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A Problem with Wear Involving Thermo-Electro-viscoelastic Materials

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Abstract. In this paper, we consider a mathematical model of a contact problem in thermo-electro-viscoelasticity. The body is in contact with an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments. We present a variational formulation of the problem, and we prove the existence and uniqueness of the weak solution.

Keywords: piezoelectric, temperature, thermo- electro-viscoelastic, variational inequality, wear.

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1. Introduction

In the recent years, piezoelectric contact problems have been of great interest to modern engineering. General models of electroelastic characteristics of piezoelectric materials can be found in [2, 7]. The problems of piezo- viscoelastic materials have been studied with different contact conditions within linearized elasticity in [1, 4] and with in nonlinear viscoelasticity in [9]. The modeling of these problems does not take into account the thermic effect. Mindlin [8] was the first to propose the thermo- piezoelectric model. The mathematical model which describes the frictional contact between a thermo-piezo- electric body and a conductive foundation is already addressed in the static case in [3]. Sofonea et al. considered in [6] the modeling of quasistatic viscoelastic problem with normal compliance friction and damage, they proved the existence and uniqueness of the weak solution,and they derived error estimates on the approximate solutions. In this paper,we consider a dynamic contact problem between a thermo-electro viscoelastic body and an electrically and thermally conductive rigid foundation which results in the wear of the contacting surface.

2. Problem statement

Problem P: Find a displacement field $u : \Omega \times [0.T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0.T] \to \mathbb{R}^d$,

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a temperature field $\theta : \Omega \times [0.T] \to \mathbb{R}$, and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\sigma = \mathscr{A}(\varepsilon(u(t))) + \mathscr{G}(\varepsilon(\dot{u}(t))) - \xi^* E(\varphi) - \theta \mathscr{M} \quad \text{in } \Omega \times [0.T], \tag{2.1}$$

$$D = \beta E(\varphi) + \xi \varepsilon(u) - (\theta - \theta^*)\mathfrak{p} \quad \text{in } \Omega \times [0.T], \tag{2.2}$$

$$\rho \ddot{u} = Div\ \sigma + f_0 \quad \text{in } \Omega \times [0.T], \tag{2.3}$$

$$\text{div } D = q_0 \quad \text{in } \Omega \times [0.T], \tag{2.4}$$

$$\dot{\theta} - div(K \nabla \theta) = -\mathscr{M}.\nabla \dot{u} + q_1 \quad \text{in } \Omega \times [0.T], \tag{2.5}$$

$$u = 0 \text{ on } \Gamma_1 \times [0.T], \tag{2.6}$$

$$\sigma \nu = h \text{ on } \Gamma_2 \times [0.T], \tag{2.7}$$

$$\begin{cases} \sigma_\nu = -\alpha \left| \dot{u}_\nu \right|, \ \ |\sigma_\tau| = -\mu \sigma_\nu, \\ \sigma_\tau = -\lambda \left(\dot{u}_\tau - v^* \right), \ \lambda \geqslant 0, \ \dot{\omega} = -k v^* \sigma_\nu, \ k > 0 \ \text{ on } \ \Gamma_3 \times [0.T], \end{cases} \tag{2.8}$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0.T], \tag{2.9}$$

$$D\nu = q_2 \text{ on } \Gamma_b \times [0.T], \tag{2.10}$$

$$-k_{ij} \frac{\partial \theta}{\partial \nu} \nu_j = k_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|) \text{ on } \Gamma_3 \times [0.T], \tag{2.11}$$

$$\theta = 0 \ \text{ in } \Gamma_1 \cup \Gamma_2 \times [0.T], \tag{2.12}$$

$$u(0) = u_0, \ v(0) = v_0, \ \theta(0) = \theta_0, \ \omega(0) = \omega_0 \ \text{ in } \ \Omega. \tag{2.13}$$

Where (2.1), (2.2) are represent the thermo- electro-viscoelastic constitutive law of the material in which $\sigma = (\sigma_{ij})$ denotes the stress tensor, we denote $\varepsilon(u)$ (respectively; $E(\varphi) = -\nabla \varphi$, $\mathscr{A}, \mathscr{G}$, ξ, ξ^*, β, $\mathscr{M} = (m_{ij})$, $\mathfrak{p} = (p_i)$) the linearized strain tensor (respectively; electric field, the elasticity tensor, the viscosity nonlinear tensor, the third order piezoelectric tensor and its transpose, the electric permittivity tensor thermal expansion, pyroelectric tensor),the constant θ^*represents the reference temperature; (2.3) is represents the equation of motion where ρ represents the mass density; (2.4) is represents the equilibrium equation, we mention that $Div\sigma$, $divD$ are the divergence operators; (2.5) is represents the evolution equation of the heat field; (2.6) and (2.7) are are the displacement and traction boundary conditions; (2.8) is describes the frictional bilateral contact with wear described above on the potential contact surface Γ_3; (2.9), (2.10) are represent the electric boundary conditions; (2.11) is pointwise heat exchange condition on the contact surface, where k_{ij} are the components of the thermal conductivity tensor, v_j are the normal components of the outward unit normal v; k_e is the heat exchange coefficient, θ_R is the known temperature of the foundation; (2.12) represents the temperature boundary conditions. Finally, (2.13) is the initial data.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we itroduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g: $u_{i.j} = \dfrac{\partial u_i}{\partial x_j}$. We also use the following notations

$$H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\},$$
$$\mathscr{H} = \left\{\sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\right\},$$
$$H_1 = \{u = (u_i)/\varepsilon(u) \in \mathscr{H}\} = H^1(\Omega)^d,$$
$$\mathscr{H}_1 = \{\sigma \in \mathscr{H}/Div\sigma \in H\}.$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathscr{H}, H_1$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(u,v)_H &= \int_\Omega u_i v_i dx \ \forall\, u,v \in H,\\
(\sigma,\tau)_{\mathscr{H}} &= \int_\Omega \sigma_{ij}\tau_{ij} dx \ \forall\, \sigma,\tau \in \mathscr{H},\\
(u,v)_{H_1} &= (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathscr{H}} \ \forall\, u,v \in H_1,\\
(\sigma,\tau)_{\mathscr{H}_1} &= (\sigma,\tau)_{\mathscr{H}} + (Div\sigma, Div\tau)_H,\ \sigma,\tau \in \mathscr{H}_1.
\end{aligned}$$

We denote by $|\cdot|_H$ (respectively;$|\cdot|_{\mathscr{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathscr{H}1}$) the associated norm on the space H (respectively; $\mathscr{H}$, H_1 and $\mathscr{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular (say $\mathscr{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote it trace on Γ. We recall that the summation convention applies to a repeated index.

For the electric displacement field we use two Hilbert spaces

$$\mathscr{W} = \mathbb{L}^2(\Omega)^d, \quad \mathscr{W}_1 = \left\{D \in \mathscr{W}, \operatorname{div} D \in \mathbb{L}^2(\Omega)\right\}.$$

Endowed with the inner products

$$(D,E)_{\mathscr{W}} = \int_\Omega D_i E_i dx, \quad (D,E)_{\mathscr{W}_1} = (D,E)_{\mathscr{W}} + (\operatorname{div} D, \operatorname{div} E)_{\mathbb{L}^2(\Omega)}.$$

And the associated norm $|.|_{\mathscr{W}}$(respectively; $|.|_{\mathscr{W}_1}$). The electric potential field is to be found in

$$W = \left\{\psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_a\right\}.$$

Since $meas\ (\Gamma_a) > 0$, the following Friedrichs–Poincaré's inequality holds, thus

$$|\nabla\psi|_{\mathscr{W}} \geqslant c_F\, |\psi|_{H^1(\Omega)}\, \forall\, \psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by

$$(\varphi,\psi)_W = (\nabla\varphi, \nabla\psi)_{\mathscr{W}},$$

and let $|.|_W$ be the associated norm. It follows from (3.1) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace Theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\psi|_{\mathbb{L}^2(\Gamma_3)} \leqslant \widetilde{c}_0\, |\psi|_W \ \ \forall\psi \in W. \tag{3.2}$$

We define the space

$$E = \{\gamma \in H^1(\Omega)/\gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{3.3}$$

We recall that when $D \in \mathscr{W}_1$ is a sufficiently regular function,the Green's type formula holds

$$(D, \nabla\psi)_{\mathscr{W}} + (\operatorname{div} D, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da\,. \tag{3.4}$$

When σ is a regular function, the following Green 's type formula holds

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1. \tag{3.5}$$

Next, we define the space

$$V = \{u \in H_1/\ u = 0 \ \text{ on } \Gamma_1\}.$$

Since $meas\ (\Gamma_1) > 0$, the following Korn's inequality holds

$$|\varepsilon(u)|_{\mathscr{H}} \geqslant c_K\, |v|_{H_1}\, \forall v \in V, \tag{3.6}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \tag{3.7}$$

let $|.|_V$ be the associated norm. It follows by (3.6) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leqslant c_0\, |v|_V\, \forall v \in V. \tag{3.8}$$

Finally, for a real Banach space $(X, |.|_X)$ we use the usual notation for the space $\mathbb{L}^p\,(0.T, X)$ and $W^{k.p}\,(0.T, X)$, where $1 \leqslant p \leqslant \infty$, $k = 1, 2, \ldots$; we also denote by $C\,(0.T, X)$ and $C^1\,(0.T, X)$ the spaces of continuous and continuously differentiable function on $[0.T]$ with values in X, with the respective norms:

$$|x|_{C(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X\,,$$

$$|x|_{C^1(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X + \max_{t\in[0.T]} |\dot{x}\,(t)|_X\,.$$

In what follows,we assume the following assumptions on the problem P.
The elasticity operator $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} (a)\ \exists\ L_{\mathscr{A}} > 0 \text{ such that } : |\mathscr{A}\,(x, \varepsilon_1) - \mathscr{A}\,(x, \varepsilon_2)| \leqslant L_{\mathscr{A}}\, |\varepsilon_1 - \varepsilon_2|\ \ \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \quad a.\ e.\ x \in \Omega, \\ (c)\ \text{ the mapping } x \to \mathscr{A}\,(x, \varepsilon) \text{ is lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d)\ \text{ the mapping } x \to \mathscr{A}\,(x, 0) \in \mathscr{H}. \end{cases} \tag{3.9}$$

The viscosity operator $\mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \exists\ L_{\mathscr{G}} > 0 : |\mathscr{G}(x, \varepsilon_1) - \mathscr{G}(x, \varepsilon_2)| \leqslant L_{\mathscr{G}} |\varepsilon_1 - \varepsilon_1|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ p.p.\ x \in \Omega, \\ (b)\ \exists m_{\mathscr{G}} > 0 : (\mathscr{G}\,(x, \varepsilon_1) - \mathscr{G}\,(x, \varepsilon_2)\,, \varepsilon_1 - \varepsilon_2) \geqslant m_{\mathscr{G}}\, |\varepsilon_1 - \varepsilon_2|^2\,, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ (c)\ \text{the mapping } x \to \mathscr{G}\,(x, \varepsilon) \text{ is lebesgue measurable in } \Omega \text{ fo rall } \varepsilon \in \mathbb{S}^d, \\ (d)\ \text{the mapping } x \mapsto \mathscr{G}(x, 0) \in\ \mathscr{H}. \end{cases} \tag{3.10}$$

The thermal expansion tensor $\mathscr{M} = (m_{ij}) : \Omega \times \mathbb{R} \to \mathbb{R}$, and the pyroelectric tensor $\mathfrak{p} = (p_i) : \Omega \to \mathbb{R}^d$ satisfy,

$$\begin{cases} (a) \quad m_{ij} = m_{ji} \in \mathbb{L}^\infty(\Omega), \\ (b) \quad p_i \in \mathbb{L}^\infty(\Omega). \end{cases} \tag{3.11}$$

The piezoelectric tensor $\xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ \xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d, \\ (b)\ \xi(x,\tau) = (e_{ijk}(x)\,\tau_{jk})\ \ \forall \tau = (\tau_{ij}) \in S^d,\ a.e.\, x \in \Omega, \\ (c)\ e_{ijk} = e_{ikj} \in \mathbb{L}^\infty(\Omega). \end{cases} \tag{3.12}$$

The electric permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} (a)\ \beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ (b)\ \beta(x,E) = (b_{ij}(x)\,E_j)\ \forall E = (E_i) \in \mathbb{R}^d,\ a.e.\, x \in \Omega, \\ (c)\ b_{ij} = b_{ji} \in \mathbb{L}^\infty(\Omega), \\ (d)\ \exists\ m_\beta > 0 \text{ such that : } b_{ij}(x)\,E_iE_j \geqslant m_\beta\,|E|^2\ \ \forall E = (E_i) \in \mathbb{R}^d,\ x \in \Omega. \end{cases} \tag{3.13}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a)\ \ \exists\ L_\tau > 0\ :\ |h_\tau(x,r_1) - \ h_\tau(x,r_2)| \leqslant L_h|r_1 - r_2|\ \forall\, r_1, r_2 \in \mathbb{R}_+,\ \ p.p.\, x \in \Gamma_3, \\ (b)\ \ x \longrightarrow h_\tau(x,r) \in \mathbb{L}^2(\Gamma_3) \text{ is lebesgue measurable in}\, \Gamma_3\ \forall r \in \mathbb{R}_+. \end{cases} \tag{3.14}$$

The mass density ρ satisfie

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geqslant \rho^*,\ a.e.\, x \in \Omega. \tag{3.15}$$

The body forces, surface tractions,the densities of electric charges, and the functions α and μ, satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0.T, H),\ h \in \mathbb{L}^2(0.T, \mathbb{L}^2(\Gamma_2)^d), \\ q_0 \in L^2(0.T, \mathbb{L}^2(\Omega)),\ q_2 \in L^2(0.T, \mathbb{L}^2(\Gamma_b)),\ q_1 \in L^2(0.T, \mathbb{L}^2(\Omega)),\ k_e \in \mathbb{L}^\infty(\Omega, \mathbb{R}_+). \\ \begin{cases} K = (k_{i,j});\ k_{ij} = k_{ji} \in \mathbb{L}^\infty(\Omega), \\ \forall c_k > 0, \forall (\xi_i) \in \mathbb{R}^d,\ k_{ij}\xi_i\xi_j \geqslant c_k\xi_i\xi_i. \end{cases} \\ \alpha \in \mathbb{L}^\infty(\Gamma_3),\ \alpha(x) \geqslant \alpha^* > 0,\ a.e. \text{ on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3),\ \mu(x) > 0,\ a.e. \text{ on } \Gamma_3. \end{cases} \tag{3.16}$$

The initial data satisfy

$$u_0 \in V, \theta_0 \in \mathbb{L}^2(\Omega),\ \ \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{3.17}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u,v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}\, \forall\, u, v \in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}\ \forall v \in H.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u,v)_{V' \times V} = ((u,v))\, \forall u \in H\, \forall v \in V.$$

It follows from assumption (3.15) that$\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$.

We define the function $f(t) \in V$ and $q : [0.T] \to W$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da \, \forall v \in V, \;\; t \in [0.T].$$

$$(q(t), \psi)_W = -\int_\Omega q_0(t) \psi dx + \int_{\Gamma_b} q_2(t) \psi da \, \forall \psi \in W, \;\; t \in [0.T].$$

for all $u,\, v \in V, \;\; \psi \in W$ and $t \in [0.T]$, and note that condition (3.14) imply that

$$f \in \mathbb{L}^2(0.T, V'), \quad q \in \mathbb{L}^2(0.T, W), \tag{3.18}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u, v) = \int_{\Gamma_3} \alpha \left|u_\nu\right| (\mu \left|v_\tau - v^*\right|) da \tag{3.19}$$

Finally, We consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u, v) = \int_{\Gamma_3} \alpha \left|u_\nu\right| v_\nu da \, \forall v \in V. \tag{3.20}$$

We define for all $\varepsilon > 0$

$$j_\varepsilon(g, v) = \int_{\Gamma_3} \alpha \left|g_\nu\right| (\mu \sqrt{|v_\tau - v^*|^2 + \varepsilon^2}) da \, \forall v \in V.$$

We define $Q : [0, T] \to E'$; $K : E \to E'$ and $R : V \to E'$ by

$$(Q(t), \mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \mu ds + \int_\Omega q \mu dx \;\; \forall \mu \in E, \tag{3.21}$$

$$(K\tau, \mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu ds \;\; \forall \mu \in E, \tag{3.22}$$

$$(Rv, \mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu dx - \int_\Omega (\mathscr{M}.\nabla v) \mu dx \;\; \forall v \in V, \tau, \mu \in E. \tag{3.23}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV : Find a displacement field $u : \Omega \times [0.T] \to V$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0.T]\mathbb{R}^d$, a temperature field θ: $\Omega \times [0.T] \to \mathbb{R}$, and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\begin{aligned}(\ddot{u}(t), w - \dot{u}(t))_{V' \times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathscr{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \phi(\dot{u}, w) - \phi(\dot{u}, \dot{u}(t)) \geqslant \\ \geqslant \big(f(t), w - \dot{u}(t)\big) \;\forall\, u, w \in V,\end{aligned} \tag{3.24}$$

$$(D(t), \nabla \psi)_{\mathbb{L}^2(\Omega)^d} + (q(t), \psi)_W = 0 \,\forall\, \psi \in W, \tag{3.25}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \text{ on } E', \tag{3.26}$$

$$\dot{\omega} = -kv^* \sigma_\nu. \tag{3.27}$$

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem are the following.

Theorem 4.1. *Let the assumptions* (3.9) − (3.17) *hold. Then, Problem PV has a unique solution* $(u, \sigma, \varphi, D, \omega)$ *which satisfies*

$$u \in C^1(0.T, H) \cap W^{1.2}(0.T, V) \cap W^{2.2}(0.T, V'), \tag{4.1}$$

$$\sigma \in \mathbb{L}^2(0.T, \mathscr{H}_1), \;\; Div\sigma \in \mathbb{L}^2(0.T, V'), \tag{4.2}$$

$$\varphi \in W^{1.2}(0.T, W), \tag{4.3}$$

$$D \in W^{1.2}(0.T, \mathscr{W}_1), \tag{4.4}$$

$$\theta \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)), \tag{4.5}$$

$$\omega \in C^1(0.T, \mathbb{L}^2(\Gamma_3)). \tag{4.6}$$

We conclude that under the assumptions (3.9)–(3.17), the mechanical problem (2.1)–(2.13) has a unique weak solution with the regularity (4.1)–(4.6).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0.T; V)$ and $\eta \in \mathbb{L}^2(0.T; V')$ are given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta} : [0.T] \to V$ such that

$$\begin{cases} \big(\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t)\big)_{V'\times V} + \big(\mathscr{G}\varepsilon(\dot{u}_{g\eta}(t)), w - \dot{u}_{g\eta}(t)\big)_{V'\times V} + (\eta(t), w - \dot{u}_{g\eta})_{V'\times V} j(g, w) - \\ \quad -j(g, \dot{u}_{g\eta}(t)) \geqslant (f(t), \, w - \dot{u}_{g\eta}(t))_{V'\times V} \;\; \forall \, w \in V, \;\; t \in [0.T], \\ u_{g\eta}(0) = u_0, \;\; \dot{u}_{g\eta}(0) = u_1. \end{cases} \tag{4.7}$$

We define $f_\eta(t) \in V$ for $a.e. \, t \in [0.T]$ by

$$(f_\eta(t), w)_{V'\times V} = (f(t) - \eta(t), w)_{V'\times V} \, \forall w \in V. \tag{4.8}$$

From (3.18), we deduce that

$$f_\eta \in \mathbb{L}^2(0.T, V'). \tag{4.9}$$

Let now $u_{g\eta} : [0.T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s) ds + u_0 \, \forall t \in [0.T]. \tag{4.10}$$

We define the operator $G : V' \to V$ by

$$(Gv, w)_{V'\times V} = (\mathscr{G}\varepsilon(v(t)), \varepsilon(w))_{\mathscr{H}} \, \forall \, v, w \in V. \tag{4.11}$$

Lemma 4.2. *For all* $g \in \mathbb{L}^2(0.T, V)$ *and* $\eta \in \mathbb{L}^2(0.T, V')$, $PV_{g\eta}$ *has a unique solution with the regularity*

$$v_{g\eta} \in C(0.T, H) \cap \mathbb{L}^2(0.T, V) \;\; and \;\; \dot{v}_{g\eta} \in \mathbb{L}^2(0.T, V'). \tag{4.12}$$

Proof. The proof from nonlinear first order evolution inequalities, given in Refs (see [5, 10]). □

In the second step, we use the displacement field $u_{g\eta}$ to consider the following variational problem.

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.

Problem $P_{\theta g\eta}$: Find $\theta_{g\eta} \in E$ such that

$$\dot{\theta}_{g\eta}(t) + K\theta_{g\eta}(t) = R\dot{u}_{g\eta}(t) + Q(t) \text{ on } E^{'}. \tag{4.13}$$

Lemma 4.3. *Under the assumptions* (3.9)–(3.17)*, the problem* $P_{\theta g\eta}$ *has a unique solution*

$$\theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Proof. Since we have the Gelfand triple $E \subset \mathbb{L}^2(\Omega) \subset E^{'}$. We use a classical result on first order evolution equations given in [11] to prove the unique solvability of (4.13). Now, we have $\theta_0 \in \mathbb{L}^2(\Omega)$. The operator K is a linear and continuous, so $a(\tau, \mu) = (K\tau, \mu)_{E'\times E}$ is bilinear, continuous and coercive, we use the continuity of $a(.,.)$ and from (3.16), we deduce that

$$a(\tau, \mu) = (K\tau, \mu)_{E'\times E} \leqslant |k|_{\mathbb{L}^\infty(\Omega)^{d\times d}} |\nabla\tau|_E |\nabla\mu|_E + |k_e|_{\mathbb{L}^\infty(\Gamma_3)} |\tau|_{\mathbb{L}^2(\Gamma_3)} |\mu|_{\mathbb{L}^2(\Gamma_3)} \leqslant C |\tau|_E |\mu|_E .$$

We have

$$a(\tau, \tau) = (K\tau, \tau)_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \tau}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau^2 ds.$$

By (3.16) there exists a constants $C > 0$ such that

$$(K\tau, \tau)_{E'\times E} \geqslant C |\tau|_E^2 .$$

We have $\theta_0 \in \mathbb{L}^2(\Omega)$. Let

$$F(t) \in E' : \ (F(t), \tau)_{E'\times E} = (R\dot{u}_{g\eta}(t) + Q(t), \tau) \ \ \forall \tau \in E.$$

Under the assumptions(3.14), (3.16) we have

$$\int_0^T |R\dot{u}|_{E'}^2 \, dt < \infty, \quad \int_0^T |Q(t)|_{\mathbb{E}'}^2 \, dt < \infty, \quad \int_0^T |F|_{E'}^2 \, dt < \infty.$$

We find

$$F \in \mathbb{L}^2(0.T, E^{'}).$$

By a classical result on first order evolution equations

$$\exists\, !\theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

□

Third step: We use the displacement field $u_{g\eta}$ and the temperature field $\theta_{g\eta}$ to consider the following variational problem.

Problem $PV_{g\eta}^{\varphi}$: Find an electric potential field $\varphi_{g\eta} : \Omega \times [0.T] \to W$ such that

$$\begin{aligned}(\beta\nabla\varphi_{g\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - \big((\theta_{g\eta}(t) - \theta^*_{g\eta}(t))\mathfrak{p}_{\mathrm{i}}, \nabla\psi\big)_{\mathbb{L}^2(\Omega)^d} = \\ = (q(t), \psi)_W \ \forall\psi \in W, \ t \in [0.T].\end{aligned} \tag{4.14}$$

We have the following result for $PV_{g\eta}^{\varphi}$

Lemma 4.4. *There exists a unique solution* $\varphi_{g\eta} \in W^{1.2}(0.T, W)$ *satisfies* (4.14), *moreover if* φ_1 *and* φ_2 *are two solutions to* (4.14). *Then, there exists a constants* $c > 0$ *sach that*

$$|\varphi_1(t) - \varphi_2(t)|_W \leqslant c\left(|u_1(t) - u_2(t)|_V + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}\right) \ \forall t \in [0.T]. \tag{4.15}$$

Proof. Let $t \in [0.T]$, we use the Riesz-fréchet representation theorem to define the operator $A_{g\eta} : W \to W$ by

$$\begin{aligned}(A_{g\eta}(t)\varphi, \psi)_W = (\beta\nabla\varphi_{g\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - \\ - \left((\theta_{g\eta}(t) - \theta^*_{g\eta}(t))\mathfrak{p}_{\mathrm{i}}, \nabla\psi\right)_{\mathbb{L}^2(\Omega)^d} \ \forall\, t \in [0.T]\,.\end{aligned} \tag{4.16}$$

For all φ, $\psi \in W$. Let φ_1, $\varphi_2 \in W$, then assumptions (3.11)–(3.13) imply

$$(A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_1, \varphi_1 - \varphi_2)_W \geqslant m_\beta\, |\varphi_1 - \varphi_2|^2_W\,. \tag{4.17}$$

In other hand, from (3.11)–(3.13), it results

$$(A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_1, \psi)_W \leqslant c_\beta\, |\varphi_1 - \varphi_2|_W\, |\psi|\, t_W,$$

where c_β is a positive constant which depends on β.

Thus

$$|A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_2|_W \leqslant c_\beta\, |\varphi_1 - \varphi_2|_W\,. \tag{4.18}$$

Inequalities (4.17) and (4.18) show that the operator $A_{g\eta}(t)$ is a strongly monotone, Lipschits continuous operator on W and, therefore, there exists a unique element $\varphi_{g\eta}\,(t) \in W$ such that

$$A_{g\eta}\varphi_{g\eta}(t) \ = q\,(t)\,. \tag{4.19}$$

We combine (4.16) and (4.17) and find that $\varphi_{g\eta}(t) \in W$ is the unique solution of the nonlinear variational equation (4.14).

We show that $\varphi_{g\eta} \in W^{1.2}(0.T, W)$. To this end, let $t_1, t_2 \in [0.T]$ and, for the sake of simplicity, we write $\varphi_{g\eta}\,(t_i) = \varphi_i$, $u_{g\eta}\,(t_i) = u_i$, $\theta_{g\eta}(t_i) = \theta_i$, $q\,(t_i) = q_i$, for $i = 1.2$.

From (4.14), (3.11)–(3.13) it results

$$\begin{aligned}m_\beta\, |\varphi_1 - \varphi_2|^2_W \leqslant \\ \leqslant c(|u_1 - u_2|_V\, |\varphi_1 - \varphi_2|_W + |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)}\, |\varphi_1 - \varphi_2|_W + |q_1 - q_2|_W\, |\varphi_1 - \varphi_2|_W).\end{aligned} \tag{4.20}$$

We find

$$\begin{aligned}|\varphi_1\,(t) - \varphi_2\,(t)|_W \leqslant \\ \leqslant c\left(|u_1\,(t) - u_2\,(t)|_V + |\theta_1\,(t) - \theta_2\,(t)|_{\mathbb{L}^2(\Omega)} + |q_1\,(t) - q_2\,(t)|_W\right) \ \forall t \in [0.T]\,.\end{aligned} \tag{4.21}$$

We also note that assumption (3.16), combined with definition imply that $q \in W^{1.2}\,(0.T, W)$. Since $u_{g\eta} \in C^1\,(0.T, V)$, $\theta_{g\eta} \in C^1\,(0.T, E)$, inequality (4.21) implies that $\varphi_{g\eta} \in W^{1.2}\,(0.T, W)$.

Let: $\eta_1, \eta_2 \in C\,(0.T, V')$, $g_1, g_2 \in C\,(0.T, V)$ and let $\varphi_{g\eta}\,(t_i) = \varphi_i$, $u_{g\eta}\,(t_i) = u_i$, we use (4.20) and arguments similar to those used in the proof of (4.15) to obtain

$$m_\beta\, |\varphi_1 - \varphi_2|_W \leqslant c(|u_1 - u_2|_V + |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)}).$$

For all $t \in [0.T]$. This inequality leads to (4.15) which concludes the proof. □

Consider the operator

$$\begin{aligned}
&\Lambda : \mathbb{L}^2(0.T, V\times V') \to \mathbb{L}^2(0.T, V\times V')\\
&\Lambda(g,\eta) = (\Lambda_1(g), \Lambda_2(\eta)), \forall g \in \mathbb{L}^2(0.T, V), \forall \eta \in \mathbb{L}^2(0.T, V'),\\
&\Lambda_1(g) = v_{g\eta},\\
&(\Lambda_2(\eta), w)_{V'\times V} = (\mathscr{A}(\varepsilon(u(t))) - \theta\mathscr{M} - \xi^* E(\varphi), \varepsilon(w))_{\mathscr{H}} + \phi(g,w),\\
&|\Lambda(g_2,\eta_2) - \Lambda(g_1,\eta_1)|^2_{\mathbb{L}^2(0.T;V\times V')} = |(\Lambda_1(g_2),\Lambda_2(\eta_2)) - (\Lambda_1(g_1),\Lambda_2(\eta_1))|^2_{\mathbb{L}^2(0.T;V\times V')} =\\
&= |\Lambda_1(g_2) - \Lambda_1(g_1)|^2_{\mathbb{L}^2(0.T;V\times V')} + |\Lambda_2(\eta_2) - \Lambda_2(\eta_1)|^2_{\mathbb{L}^2(0.T;V\times V')}.
\end{aligned} \tag{4.22}$$

We have the following result.

Lemma 4.5. *The mapping* $\Lambda : \mathbb{L}^2(0.T, V\times V') \to \mathbb{L}^2(0.T, V\times V')$ *has a unique element* $(g^*,\eta^*) \in \mathbb{L}^2(0.T, V\times V')$*, such that*

$$\Lambda(g^*,\eta^*) = (g^*,\eta^*). \tag{4.23}$$

Proof. Let $(g_i,\eta_i) \in \mathbb{L}^2(0.T, V\times V')$. We use the notation (u_i,φ_i). For $(g,\eta) = (g_i,\eta_i)$, $i = 1.2$. Let $t \in [0.T]$.

We have

$$\Lambda_1(g) = v_{g\eta}. \tag{4.24}$$

So

$$|g_1(t) - g_2(t)|^2_V \leqslant |v_1(t) - v_2(t)|^2_V. \tag{4.25}$$

It follows that

$$\begin{aligned}
&(\dot v_1(t) - \dot v_2(t), v_1(t) - v_2(t)) + (\mathscr{G}\varepsilon(v_1(t)) - \mathscr{G}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) +\\
&+ (\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \leqslant 0
\end{aligned} \tag{4.26}$$

From the definition of the functional j given by (3.19), and using (3.8), (3.16) we have

$$j(g_2, v_2(t)) - j(g_2, v_1(t)) - j(g_1, v_2(t)) + j(g_1, v_1(t)) \leqslant C\,|g_1 - g_2|_V\,|v_1 - v_2|_V \tag{4.27}$$

Integrating the (4.26) inequality with respect to time, using the initial conditions $v_2(0) = = v_1(0) = v_0$, using (3.8), (3.10), (4.26) using Cauchy–Schwartz's inequality and the inequality $2ab \leqslant \dfrac{C}{m_{\mathscr{G}}}a^2 + \dfrac{m_{\mathscr{G}}}{C}b^2$ et $2ab \leqslant \dfrac{1}{m_{\mathscr{G}}}a^2 + m_{\mathscr{G}}b^2$, by Gronwall's inequality we find

$$|v_1(t) - v_2(t)|^2_V \leqslant C\bigg(\int_0^t |g_1(s) - g_2(s)|^2_V\,ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'}\,ds\bigg). \tag{4.28}$$

So

$$|g_1 - g_2|^2_V \leqslant C\bigg(\int_0^t |g_1(s) - g_2(s)|^2_V\,ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'}\,ds\bigg). \tag{4.29}$$

And, we have

$$(\Lambda_2(\eta), w)_{V'\times V} = \bigg(\mathscr{A}(\varepsilon(u(t))) - \theta\mathscr{M} - \xi^* E(\varphi), \varepsilon(w)\bigg)_{\mathscr{H}} + \phi(g,w). \tag{4.30}$$

From the definition of the functional ϕ given by (3.20), and using (3.8), (3.16) we have

$$\phi(g_1, v_2(t)) - \phi(g_1, v_1(t)) - \phi(g_2, v_2(t)) + \phi(g_2, v_1(t)) \leqslant C\,|g_1 - g_2|_V\,|v_1 - v_2|_V. \tag{4.31}$$

So

$$\begin{aligned}
|\eta_1(t) - \eta_2(t)|^2_{V'} \leqslant C(&|u_1(t) - u_2(t)|^2_V + \textstyle\int_0^t |u_1(s) - u_2(s)|^2_V\,ds +\\
&+ |\varphi_1(t) - \varphi_2(t)|^2_W + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} + |g_1(t) - g_2(t)|^2_V).
\end{aligned} \tag{4.32}$$

By (4.26), using the inequality $2ab \leqslant \frac{2C}{m_{\mathscr{G}}}a^2 + \frac{m_{\mathscr{G}}}{2C}b^2$ and $2ab \leqslant \frac{2}{m_{\mathscr{G}}}a^2 + \frac{m_{\mathscr{G}}}{2}b^2$, we find

$$\begin{aligned}&\frac{1}{2}\left|v_1(t)-v_2(t)\right|_V^2 + m_{\mathscr{G}}\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds \leqslant \frac{1}{m_{\mathscr{G}}}\int_0^t \left|\eta_1(s)-\eta_2(s)\right|_{V'}^2 ds + \\ &+ \frac{m_{\mathscr{G}}}{4}\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds + C\times\frac{C}{m_{\mathscr{G}}}\int_0^t \left|g_1(s)-g_2(s)\right|_V^2 ds + \\ &+ C\times\frac{m_{\mathscr{G}}}{4C}\textstyle\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds.\end{aligned} \tag{4.33}$$

So

$$\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds \leqslant C\Big(\int_0^t \left|\eta_1(s)-\eta_2(s)\right|_{V'}^2 ds + \int_0^t \left|g_1(s)-g_2(s)\right|_V^2 ds\Big). \tag{4.34}$$

By (3.26), we find

$$\begin{aligned}\Big(\dot{\theta}_1(t)-\dot{\theta}_2(t), \theta_1(t)-\theta_2(t)\Big)_{E'\times E} + (K(\theta_1)-K(\theta_2), \theta_1(t)-\theta_2(t))_{E'\times E} = \\ = (R(v_1)-R(v_2), \theta_1(t)-\theta_2(t))_{E'\times E}.\end{aligned} \tag{4.35}$$

We integrate (4.35) over [0.T] we use the initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$, and we use the coercivety of K and the Lipschitz continuity of R to deduce that

$$\begin{aligned}&\frac{1}{2}\left|\theta_1(t)-\theta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 + C\int_0^t \left|\theta_1(s)-\theta_2(s)\right|_{\mathbb{L}^2(\Omega)}^2 ds \leqslant \\ &\leqslant C\left(\int_0^t \left|v_1(s)-v_2(s)\right|_V \left|\theta_1(s)-\theta_2(s)\right|_{\mathbb{L}^2(\Omega)} ds\right).\end{aligned}$$

using the inequality $2ab \leqslant \frac{1}{2}a^2 + 2b^2$, we find

$$\begin{aligned}&\frac{1}{2}\left|\theta_1(t)-\theta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 + C\int_0^t \left|\theta_1(s)-\theta_2(s)\right|_{\mathbb{L}^2(\Omega)}^2 ds \leqslant \\ &\leqslant \frac{C}{4}\int_0^t \left|v_1(s)-v_2(s)\right|_V ds + C\left|\theta_1(s)-\theta_2(s)\right|_{\mathbb{L}^2(\Omega)} ds.\end{aligned}$$

Also

$$\left|\theta_1(t)-\theta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 \leqslant C\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 ds. \tag{4.36}$$

By (4.34), we find

$$\left|\theta_1(t)-\theta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 \leqslant C\left(\int_0^t \left|\eta_1(s)-\eta_2(s)\right|_{V'}^2 ds + \int_0^t \left|g_1(s)-g_2(s)\right|_V^2 ds\right). \tag{4.37}$$

Also

$$\begin{aligned}&\left|u_1(t)-u_2(t)\right|_V^2 + \int_0^t \left|u_1(s)-u_2(s)\right|_V^2 ds \leqslant \\ &\leqslant C\Big(\int_0^t \left|v_1(s)-v_2(s)\right|_V^2 + \int_0^t \left|u_1(s)-u_2(s)\right|_V^2\Big)ds.\end{aligned} \tag{4.38}$$

And

$$\left|u_1(t)-u_2(t)\right|_V^2 \geqslant 0.$$

$$\int_0^t\int_0^s \left|u_1(r)-u_2(r)\right|_V^2 dr ds \geqslant 0.$$

So

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leqslant$$
$$\leqslant C \int_0^t (|v_1(s) - v_2(s)|_V^2 + |u_1(s) - u_2(s)|_V^2) ds + \int_0^t \int_0^s |u_1 - u_2|_V^2 \, dr ds,$$

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leqslant$$
$$\leqslant C \int_0^t (|v_1(s) - v_2(s)|_V^2 + |u_1(s) - u_2(s)|_V^2 + \int_0^s |u_1(r) - u_2(r)|_V^2 \, dr) ds$$

by Gronwall's inequality, and using (4.34) we have

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds \leqslant C \Big(\int_0^t |\eta_1(s) - \eta_2(s)|_{V'}^2 \, ds + \int_0^t |g_1(s) - g_2(s)|_V^2 \, ds \Big) \tag{4.39}$$

and using (4.29) and (4.32) we find

$$|\Lambda(g_1, \eta_1) - \Lambda(g_2, \eta_2)|_{\mathbb{L}^2(0.T; V \times V')}^2 \leqslant C \int_0^t |(g_1, \eta_1) - (g_2, \eta_2)|_{V \times V'}^2 \, ds. \tag{4.40}$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathbb{L}^2(0.T, V \times V')$ and so Λ has a unique fixed point in this Banach space. □

We consider the operator $\mathscr{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$,

$$\mathscr{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s) ds \, \forall t \in [0.T]. \tag{4.41}$$

Lemma 4.6. *The operator $\mathscr{L} : C(0.T, \mathbb{L}^2(\Gamma_3)) \to C(0.T, \mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$, such that*

$$\mathscr{L}\omega^* = \omega^*.$$

Proof. Using (4.41), we have

$$|\mathscr{L}\omega_1(t) - \mathscr{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \leqslant kv^* \int_0^t |\sigma_1(s) - \sigma_2(s)|_{\mathscr{H}}^2 \, ds. \tag{4.42}$$

From (2.1),we have

$$\begin{aligned} |\mathscr{L}\omega_1(t) - \mathscr{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \leqslant C \int_0^t (|u_1(t) - u_2(t)|_V^2 + \int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + \\ + |\varphi_1(s) - \varphi_2(s)|_W^2 + |\theta_1(s) - \theta_2(s)|_{\mathbb{L}^2(\Omega)}^2) dt. \end{aligned} \tag{4.43}$$

By (4.15) and (4.36), we find

$$\begin{aligned} |u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds + |\varphi_1(t) - \varphi_2(t)|_{\mathbb{L}^2(\Omega)}^2 + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant \\ \leqslant \int_0^t |v_1(s) - v_2(s)|_V^2 \, ds. \end{aligned} \tag{4.44}$$

So

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds + |\varphi_1(t) - \varphi_2(t)|_W^2 + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant$$
$$\leqslant C\left(\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds + |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \right).$$

So, we have

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2 \, ds + |\varphi_1(t) - \varphi_2(t)|_W^2 + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant$$
$$\leqslant C \, |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 . \tag{4.45}$$

By (4.43), we find

$$|\mathscr{L}\omega_1(t) - \mathscr{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leqslant C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} \, ds.$$

Thus, for m sufficiently large, $\mathscr{L}^m$ is a contraction on $C(0.T, \mathbb{L}^2(\Gamma_3))$ and so $\mathscr{L}$ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.

Existence

Let $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ be the fixed point of Λ defined by (4.22), let $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathscr{L}\omega^*$ defined by (4.41), and let $(u, \theta, \varphi) = (u_{g^*\eta^*}, \theta_{g^*\eta^*}, \varphi_{g^*\eta^*})$ be the solutions of Problems $PV_{g^*\eta^*}$, $P_{\theta g\eta}$ and $PV^{\varphi}_{g^*\eta^*}$. It results from (4.7), (4.13) and (4.16) that $(u_{g^*\eta^*}, \theta_{g^*\eta^*}, \varphi_{g^*\eta^*})$ is the solutions of Problems PV. Properties (4.1)–(4.6) follow from Lemmas 4.2, 4.3 and 4.4 .

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators $\Lambda, \mathscr{L}$ defined by (4.22), (4.41), and the unique solvability of the Problem $PV_{g\eta}$, $P_{\theta g\eta}$ and $PV^{\varphi}_{g\eta}$ which completes the proof.

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Проблема износа термоэлектровязкоупругих материалов

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Аннотация. В данной работе рассматривается математическая модель контактной задачи термоэлектровязкоупругости. Тело соприкасается с препятствием. Контакт фрикционный и двусторонний с подвижным жестким основанием, что приводит к износу контактирующей поверхности. Устанавливается вариационная формулировка модели и доказывается существование единственного слабого решения задачи. Доказательство основано на классическом факте существования и единственности параболических неравенств, дифференциальных уравнений и аргументов с фиксированной точкой. Приводится вариационная постановка задачи, доказывается существование и единственность слабого решения.

Ключевые слова: пьезоэлектрик, температура, термоэлектровязкоупругость, вариационное неравенство, износ.