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Algorithm of the Regularization Method for a Singularly Perturbed Integro-differential Equation with a Rapidly Decreasing Kernel and Rapidly Oscillating Inhomogeneity

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Abstract. In this paper, we consider a singularly perturbed integro-differential equation with a rapidly oscillating right-hand side, which includes an integral operator with a rapidly varying kernel. The main goal of this work is to generalize the Lomov's regularization method and to reveal the influence of the rapidly oscillating right-hand side and a rapidly varying kernel on the asymptotics of the solution to the original problem.

Keywords: singular perturbation, integro-differential equation, rapidly oscillating right-hand side, rapidly varying kernel, regularization, solvability of iterative problems.

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Various applied problems related to the study of the properties of media with a periodic structure can be described using differential equations with rapidly oscillating inhomogeneities. The presence of rapidly oscillating components creates serious problems for the numerical integration of such equations. Therefore, asymptotic methods are usually applied to equations of this type, the most famous of which are the Feshchenko–Shkil–Nikolenko splitting method [1–3] and the Lomov regularization method [4–17]. However, both of these methods were developed mainly for singularly perturbed differential equations that do not contain an integral operator. Note that, as far as we know, the splitting method has not been applied to integro-differential equations, and the transition from differential equations to integro-differential equations with rapidly oscillating inhomogeneities requires a significant revision of the algorithm of the regularization method itself. The integral term gives rise to new types of singularities in solutions that differ from the previously known ones, which complicates the development of the algorithm of

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the regularization method [4]. Moreover, in the problem considered below, the integral operator contains a rapidly decreasing factor. Problems of this type have been studied only in the presence of slowly varying inhomogeneities (see, for example, [12–14]. An analysis of the influence of a rapidly decreasing kernel on the asymptotic solution of problems with fast oscillations has not been performed before and is the subject of our study.

1. Problem statement

Consider the following integro-differential equation:

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon \frac{dy}{dt} - A(t)y - \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds = \\ = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon}, \quad y(0,\varepsilon)=y^0, \quad t\in[0,T], \tag{1}$$

where $A(t)$, $\mu(t)$, $h_1(t)$, $h_2(t)$, $\beta(t)$ are scalar functions, $\beta'(t) > 0$ is the frequency of a rapidly oscillating sine, y^0 is a constant number, $\varepsilon > 0$ is a small parameter. The function $\lambda_1(t) = A(t)$ is the eigenvalue of the limiting operator $A(t)$, the functions $\lambda_2(t) = -i\beta'(t)$ and $\lambda_3(t) = +i\beta'(t)$ are related to the presence of a rapidly oscillating sine in eqation (1), the function $\lambda_4(t) = \mu(t)$ characterizes the rapid change in the kernel of the integral operator.

Problem (1) will be considered under the following conditions:

1) $A(t), \mu(t)\beta(t) \in C^\infty([0,T],\mathbb{R}),\ h_1(t), h_2(t) \in ([0,T],\mathbb{C})$,

$K(t,s) \in C^\infty(\{0 \leqslant s \leqslant t \leqslant T\}, \mathbb{C})$;

2) $A(t) < 0\ \ \forall t \in [0,T]$.

Let us develop an algorithm for the regularization method under the specified conditions.

2. Solution space and regularization of problem (1)

We introduce the regularizing variables (см. [4])

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \quad j=\overline{1,4}, \tag{2}$$

and instead of the problem (1) consider the problem

$$L_\varepsilon \tilde{y}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{y}}{\partial t} + \sum_{j=1}^4 \lambda_j(t)\frac{\partial\tilde{y}}{\partial\tau_j} - \lambda_1(t)\tilde{y} - \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\tilde{y}\Big(s,\frac{\psi(s)}{\varepsilon},\varepsilon\Big)ds = \\ = h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2} - e^{\tau_3}\right), \ \ \tilde{y}(0,0,\varepsilon) = y^0, \quad t\in[0,T] \tag{3}$$

for the function $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$, where (according to (2)) $\tau = (\tau_1,\tau_2,\tau_3,\tau_4)$, $\psi = (\psi_1,\psi_2,\psi_3,\psi_4)$. It is clear that if $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$ is the solution of the problem (3), then the vector function $\tilde{y} = \tilde{y}\left(t,\dfrac{\psi(t)}{\varepsilon},\varepsilon\right)$ is an exact solution of the problem (1); therefore, problem (3) is extended with respect to problem (1). However, it cannot be considered fully regularized, since the integral term

$$J\tilde{y} \equiv J\left(\tilde{y}(t,\tau,\varepsilon)\,|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\,\tilde{y}\Big(s,\frac{\psi(s)}{\varepsilon},\varepsilon\Big)ds$$

has not been regularized in it.

To regularize the operator J, we introduce a class M_ε that is asymptotically invariant with respect to the operator $J\tilde{y}$ (see [4], p. 62). Let us first consider the space U of vector functions $y\left(t,\tau\right)$, representable by the sums

$$y\left(t,\tau\right)=y_0\left(t\right)+\sum_{j=1}^{4}y_j\left(t\right)e^{\tau_j},\quad y_j\left(t\right)\in C^\infty\left(\left[0,T\right],\mathbb{C}^1\right),\ j=\overline{0,4}. \tag{4}$$

Let us show that the class $M_\varepsilon=U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant under the operator Jy. The image of the operator on the element (4) of the space U has the form:

$$\begin{aligned}Jy\left(t,\tau\right)&=\int_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_4(\theta)d\theta}K\left(t,s\right)y_0\left(s\right)ds+\sum_{j=1}^{4}\int_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_4(\theta)d\theta}K\left(t,s\right)y_j\left(s\right)e^{\frac{1}{\varepsilon}\int_0^s\lambda_j(\theta)d\theta}ds=\\&=\int_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\lambda_4(\theta)d\theta}K\left(t,s\right)y_0\left(s\right)ds+e^{\frac{1}{\varepsilon}\int_0^t\lambda_4(\theta)d\theta}\int_0^t K\left(t,s\right)y_4\left(s\right)ds+\\&+\sum_{j=1,j\neq4}^{4}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_4(\theta)d\theta}\int_0^t K\left(t,s\right)y_j\left(s\right)e^{\frac{1}{\varepsilon}\int\limits_0^s(\lambda_j(\theta)-\lambda_4(\theta))d\theta}ds.\end{aligned}$$

By applying the operation of integration by parts, we find that the image of the operator J on an element (4) of the space U can be represented as a series

$$\begin{aligned}Jy\left(t,\tau\right)&=e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_4(\theta)d\theta}\int_0^t K\left(t,s\right)y_4\left(s\right)ds+\\&+\sum_{\nu=0}^{\infty}\left(-1\right)^\nu\varepsilon^{\nu+1}\left[\left(I_0^\nu\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_4(\theta)d\theta}-\left(I_0^\nu\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=0}\right]+\\&+\sum_{j=1,j\neq4}^{4}\sum_{\nu=0}^{\infty}\left(-1\right)^\nu\varepsilon^{\nu+1}\left[\left(I_j^\nu\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_j(\theta)d\theta}-\right.\\&\left.-\left(I_j^\nu\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=0}e^{\frac{1}{\varepsilon}\int\limits_0^t\lambda_4(\theta)d\theta}\right],\quad\tau=\psi\left(t\right)/\varepsilon\end{aligned}$$

where it is indicated:

$$\begin{aligned}&I_0^0=\frac{1}{-\lambda_4\left(s\right)},\quad I_0^\nu=\frac{1}{-\lambda_4\left(s\right)}\frac{\partial}{\partial s}I_0^{\nu-1},\\&I_j^0=\frac{1}{\lambda_j\left(s\right)-\lambda_4\left(s\right)},\quad I_j^\nu=\frac{1}{\lambda_j\left(s\right)-\lambda_4\left(s\right)}\frac{\partial}{\partial s}I_j^{\nu-1},\ j=\overline{1,3},\ \nu\geqslant1.\end{aligned}$$

It is easy to show (see, for example, [18], pp. 291—294), that this series converges asymptotically when $\varepsilon\to+0$ (uniformly over $t\in[0,T]$). This means that the class M_ε asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

Let us introduce operators $R_\nu:U\to U$, acting on each element $y\left(t,\tau\right)\in U$ of the form (4) according to the law:

$$R_0y\left(t,\tau\right)=e^{\tau_4}\int_0^t K\left(t,s\right)y_4\left(s\right)ds, \tag{5_0}$$

$$\begin{aligned}R_1y\left(t,\tau\right)&=\left[\left(I_0^0\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=t}e^{\tau_4}-\left(I_0^0\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=0}\right]+\\&+\sum_{j=1}^{3}\left[\left(I_j^0\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\tau_j}-\left(I_j^0\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=0}e^{\tau_4}\right],\end{aligned} \tag{5_1}$$

$$
\begin{aligned}
&R_{\nu+1} y\left(t,\tau\right)=(-1)^{\nu}\left[\left(I_0^{\nu}\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=t}e^{\tau_4}-\left(I_0^{\nu}\left(K\left(t,s\right)y_0\left(s\right)\right)\right)_{s=0}\right]+\\
&+\sum_{j=1}^{3}(-1)^{\nu}\left[\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\tau_j}-\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=0}e^{\tau_4}\right],\quad \nu\geqslant 1.
\end{aligned}
\tag{$5_{\nu+1}$}
$$

Now let $\tilde{y}\left(t,\tau,\varepsilon\right)$ be arbitrary continuous in $(t,\tau)\in[0,T]\times\left\{\tau:\mathrm{Re}\tau_j\leqslant 0,\ \ j=\overline{1,4}\right\}$ function with asymptotic expansion

$$
\tilde{y}\left(t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^k y_k\left(t,\tau\right),\quad y_k\left(t,\tau\right)\in U \tag{6}
$$

converging as $\varepsilon\to+0$ (uniformly in $(t,\tau)\in[0,T]\times\left\{\tau:\mathrm{Re}\tau_j\leqslant 0, j=\overline{1,4}\right\}$). Then image $J\tilde{y}\left(t,\tau,\varepsilon\right)$ of this function expands into an asymptotic series

$$
J\tilde{y}\left(t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^k Jy_k\left(t,\tau\right)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\left(t,\tau\right)|_{\tau=\psi(t)/\varepsilon}.
$$

This equality is the basis for introducing an extension of the operator J on series of the form (6):

$$
\tilde{J}\tilde{y}\left(t,\tau,\varepsilon\right)\equiv\tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k\left(t,\tau\right)\right)\overset{def}{=}\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\left(t,\tau\right). \tag{7}
$$

Although the operator $\tilde{J}$ is defined formally, its usefulness is obvious, since in practice one usually constructs an N-th approximation of the asymptotic solution of the problem (1), in which only N-th partial sums of the series (7) that have not formal, but true meaning. Now we can write the problem completely regularized with respect to the original problem (2):

$$
\begin{aligned}
L_{\varepsilon}\tilde{y}(t,\tau,\varepsilon)&\equiv\varepsilon\frac{\partial\tilde{y}}{\partial t}+\sum_{j=1}^{4}\lambda_j\left(t\right)\frac{\partial\tilde{y}}{\partial\tau_j}-\lambda_1(t)\tilde{y}-\tilde{J}\tilde{y}=\\
&=h_1(t)-\frac{1}{2i}h_2(t)\left(e^{\tau_2}-e^{\tau_3}\right),\quad \tilde{y}(t,\tau,\varepsilon)|_{t=0,\tau=0}=y^0,\quad t\in[0,T],
\end{aligned}
\tag{8}
$$

where the operator $\tilde{J}$ has the form (7).

3. Iterative problems and their solvability in the space U

Substituting series (6) into (8) and equating the coefficients at the same powers of ε, we obtain the following iterative problems:

$$
Ly_0\left(t,\tau\right)\equiv\sum_{j=1}^{4}\lambda_j(t)\frac{\partial y_0}{\partial\tau_j}-\lambda_1(t)y_0-R_0y_0=h_1(t)-\frac{1}{2i}h_2(t)\left(e^{\tau_2}-e^{\tau_3}\right),\ y_0\left(0,0\right)=y^0; \tag{9_0}
$$

$$
Ly_1\left(t,\tau\right)=-\frac{\partial y_0}{\partial t}+R_1y_0,\ y_1\left(0,0\right)=0; \tag{9_1}
$$

$$
Ly_2\left(t,\tau\right)=-\frac{\partial y_1}{\partial t}+R_1y_1+R_2y_0,\ \ y_0\left(0,0\right)=0; \tag{9_2}
$$

$$
\cdots
$$

$$
Ly_k\left(t,\tau\right)=-\frac{\partial y_{k-1}}{\partial t}+R_ky_0+\cdots+R_1y_{k-1},\quad y_k\left(0,0\right)=0,\ \ k\geqslant 1. \tag{9_k}
$$

Each of the iterative problems (9_k) can be written as

$$Ly\,(t,\tau) \equiv \sum_{j=1}^{3} \lambda_j\,(t)\,\frac{\partial y}{\partial \tau_j} - \lambda_1(t)y - R_0 y = H\,(t,\tau)\,, \quad y\,(0,0) = y_*\,, \tag{10}$$

where $H\,(t,\tau) = H_0\,(t) + \sum\limits_{j=1}^{3} H_j\,(t)\,e^{\tau_j}$ is a well-known function of the space U, $y_* \in \mathbb{C}$ is constant, and the operator R_0 has the form (see (6_0))

$$R_0 y \equiv R_0\left(y_0\,(t) + \sum_{j=1}^{4} y_j\,(t)\,e^{\tau_j}\right) = e^{\tau_4}\int_0^t K\,(t,s)\,y_4\,(s)\,ds.$$

We introduce a scalar (for each $t \in [0,T]$) product in the space U :

$$< z, w > \equiv < z_0\,(t) + \sum_{j=1}^{4} z_j\,(t)\,e^{\tau_j},\, w_0\,(t) + \sum_{j=1}^{4} w_j\,(t)\,e^{\tau_j} > \stackrel{def}{=}$$
$$\stackrel{def}{=} (z_0\,(t)\,, w_0\,(t)) + \sum_{j=1}^{4} (z_j\,(t)\,, w_j\,(t))$$

where $(*, *)$ denotes the usual scalar product in the complex space $\mathbb{C}$. Let us prove the following statement.

Theorem 1. *Let the conditions 1) and 2) hold and right-hand part* $H\,(t,\tau) = H_0\,(t) + \sum\limits_{j=1}^{4} H_j\,(t)\,e^{\tau_j}$ *of the equation (10) belongs to the space U. Then for the solvability of equation (10) in U it is necessary and sufficient that the identity*

$$< H_1\,(t,\tau)\,, e^{\tau_1} > \equiv 0 \Leftrightarrow H_1\,(t) \equiv 0 \;\forall t \in [0,T] \tag{11}$$

is fulfilled.

Proof. We will define the solution of the equation (10) as an element (4) of the space U:

$$y\,(t,\tau) = y_0\,(t) + \sum_{j=1}^{3} y_j\,(t)\,e^{\tau_j}. \tag{12}$$

Substituting (12) into the equation (10), we will have

$$\sum_{j=1}^{4} [\lambda_j\,(t) - \lambda_1(t)]\,y_j\,(t)\,e^{\tau_j} - \lambda_1(t)y_0\,(t) - e^{\tau_4}\int_0^t K\,(t,s)\,y_4\,(s)\,ds = H_0\,(t) + \sum_{j=1}^{4} H_j\,(t)\,e^{\tau_j}.$$

Equating here separately the free terms and coefficients at the same exponents, we obtain the following equations:

$$-\lambda_1\,(t)\,y_0\,(t) = H_0\,(t)\,, \tag{13_0}$$

$$[\lambda_j\,(t) - \lambda_1\,(t)]\,y_j\,(t) = H_j\,(t)\,, j = \overline{1,3}, \tag{13_j}$$

$$[\lambda_4\,(t) - \lambda_1\,(t)]\,y_4\,(t) - \int_0^t K\,(t,s)y_4\,(s)\,ds = H_4(t). \tag{13_4}$$

Due to the fact that the function $\lambda_1(t) \neq 0\ \forall t \in [0,T]$, the equation (13_0) has a unique solution $y_0(t) = -\lambda_1^{-1}(t)\,H_0(t)$. Since the function $[\lambda_4(t) - \lambda_1(t)] \neq 0\ \forall t \in [0,T]$, then the equation (13_4) can be written as

$$y_4(t) = \int_0^t \Big([\lambda_4(t) - \lambda_1(t)]^{-1} K(t,s)\Big) y_4(s)\,ds - [\lambda_4(t) - \lambda_1(t)]^{-1} H_4(t)\,. \tag{14}$$

Due to the smoothness of the kernel $\Big([\lambda_4(t) - \lambda_1(t)]^{-1} K(t,s)\Big)$ and heterogeneity $[\lambda_4(t) - \lambda_1(t)]^{-1} H_4(t)$ this Volterra integral equation has a unique solution $y_4(t) \in C^\infty([0,T], \mathbb{C})$.

Since $\lambda_{2,3}(t) = \pm i\beta'(t)$ are purely imaginary functions, and the function $\lambda_1(t)$ is real, then the equation (13_j) при $j = 2,3$ solvable in the space $C^\infty([0,T],\mathbb{C})$. The equation (13_1) is solvable in the space $C^\infty([0,T],\mathbb{C})$ if and only if the identity $H_1(t) \equiv 0\ \ \forall t \in [0,T]$ holds. It is easy to see that this identity coincides with the identity (11). Thus, condition (11) is necessary and sufficient for the solvability of equation (10) in the space U. The theorem is proved. □

Remark 1. If identity (11) holds, then under conditions 1) and 2) the equation (10) has the following solution in the space U :

$$y(t,\tau) = y_0(t) + \alpha_1(t)\,e^{\tau_1} + H_{21}(t)e^{\tau_2} + H_{31}(t)e^{\tau_3} + y_4(t)\,e^{\tau_4}, \tag{15}$$

where $\alpha_1(t) \in C^\infty([0,T],\mathbb{C})$ is an arbitrary function, $y_0(t) = -\lambda_1^{-1}(t)\,H_0(t)$, $y_4(t)$ is the solution of the integral equation (14) and introduced the notation:

$$H_{21}(t) \equiv \frac{H_2(t)}{\lambda_2(t) - \lambda_1(t)},\ \ H_{31}(t) \equiv \frac{H_3(t)}{\lambda_3(t) - \lambda_1(t)}.$$

4. Unique solvability of a general iterative problem in the space U. Remainder term theorem

As can be seen from (15), the solution to the equation (10) is determined ambiguously. However, if it is subject to additional conditions:

$$\begin{aligned} & y(0,0) = y_*, \\ & < -\frac{\partial y}{\partial t} + R_1 y + Q(t,\tau)\,,\, e^{\tau_1} > \equiv 0\ \forall t \in [0,T] \end{aligned} \tag{16}$$

where $Q(t,\tau) = Q_0(t) + \sum\limits_{j=1}^{4} Q_i(t)\,e^{\tau_j}$ is a known function of the space U, y_* is a constant number of the complex space $\mathbb{C}$, then problem (10) will be uniquely solvable in the space U. More precisely, the following result takes place.

Theorem 2. *Let conditions 1) and 2) hold, the right-hand side $H(t,\tau)$ of the equation (10) belongs to the space U and satisfies the orthogonality condition (11). Then equation (10) under additional conditions (16) is uniquely solvable in U.*

Proof. Under condition (11), the equation (10) has the solution (15) in the space U, where the function $\alpha_1(t) \in C^\infty([0,T],\mathbb{C})$ so far arbitrary. Subordinate (15) to the initial condition $y(0,0) = y_*$. Will have

$$\begin{aligned} & y_* = y_0(0) + \alpha_1(0) + H_{21}(0) + H_{31}(0) - \frac{H_4(0)}{\lambda_4(0) - \lambda_1(0)} \quad \Leftrightarrow \\ & \Leftrightarrow \ \alpha_1(0) = y_* + \lambda_1^{-1}(0) H_0(0) - H_{21}(0) - H_{31}(0) + \frac{H_4(0)}{\lambda_4(0) - \lambda_1(0)}. \end{aligned} \qquad (17))$$

Let us now subordinate the solution (15) to the second condition (16). The right-hand side of this equation is

$$\begin{aligned} & -\frac{\partial y_0}{\partial t} + R_1 y_0 + Q(t,\tau) = -\dot{y}_0(t) - \dot{\alpha}_1(t) e^{\tau_1} - \\ & -\sum_{j=2}^{3} \left(\frac{H_{j1}(t)}{\lambda_j(t) - \lambda_1(t)} \right)^{\bullet} e^{\tau_j} + \dot{y}_4(t) e^{\tau_4} + \frac{K(t,t)\alpha_1(t)}{\lambda_1(t) - \lambda_4(t)} e^{\tau_1} - \frac{K(t,0)\alpha_1(0)}{\lambda_j(0) - \lambda_4(0)} + \\ & + \sum_{j=2}^{4} \left[\frac{K(t,t) y_j(t)}{\lambda_j(t) - \lambda_4(t)} e^{\tau_j} - \frac{K(t,0) y_j(0)}{\lambda_j(0) - \lambda_4(0)} \right] + Q(t,\tau). \end{aligned} \qquad (18)$$

Now multiplying (18) scalarly by e^{τ_1}, we obtain the differential equation

$$-\dot{\alpha}_1(t) + \frac{K(t,t)\alpha_1(t)}{[\lambda_1(t) - \lambda_4(t)]} + Q_1(t) = 0.$$

Adding the initial condition (17) to it, we uniquely find the function $\alpha_1(t)$, and, therefore, construct the solution (15) of the problem (10) in the space U uniquely. The theorem is proved.

Applying Theorems 1 and 2 to iterative problems (9_k), we find uniquely their solutions in the space U and construct series (6). Just as in [4], we prove the following statement.

Theorem 3. *Let conditions 1)–2) be satisfied for the equation (1). Then at* $\varepsilon \in (0, \varepsilon_0](\varepsilon_0 > 0$ *is small enough) equation (1) has a unique solution* $y(t,\varepsilon) \in C^1([0,T],\mathbb{C})$; *in this case, the estimate*

$$||y(t,\varepsilon) - y_{\varepsilon N}(t)||_{C[0,T]} \leqslant c_N \varepsilon^{N+1}, \quad N = 0, 1, 2, \ldots,$$

holds true; here $y_{\varepsilon N}(t)$ *is narrowing* $\left(at\ \tau = \dfrac{\psi(t)}{\varepsilon} \right)$ *of the* N*-th partial sum of the series (6) (with coefficients* $y_k(t,\tau) \in U$ *satisfying iterative problems* (9_k)*, and the constant* $c_N > 0$ *does not depend on* ε *at* $\varepsilon \in (0, \varepsilon_0]$.

5. Construction of a solution of the first iterative problem

Using Theorem 1, we will try to find a solution for the first iterative problem (9_k). Since the right-hand side $h_1(t)$ to the equation (9_0), satisfies condition (11), this equation has (according to (15)) the solution in the space U in the form

$$y_0(t,\tau) = y_0^{(0)}(t) + \alpha_1^{(0)}(t) e^{\tau_1} + h_{21}(t) \sigma_1 e^{\tau_2} + h_{31}(t) \sigma_2 e^{\tau_3}, \qquad (19)$$

where $\alpha_1^{(0)}(t) \in C^\infty([0,T],\mathbb{C})$ is an arbitrary function,$y_0^{(0)}(t) = -\lambda_1^{-1}(t) h_1(t)$ and introduced the notations:

$$h_{21}(t) = -\frac{1}{2i} \frac{h_2(t)}{\lambda_2(t) - \lambda_1(t)}, \quad h_{31}(t) = \frac{1}{2i} \frac{h_2(t)}{\lambda_3(t) - \lambda_1(t)}.$$

Submitting (19) to the initial condition $y_0(0,0) = y^0$, will have

$$\begin{aligned} & y_0^{(0)}(0) + \alpha_1^{(0)}(0) + h_{21}(0) + h_{31}(0) = y^0 \ \Leftrightarrow \\ & \Leftrightarrow \ \alpha_1^{(0)}(0) = y^0 + \lambda_1^{-1}(0) h_1(0) - h_{21}(0) - h_{31}(0). \end{aligned} \qquad (20)$$

For the complete computation of the function $\alpha_1^{(0)}(t)$, we proceed to the next iterative problem (9_1). Substituting solution (19) of the equation (9_0) into it, we arrive at the following equation:

$$Ly_1(t,\tau) = -\frac{d}{dt}y_0^{(0)}(t) - \frac{d}{dt}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} + \left[\frac{K(t,t)\,\alpha_1^{(0)}(t)}{\lambda_1(t)-\lambda_4(t)}e^{\tau_1} - \frac{K(t,0)\,\alpha_1^{(0)}(0)}{\lambda_1(0)-\lambda_4(0)}\right].$$

Performing scalar multiplication here, we obtain the ordinary differential equation

$$-\frac{d\alpha_1^{(0)}(t)}{dt} + \frac{K(t,t)}{\lambda_1(t)-\lambda_4(t)}\alpha_1^{(0)}(t) = 0.$$

Adding the initial condition (20) to this equation, we find $\alpha_1^{(0)}(t)$:

$$\alpha_1^{(0)}(t) = \left[y^0 + \lambda_1^{-1}(0)\,h_1(0) - h_{21}(0) - h_{31}(0)\right]\exp\left\{\int_0^t \frac{K(\theta,\theta)}{\lambda_1(\theta)-\lambda_4(\theta)}d\theta\right\}$$

and hence the solution (19) to problem (9_0) will be found uniquely in the space U. In this case, the leading term of the asymptotics is as follows:

$$\begin{aligned} y_{\varepsilon 0}(t) &= y_0^{(0)}(t) + h_{21}(t)\,e^{-\frac{i}{\varepsilon}\int\limits_0^t \beta'(\theta)d\theta} + h_{31}(t)\,e^{+\frac{i}{\varepsilon}\int\limits_0^t \beta'(\theta)d\theta} + \\ &+ \left[y^0 + \lambda_1^{-1}(0)\,h_1(0) - h_{21}(0) - h_{31}(0)\right]e^{\int\limits_0^t \frac{K(\theta,\theta)}{\lambda_1(\theta)-\lambda_4(\theta)}d\theta + \frac{1}{\varepsilon}\int\limits_0^t \lambda_1(\theta)d\theta}. \end{aligned} \tag{21}$$

From the expression (21) for $y_{\varepsilon 0}(t)$ it is seen that the construction of the leading term of the asymptotics of the solution to problem (1) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

Example. Consider the integro-differential problem

$$\varepsilon\frac{dy}{dt} = -2y + \int_0^t e^{-\frac{t-s}{\varepsilon}}\left(t - t^2 + s^2\right)y(s,\varepsilon)\,ds + t^2 + t\sin\frac{t}{\varepsilon}, \quad y(0,\varepsilon) = y^0, \quad t \in [0,T]. \tag{22}$$

Here:

$$K(t,s) = t - t^2 + s^2, \quad h_1(t) = t^2, \quad h_2(t) = t, \quad \beta(t) = t,$$

$$\lambda_1 = -2, \quad \lambda_2 = -i, \quad \lambda_3 = +i, \quad \lambda_4 = -1, \quad K(t,t) = t.$$

Using formula (21), we calculate the leading term of the asymptotic solution to problem (22):

$$\begin{aligned} y_{\varepsilon 0}(t) &= \frac{t^2}{2} - \frac{t}{2i(-i+2)}e^{-\frac{it}{\varepsilon}} + \frac{t}{2i(i+2)}e^{\frac{it}{\varepsilon}} + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}} = \\ &= \frac{t^2}{2} + \frac{1}{5}t\left(2\,\sin\left(\frac{t}{\varepsilon}\right) - \cos\left(\frac{t}{\varepsilon}\right)\right) + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}. \end{aligned} \tag{23}$$

If there were no rapidly oscillating term $t\sin\dfrac{t}{\varepsilon}$ in the right-hand side of equation (22), then the leading term of the asymptotics would have the form

$$\hat{y}_{\varepsilon 0}(t) = \frac{t^2}{2} + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}$$

and the limiting solution of problem (22) would be the function $\bar{\bar{y}}(t) = \dfrac{t^2}{2}$. In the presence of a rapidly oscillating inhomogeneity, as can be seen from formulas (23), the exact solution $y(t,\varepsilon)$ of the problem (22), leaving the value y^0 at $t = 0$, performs (when $\varepsilon \to +0$) fast oscillations around the function $\bar{\bar{y}}(t) = \dfrac{t^2}{2}$. The formation of a term $y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}$ is influenced by the kernel $K(t,s)$ of the integral operator (more precisely: $K(t,t)$). In its absence, the specified term would have the form $y^0 e^{-\frac{2t}{\varepsilon}}$. Influence of a rapidly decreasing factor $e^{-\frac{t-s}{\varepsilon}}$ on the leading term of the asymptotics does not affect. It will be found when constructing the following asymptotic solution $y_{\varepsilon 1}(t)$.

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Алгоритм метода регуляризации для сингулярно возмущенного интегро-дифференциального уравнения с быстро убывающим ядром и с быстро осциллирующей неоднородностью

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Аннотация. В настоящей работе рассматривается сингулярно возмущенное интегро-дифференциальное уравнение с быстро осциллирующей правой частью, которое включает интегральный оператор с быстро меняющимся ядром. Основная цель данной работы — обобщить метод регуляризации Ломова и выявить влияние быстро осциллирующей правой части и быстро меняющегося ядра на асимптотику решения исходной задачи.

Ключевые слова: сингулярное возмущение, интегро-дифференциальное уравнение, быстро осциллирующая неоднородность, быстро меняющееся ядро, регуляризация, разрешимость итерационных задач.