) + \lambda^2 U_{20}(z) + \dots + \lambda^m U_{m0}(z) + \dots,$$

convergent for $|\lambda| < \varepsilon$, $z \in D$ (see [2; 3]).

Inserting this series in (6), we obtain the infinite system of differential equations in vector-matrix form

$$\begin{aligned} U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0U(z) &= 0, \\ U_{10}^{(3)}(z) + Q_0(z)U_{10}^{(1)}(z) + R_0U_{10}(z) &= q(z)U^{(1)}(z), \\ U_{20}^{(3)}(z) + Q_0(z)U_{20}^{(1)}(z) + R_0U_{20}(z) &= q(z)U_{10}^{(1)}(z), \\ \dots \\ U_{n0}^{(3)}(z) + Q_0(z)U_{n0}^{(1)}(z) + R_0U_{n0}(z) &= q(z)U_{n-1,0}^{(1)}(z), \\ \dots \end{aligned}$$

Theorem 1. *The solution vector*

$$U^{(3)}(z) + (Q_0(z) - \lambda q(z))U^{(1)}(z) + R_0(z)U(z) = 0$$

with condition (5) on a compact Riemann surface F of genus $g \geq 2$ satisfies the explicit variational formula

$$U(z; \lambda; 0) = \left[E + \lambda A_0(z) + \lambda^2 A_1(z) + \dots + \lambda^n A_{n-1}(z) + \dots \right] U(z),$$

where $z \in D$, $|\lambda| < \varepsilon$,

$$A(x) = q(x)U^{(1)}(x)V(x), D(x) = q(x)U(x)V(x), A_0(z) = \int_{z_0}^z A(x)dx,$$

$$\begin{aligned} A_n(z) = \int_{z_0}^z \left[A(x)D^n(x) + A_0(x)A(x)D^{n-1}(x) + A_1(x)A(x)D^{n-2}(x) \right. \\ \left. + \dots + A_{n-2}(x)A(x)D(x) + A_{n-1}(x)A(x) \right] dx, \end{aligned}$$

and E is the identity matrix of order 3.

Proof. Find the solution to the second equation of the system by Lagrange's method of variation of constants:

$$U_{10}(z) = \int_{z_0}^z q(x)U^{(1)}(x)V(x)dxU(z).$$

If $n = 1$ then $U_{10}(z) = A_0(z)U(z)$.

For $n > 1$, denote by $U_{n0}(z) = A_{n-1}(z)U(z)$, where

$$A_{n-1}(z) = \int_{z_0}^z q(x)U_{n-1,0}^{(1)}(x)V(x)dx.$$

For $n = 2$, we have $U_{20}(z) = A_1(z)U(z)$. On the other hand,

$$U_{20}(z) = A_1(z)U(z) = \int_{z_0}^z q(x)U_{10}^{(1)}(x)V(x)dxU(z) = \int_{z_0}^z q(x)[A_0(x)U(x)]'_x V(x)dxU(z).$$

It follows that

$$\begin{aligned} A_1(z) &= \int_{z_0}^z q(x)[A_0(x)U(x)]'_x V(x)dx = \\ &= \int_{z_0}^z q(x) \left[A_0'(x)U(x) + A_0(x)U^{(1)}(x) \right] V(x)dx = \\ &= \int_{z_0}^z q(x) \left[A(x)U(x) + A_0(x)U^{(1)}(x) \right] V(x)dx = \end{aligned}$$

$$\begin{aligned}
&= \int_{z_0}^z A(x)q(x)U(x)V(x)dx + \int_{z_0}^z A_0(x)q(x)U^{(1)}(x)V(x)dx = \\
&= \int_{z_0}^z A(x)D(x)dx + \int_{z_0}^z A_0(x)A(x)dx.
\end{aligned}$$

Thus,

$$U_{20}(z) = \left(\int_{z_0}^z A(x)D(x)dx + \int_{z_0}^z A_0(x)A(x)dx \right) U(z).$$

For $n = 3$, we have the equality $U_{30}(z) = A_2(z)U(z)$. On the other hand,

$$U_{30}(z) = A_2(z)U(z) = \int_{z_0}^z q(x)U_{20}^{(1)}(x)V(x)dxU(z) = \int_{z_0}^z q(x) \left[A_1(x)U(x) \right]'_x V(x)dxU(z),$$

where

$$\begin{aligned}
A_2(z) &= \int_{z_0}^z q(x) \left[A_1(x)U(x) \right]'_x V(x)dx = \int_{z_0}^z q(x) \left[A_1'(x)U(x) + A_1(x)U^{(1)}(x) \right] V(x)dx = \\
&= \int_{z_0}^z q(x) \left[A(x)D(x)U(x)V(x)dx \right] + \int_{z_0}^z q(x) \left[A_0(x)A(x)U(x)V(x)dx \right] + \\
&+ \int_{z_0}^z q(x) \left[A_1(x)U^{(1)}(x)V(x)dx \right] = \int_{z_0}^z \left[A(x)D^2(x) + A_0(x)A(x)D(x) + A_1(x)A(x) \right] dx.
\end{aligned}$$

Therefore,

$$A_2(z) = \int_{z_0}^z \left[A(x)D^2(x) + A_0(x)A(x)D(x) + A_1(x)A(x) \right] dx$$

and

$$U_{30}(z) = \left(\int_{z_0}^z \left[A(x)D^2(x) + A_0(x)A(x)D(x) + A_1(x)A(x) \right] dx \right) U(z).$$

By the induction assumption, for $n = m$ we have the equality

$$A_m(z) = \int_{z_0}^z \left[A(x)D^m(x) + A_0(x)A(x)D^{m-1}(x) + A_1(x)A(x)D^{m-2}(x) + \dots + A_{m-1}(x)A(x) \right] dx.$$

Prove this assertion for the case $n = m + 1$. We have

$$U_{m+1,0}(z) = A_m(z)U(z) = \int_{z_0}^z q(x)U_{m0}^{(1)}(x)V(x)dxU(z),$$

where

$$\begin{aligned}
A_m(z) &= \int_{z_0}^z q(x) \left[A_{m-1}(x)U(x) \right]'_x V(x)dx = \\
&= \int_{z_0}^z q(x) \left[A_{m-1}'(x)U(x) + A_{m-1}(x)U^{(1)}(x) \right] V(x)dx = \\
&= \int_{z_0}^z q(x) \left[A(x)D^{m-1}(x) + A_0(x)A(x)D^{m-2}(x) + \right. \\
&+ \left. A_1(x)A(x)D^{m-3}(x) + \dots + A_{m-2}(x)A(x) \right] U(x)V(x)dx + \int_{z_0}^z q(x)A_{m-1}(x)U^{(1)}(x)V(x)dx =
\end{aligned}$$

$$\begin{aligned}
&= \int_{z_0}^z q(x)A(x)D^{m-1}(x)U(x)V(x)dx + \int_{z_0}^z q(x)A_0(x)A(x)D^{m-2}(x)U(x)V(x)dx + \\
&\quad + \int_{z_0}^z q(x)A_1(x)A(x)D^{m-3}(x)U(x)V(x)dx + \\
&\quad + \cdots + \int_{z_0}^z q(x)A_{m-2}(x)A(x)U(x)V(x)dx + \int_{z_0}^z A_{m-1}(x)A(x)dx = \\
&\quad = \int_{z_0}^z \left[A(x)D^m(x) + A_0(x)A(x)D^{m-1}(x) + \right. \\
&\quad \left. + A_1(x)A(x)D^{m-2}(x) + \cdots + A_{m-2}(x)A(x)D(x) + A_{m-1}(x)A(x) \right] dx.
\end{aligned}$$

Consequently, by induction, we have proved the formula for the matrix A_n for any n .

Let us now introduce the explicit variational formula with respect to λ for the solution vector:

$$\begin{aligned}
U(z; \lambda; 0) &= U(z) + \lambda U_{10}(z) + \lambda^2 U_{20}(z) + \cdots + \lambda^n U_{n0}(z) + \cdots \\
&= EU(z) + \lambda A_0(z)U(z) + \lambda^2 A_1(z)U(z) + \cdots + \lambda^n A_{n-1}(z)U(z) + \cdots \\
&= \left[E + \lambda A_0(z) + \lambda^2 A_1(z) + \cdots + \lambda^n A_{n-1}(z) + \cdots \right] U(z).
\end{aligned}$$

Thus, the theorem is proved.

Remark 1. This theorem gives an explicit variational formula for the solution vector, i.e., all the variational terms of any order or the whole Taylor series in λ under variation with respect to one holomorphic differential in $\Omega^2(F)$.

Proposition 1. Let $q_1(z)dz^2, \dots, q_{3g-3}(z)dz^2$ be a basis of quadratic holomorphic differentials on $F = D/\Gamma$ of genus $g \geq 2$. Then the perturbed equation

$$U^{(3)}(z) + (Q_0(z) - \sum_{j=1}^{3g-3} \lambda_j q_j(z))U^{(1)}(z) + R_0(z)U(z) = 0$$

with condition (5) satisfies the formula for the first variation of the solution vector

$$U(z; \lambda_1, \dots, \lambda_{3g-3}; 0) = \left[E + \sum_{j=1}^{3g-3} \lambda_j A_{0;e_j}(z) \right] U(z) + o(\lambda_1, \dots, \lambda_{3g-3}),$$

where $|\lambda_j| \rightarrow 0, j = 1, \dots, 3g-3, z \in D, A_{0;e_j}(z) = \int_{z_0}^z q_j(x)U_x^{(1)}V(x)dx$.

Proof. Since the coefficient at the first derivative depends holomorphically on $\lambda = (\lambda_1, \dots, \lambda_{3g-3})$, the solution vector to this equation is representable as

$$U(z; \lambda_1, \dots, \lambda_{3g-3}; 0) = U(z) + \sum_{j=1}^{3g-3} \lambda_j U_{10;e_j}(z) + o(\lambda),$$

$|\lambda_j| \rightarrow 0, j = 1, \dots, 3g-3$. Here e_j is the vector whose j th coordinate is equal to 1 and all the remaining coordinates are zero. Now, put $d = 3g-3$.

Inserting this expression in the equation, we obtain the vector equality

$$U^{(3)}(z) + \sum_{j=1}^{3g-3} \lambda_j U_{10;e_j}^{(3)}(z) + o(\lambda) + \left(Q_0(z) - \lambda_1 q_1(z) - \dots - \lambda_d q_d(z) \right) \times \\ \times \left(U^{(1)}(z) + \sum_{j=1}^{3g-3} \lambda_j U_{10;e_j}^{(1)}(z) + o(\lambda) \right) + R_0(z) \left(U(z) + \sum_{j=1}^{3g-3} \lambda_j U_{10;e_j}(z) + o(\lambda) \right) = 0.$$

Note that here the following conditions are fulfilled:

$$U_{10;e_j}(z_0) = U_{10;e_j}^{(1)}(z_0) = U_{10;e_j}^{(2)}(z_0) = 0, \quad j = 1, \dots, d.$$

Hence we obtain a system of vector linear differential equations of the form

$$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0(z)U(z) = 0, \\ U_{10;e_j}^{(3)}(z) + Q_0(z)U_{10;e_j}^{(1)}(z) + R_0(z)U_{10;e_j}(z) = q_j(z)U^{(1)}(z), \quad j = 1, \dots, d.$$

For each j , $j = 1, \dots, d$, solve the equation by Lagrange's method of variation of constants:

$$U_{10;e_j}(z) = \left[\int_{z_0}^z q_j(x)U^{(1)}(x)V(x)dx \right] U(z).$$

Put $A_j(z) = q_j(z)U^{(1)}(z)V(z)$ and

$$A_{0;e_j}(z) = \int_{z_0}^z A_j(x)dx, \quad j = 1, \dots, d.$$

This gives the equality $U_{10;e_j}(z) = A_{0;e_j}(z)U(z)$, $j = 1, \dots, d$. Therefore, we have the formula of the first variation of the solution vector:

$$U(z; \lambda_1, \dots, \lambda_{3g-3}; 0) = U(z) + \lambda_1 A_{0;e_1}(z)U(z) + \dots + \lambda_d A_{0;e_d}(z)U(z) + o(\lambda_1, \dots, \lambda_{3g-3}) = \\ = [E + \lambda_1 A_{0;e_1}(z) + \dots + \lambda_d A_{0;e_d}(z)]U(z) + o(\lambda_1, \dots, \lambda_{3g-3}),$$

$\lambda_1 \rightarrow 0, \dots, \lambda_{3g-3} \rightarrow 0$ under variation with respect to a basis of quadratic holomorphic differentials on a compact Riemann surface of genus $g > 1$.

3. Elements of the monodromy group under a variation with respect to a basis of cubic differentials

Consider the perturbed differential vector equation

$$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + \left(R_0(z) - \sum_{j=1}^m \mu_j r_j \right) U(z) = 0. \quad (9)$$

on the surface $F = D/\Gamma$, where r_1, \dots, r_m is a basis of cubic holomorphic differentials in the space $\Omega^3(F)$, $m = 5g - 5$, $\mu = (\mu_1, \dots, \mu_m)$. As above, denote by $U(z; 0; \mu) = (u(z; 0; \mu), v(z; 0; \mu), w(z; 0; \mu))^T$ three linearly independent solutions to the Cauchy problem at a point z_0 defined by the conditions

$$U(z_0; 0, \mu) = (1, 0, 0)^T; \quad U^{(1)}(z_0; 0, \mu) = (0, 1, 0)^T; \quad U^{(2)}(z_0; 0, \mu) = (0, 0, 1)^T, \quad (10)$$

for every μ . By the Poincaré's small parameter method and the Cauchy–Kovalevskaya theorem, we have the solution to (9) in vector form

$$U(z; 0; \mu_1, \dots, \mu_m) = U(z) + \sum_{k=1}^m \mu_k U_{01; \hat{e}_k}(z) + o(\mu_1, \dots, \mu_m),$$

where $\mu_1, \dots, \mu_m \rightarrow 0$.

Inserting the last expression in (9), we obtain the vector equalities

$$\begin{aligned} & U^{(3)}(z) + \mu_1 U_{01; \hat{e}_1}^{(3)}(z) + \dots + \mu_m U_{01; \hat{e}_m}^{(3)}(z) + o(\mu) + \\ & + Q_0(z)(U^{(1)}(z) + \mu_1 U_{01; \hat{e}_1}^{(1)}(z) + \dots + \mu_m U_{01; \hat{e}_m}^{(1)}(z) + o(\mu)) + \\ & + (R_0(z) - \sum_{j=1}^m \mu_j r_j(z))(U(z) + \mu_1 U_{01; \hat{e}_1}(z) + \dots + \mu_m U_{01; \hat{e}_m}(z) + o(\mu)) = 0. \end{aligned}$$

Note that the following conditions are satisfied:

$$U_{01; \hat{e}_k}(z_0) = U_{01; \hat{e}_k}^{(1)}(z_0) = U_{01; \hat{e}_k}^{(2)}(z_0) = 0, \quad k = 1, \dots, m.$$

From this we obtain the system of vector linear differential equations

$$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0(z)U(z) = 0;$$

$$U_{01; \hat{e}_k}^{(3)}(z) + Q_0(z)U_{01; \hat{e}_k}^{(1)}(z) + R_0(z)U_{01; \hat{e}_k}(z) = r_k(z)U(z), \quad k = 1, \dots, m.$$

For each $k, k = 1, \dots, m$, solve the second equation by Lagrange's method of variation of constants

$$U_{01; \hat{e}_k}(z) = \int_{z_0}^z r_k(t)U(t)V(t)dtU(z).$$

Introduce the notations

$$B_k(z) = r_k(z)U(z)V(z), \quad B_{0; \hat{e}_k}(z) = \int_{z_0}^z B_k(t)dt, \quad k = 1, \dots, m.$$

Hence, we obtain the equalities

$$U_{01; \hat{e}_k}(z) = B_{0; \hat{e}_k}U(z), \quad k = 1, \dots, m.$$

Thus,

$$\begin{aligned} U(z; 0; \mu_1, \dots, \mu_m) &= U(z) + \sum_{k=1}^m \mu_k B_{0; \hat{e}_k}(z)U(z) + o(\mu_1, \dots, \mu_m) \\ &= \left[E + \sum_{k=1}^m \mu_k B_{0; \hat{e}_k}(z) \right] U(z) + o(\mu_1, \dots, \mu_m), \end{aligned}$$

where $\mu_1, \dots, \mu_m \rightarrow 0$.

For deducing the variational formulas for the elements of the monodromy group, we must express $U_{01; \hat{e}_k}(Lz)$ through $U(z)$ and the coefficients of the equation. We infer

$$U_{01; \hat{e}_k}(Lz) = \left[\int_{z_0}^{Lz} B_k(x)dx \right] U(Lz) = \left[\int_{z_0}^{Lz_0} B_k(x)dx + \int_{Lz_0}^{Lz} B_k(x)dx \right] U(Lz) =$$

$$= B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \xi_L(z)^2\chi(L)U_{01;\widehat{e}_k}(z)$$

since

$$\begin{aligned} \int_{Lz_0}^{Lz} B_k(x)\chi(L)dx\xi_L(z)^2U(z) &= \langle x = Lt \rangle = \int_{z_0}^z B_k(Lt)\chi(L)dLt\xi_L(z)^2U(z) = \\ &= \int_{z_0}^z r_k(Lt)U(Lt)V(Lt)\chi(L)L'(t)dt\xi_L(z)^2U(z) = \\ &= \int_{z_0}^z r_k(t)L'(t)^{-3}L'(t)\chi(L)U(t)L'(t)V(t)\chi(L^{-1})\chi(L)L'(t)dt\xi_L(z)^2U(z) = \\ &= \chi(L) \int_{z_0}^z r_k(t)U(t)V(t)dt\xi_L(z)^2U(z) = \\ &= \chi(L)B_{0;\widehat{e}_k}(z)\xi_L(z)^2U(z) = \chi(L)\xi_L(z)^2U_{01;\widehat{e}_k}(z). \end{aligned}$$

Using the above-proven equality for $U_{01;\widehat{e}_k}(Lz)$, deduce the first-order variational formula for the elements of the monodromy group:

$$\begin{aligned} \xi_L(z)^2\chi(L; 0; \mu)U(z; 0; \mu) &= U(Lz; 0; \mu) = U(Lz) + \sum_{k=1}^m \mu_k U_{01;\widehat{e}_k}(Lz) + o(\mu) = \\ &= \chi(L)U(z)\xi_L(z)^2 + \sum_{k=1}^m \mu_k [B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \chi(L) \sum_{k=1}^m \mu_k U_{01;\widehat{e}_k}(z)\xi_L(z)^2 + o(\mu)] = \\ &= \chi(L) [U(z; 0; \mu)\xi_L(z)^2 - o(\mu)\xi_L(z)^2] + \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + o(\mu) = \\ &= \chi(L)U(z; 0; \mu)\xi_L(z)^2 - \chi(L)o(\mu)\xi_L(z)^2 + \\ &\quad + \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)[U(z; 0; \mu)\xi_L(z)^2 - o(1)\xi_L(z)^2] + o(\mu) = \\ &= \chi(L) [U(z; 0, \mu) - o(\mu)] \xi_L(z)^2 + \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z; 0, \mu)\xi_L(z)^2 - \\ &\quad - \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)o(1)\xi_L(z)^2 + o(\mu) = \\ &= [\chi(L) + \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)]U(z; 0, \mu)\xi_L(z)^2 - \chi(L)o(\mu)U^{-1}(z; 0, \mu)U(z; 0, \mu)\xi_L(z)^2 - \\ &\quad - \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0)\chi(L)o(1)U^{-1}(z; 0, \mu)U(z; 0, \mu)\xi_L(z)^2 + o(\mu). \end{aligned}$$

Hence we obtain a formula for the first variation of the elements of the monodromy group:

$$\chi(L; 0; \mu) = \left[E + \sum_{k=1}^m \mu_k B_{0;\widehat{e}_k}(Lz_0) \right] \chi(L) - o(\mu) - o(\mu), \quad \mu \rightarrow 0.$$

Thus, we have proved the following theorem:

Theorem 2. *The following variational formulas hold for the solution vector and the elements of the monodromy group of equation (9) perturbed with respect to the basis of holomorphic cubic differentials r_j , $j = 1, \dots, m = 5g - 5$, with normalization (10):*

$$U(z; 0; \mu_1, \dots, \mu_m) = [E + \mu_1 B_{0; \widehat{e}_1}(z) + \dots + \mu_m B_{0; \widehat{e}_m}(z)]U(z) + o(\mu_1, \dots, \mu_m)$$

and

$$\chi(L; 0; \mu) = \left[E + \sum_{k=1}^m \mu_k B_{0; \widehat{e}_k}(Lz_0) \right] \chi(L) + o(\mu),$$

$\mu_1, \dots, \mu_m \rightarrow 0$, where

$$B_k(z) = r_k(z)U(z)V(z), B_{0; \widehat{e}_k}(z) = \int_{z_0}^z B_k(t)dt, \quad k = 1, \dots, m.$$

Remark 2. These variational formulas show how the generators of the monodromy group $\chi(A_1), \dots, \chi(A_g), \chi(B_1), \dots, \chi(B_g)$ and the solution vector to the third-order equation depend of the parameters (μ_1, \dots, μ_m) under a variation with respect to a basis of cubic holomorphic differentials on F .

Now, consider the equation perturbed simultaneously with respect to $\lambda = (\lambda_1, \dots, \lambda_d)$ and to $\mu = (\mu_1, \dots, \mu_m)$,

$$U^{(3)}(z) + \left(Q_0(z) - \sum_{j=1}^{3g-3} \lambda_j q_j(z) \right) U^{(1)}(z) + \left(R_0(z) - \sum_{j=1}^{5g-5} \mu_j r_j(z) \right) U(z) = 0 \quad (11)$$

and the Cauchy problem at a point z_0 defined by the condition

$$\begin{aligned} U(z_0; \lambda; \mu) &= (1, 0, 0)^T; \quad U^{(1)}(z_0; \lambda; \mu) = (0, 1, 0)^T; \\ U^{(2)}(z_0; \lambda; \mu) &= (0, 0, 1)^T, \end{aligned} \quad (12)$$

for any μ and λ .

Corollary 1. *The solution vector to equation (11) with the Cauchy problem (12) satisfies the formulas of the first variation*

$$U(z; \lambda; \mu) = \left[E + \sum_{j=1}^{3g-3} \lambda_j A_{0; e_j}(z) + \sum_{j=1}^{5g-5} \mu_j B_{0; \widehat{e}_j}(z) \right] U(z) + o(\lambda_1, \dots, \lambda_{3g-3}) + o(\mu_1, \dots, \mu_{5g-5}),$$

$\lambda_1, \dots, \lambda_{3g-3} \rightarrow 0$, $\mu_1, \dots, \mu_{5g-5} \rightarrow 0$, where

$$\begin{aligned} A_{0; e_j}(z) &= \int_{z_0}^z q_j(x) U_x^{(1)} V(x) dx, \quad j = 1, \dots, 3g - 3, \\ B_{0; \widehat{e}_k}(z) &= \int_{z_0}^z r_k(x) U(x) V(x) dx, \quad k = 1, \dots, 5g - 5. \end{aligned}$$

Remark 3. The equality $U(Lz)(L'(z))^{-1} = \chi(L)U(z)$, $L \in \Gamma$, means that the solution vector $U(z)$ for the Cauchy problem at z_0 is the form of vector third-order Prym 1-differentials on $F = D/\Gamma$ with respect to the matrix character χ of the group Γ with values in $GL(3, \mathbb{C})$, or, more exactly, $U(z)$ is a holomorphic section of the vector bundle $\chi \otimes K^{-1}$, where K is the canonical bundle on $F = D/\Gamma$ [5].

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Вариационные формулы группы монодромии для уравнения третьего порядка на компактной римановой поверхности

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Аннотация. В данной статье выводятся явные вариационные формулы для вектор-решения и для элементов его группы монодромии обыкновенного дифференциального уравнения третьего порядка на компактной римановой поверхности рода $g \geq 2$ относительно вариации в пространствах квадратичных и кубических голоморфных дифференциалов.

Ключевые слова: римановы поверхности, уравнение третьего порядка на римановой поверхности, вариационные формулы, голоморфные дифференциалы.