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On the Multidimensional Boundary Analogue of the Morera Theorem

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Abstract. We discuss functions with the one-dimensional holomorphic extension property along complex lines and curves and also boundary multidimensional variants of the Morera theorem. We show how integral representations can be applied to the study of analytic continuation of functions, in particular to multidimensional boundary analogues of the Morera theorems.

Keywords: one-dimensional holomorphic extension property, multidimensional variants of the Morera theorem.

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This is a survey of the results related to the holomorphic extension of functions given on the boundary of a bounded domain $D \subset \mathbb{C}^n$, $n > 1$ to this domain. The subject is not new. Results about the extension of the Hartogs–Bochner type are well known and have already become classical. They are the subject of many monographs and surveys (see, for example, Aizenberg and Yuzhakov, Khenkin, Rudin, and many others [1–8]).

Here we will discuss functions with the one-dimensional holomorphic extension property along complex lines and curves and also boundary multidimensional variants of the Morera theorem. We desire to show how integral representations can be applied to the study of analytic continuation of functions, in particular to multidimensional boundary analogues of the Morera theorems.

On a complex plane $\mathbb{C}$ the results on functions with a one-dimensional holomorphic extension property are trivial, and the boundary Morera theorem is absent. Therefore most of the results of the paper are essentially multidimensional.

The so-called Morera property is weaker than the one-dimensional holomorphic extension property. The former consists in the vanishing of integrals of a given function over the intersection of the boundary of the domain with complex lines (complex planes).

1. The Bochner-Green integral representation

We consider an n-dimensional complex space $\mathbb{C}^n$ with the variables $z = (z_1, \dots, z_n)$. If z and w are points in $\mathbb{C}^n$, then we write

$$\langle z, w\rangle = z_1 w_1 + \cdots + z_n w_n, \qquad |z| = \sqrt{\langle z, \bar{z}\rangle},$$

where $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$. The topology in $\mathbb{C}^n$ is given by the metric $(z, w) \mapsto |z - w|$. If $z \in \mathbb{C}^n$, then

$$\operatorname{Re} z = (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n) \in \mathbb{R}^n, \qquad \operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n) \in \mathbb{R}^n.$$

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We write $\operatorname{Re} z_j = x_j$ and $\operatorname{Im} z_j = y_j$, i.e., $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Thus $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Orientation of $\mathbb{C}^n$ is determined by the coordinate order $(x_1, \dots, x_n, y_1, \dots, y_n)$. Accordingly, the volume form dv is given by

$$dv = dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n = dx \wedge dy = \left(\frac{i}{2}\right)^n dz \wedge d\bar{z} = \left(-\frac{i}{2}\right)^n d\bar{z} \wedge dz.$$

As usual, a function f on an open set $U \subset \mathbb{C}^n$ belongs to the space $\mathcal{C}^k(U)$, i.e., $f \in \mathcal{C}^k(U)$, if f is k times continuously differentiable in U as $0 \leqslant k \leqslant \infty$ and $\mathcal{C}^0(U) = \mathcal{C}(U)$. The space $\mathcal{O}(U)$ consists of those functions f that are holomorphic on the open set U. A function f belongs to the space $\mathcal{A}(U)$, if f is holomorphic in U and continuous on the closure $\overline{U}$, i.e., $f \in \mathcal{O}(U) \cap \mathcal{C}(\overline{U})$.

A domain D in $\mathbb{C}^n$ has a boundary ∂D of class $\mathcal{C}^k$ (we write $\partial D \in \mathcal{C}^k$), if

$$D = \{z \in \mathbb{C}^n : \rho(z) < 0\},$$

where ρ is the real-valued function of class $\mathcal{C}^k$ in some neighborhood of the closure of D, and the differential $d\rho \neq 0$ on ∂D. If $k = 1$, then we say that D is a domain with a smooth boundary. We will call the function ρ a defining function for the domain D. The orientation of the boundary ∂D is induced by the orientation of D.

A domain D with a piecewise-smooth boundary ∂D will be understood as a smooth polyhedron, that is, a domain of the form

$$D = \{z \in \mathbb{C}^n : \rho_j(z) < 0,\ j = 1, \dots, m\},$$

where the real-valued functions ρ_j are class $\mathcal{C}^1$ in some neighborhood of the closure $\overline{D}$, and for every set of distinct indices $j_1, \dots, j_s$ we have $d\rho_{j_1} \wedge \dots \wedge d\rho_{j_s} \neq 0$ on the set $\{z \in \mathbb{C}^n : \rho_{j_1}(z) = \dots = \rho_{j_s}(z) = 0\}$. It is well known that the Stokes's formula holds for such domains D and surfaces ∂D.

We denote a ball of radius $\varepsilon > 0$ with the center at the point $z \in \mathbb{C}^n$ by

$$B(z, \varepsilon) = \{\zeta \in \mathbb{C}^n : |\zeta - z| < \varepsilon\},$$

and its boundary by $S(z, \varepsilon)$, i.e., $S(z, \varepsilon) = \partial B(z, \varepsilon)$.

Consider the exterior differential form (the Bochner–Martinelli kernel) $U(\zeta, z)$ of type $(n, n-1)$ given by

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}}\, d\bar{\zeta}[k] \wedge d\zeta, \tag{1.1}$$

where $d\bar{\zeta}[k] = d\bar{\zeta}_1 \wedge \dots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \dots \wedge d\bar{\zeta}_n$, $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$. When $n = 1$, the form $U(\zeta, z)$ reduces to the Cauchy kernel $\dfrac{1}{2\pi i} \cdot \dfrac{d\zeta}{\zeta - z}$. It is clear that the form $U(\zeta, z)$ has the coefficients that are harmonic in $\mathbb{C}^n \setminus \{z\}$, and it is closed with respect to ζ, i.e., $d_\zeta U(\zeta, z) = 0$.

Let $g(\zeta, z)$ be a fundamental solution to the Laplace equation:

$$g(\zeta, z) = \begin{cases} -\dfrac{(n-2)!}{(2\pi i)^n} \cdot \dfrac{1}{|\zeta - z|^{2n-2}}, & n > 1, \\ \dfrac{1}{2\pi i} \ln |\zeta - z|^2, & n = 1. \end{cases} \tag{1.2}$$

Then

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k}\, d\bar{\zeta}[k] \wedge d\zeta = (-1)^{n-1} \partial_\zeta g \wedge \sum_{k=1}^{n} d\bar{\zeta}[k] \wedge d\zeta[k], \tag{1.3}$$

where the operator ∂ is given by

$$\partial = \sum_{k=1}^{n} d\zeta_k \frac{\partial}{\partial \zeta_k}.$$

We will write the Laplace operator Δ in the following form:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial \zeta_k \partial \bar{\zeta}_k} = \frac{1}{4} \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) = \frac{1}{4} \Delta^R.$$

If $\zeta_k = x_k + iy_k$, then

$$\frac{\partial}{\partial \zeta_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right), \qquad \frac{\partial}{\partial \bar{\zeta}_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right).$$

When $f \in \mathcal{C}^1(U)$, we define the differential form μ_f via

$$\mu_f = \sum_{k=1}^{n} (-1)^{n+k-1} \frac{\partial f}{\partial \bar{\zeta}_k} d\zeta[k] \wedge d\bar{\zeta}.$$

Theorem 1.1 (Green's formula in a complex form). *Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise-smooth boundary, and let $f \in \mathcal{C}^2(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \mu_f(\zeta) + \int_D g(\zeta, z) \Delta f(\zeta)\, d\bar{\zeta} \wedge d\zeta = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases} \tag{1.4}$$

where the integral in (1.4) converges absolutely. □

2. The Bochner-Martinelli integral representation

Let us formulate some consequences of the Bochner–Green formula (1.4) for various classes of functions f.

Corollary 2.1 (Bochner [9]). *Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise-smooth boundary, and let f be a harmonic function in D of class $\mathcal{C}^1(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \mu_f(\zeta) = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \overline{D}. \end{cases} \tag{2.1}$$

Corollary 2.2 (Koppelman [10]). *Let D be a bounded domain in $\mathbb{C}^n$ with a piecewise-smooth boundary, and let f be a function in $\mathcal{C}^1(\overline{D})$. Then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_D \bar{\partial} f(\zeta) \wedge U(\zeta, z) = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases} \tag{2.2}$$

where

$$\bar{\partial} = \sum_{k=1}^{n} d\bar{\zeta}_k \frac{\partial}{\partial \bar{\zeta}_k},$$

and the integral in (2.2) converges absolutely.

Formula (2.2) is the Bochner–Martinelli formula for smooth functions.

Corollary 2.3 (Bochner [9], Martinelli [11])**.** *If D is a bounded domain in $\mathbb{C}^n$ with a piecewise-smooth boundary, and f is a holomorphic function in D of class $\mathcal{C}(\overline{D})$, then*

$$\int_{\partial D} f(\zeta)U(\zeta,z) = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \overline{D}. \end{cases} \tag{2.3}$$

Formula (2.3) was obtained by Martinelli, and later independently by Bochner by different methods. It is the first integral representation for holomorphic functions in $\mathbb{C}^n$ where the integration is carried out over the whole boundary of the domain. By now this formula has become classical and found its place in many textbooks on multidimensional complex analysis (see, for example, [7, 8]).

Formula (2.3) reduces to Cauchy's formula when $n = 1$, but unlike to Cauchy's formula, the kernel in (2.3) is not holomorphic in z and ζ when $n > 1$. By splitting the kernel $U(\zeta, z)$ into real and imaginary parts, it is easy to show that

$$\int_{\partial D} f(\zeta)U(\zeta,z)$$

is the sum of the double-layer potential and the tangential derivative of a single-layer potential.

Consequently, the Bochner–Martinelli integral inherits some of the properties of the Cauchy integral and some of the properties of the double-layer potential. It differs from the Cauchy integral in not being a holomorphic function, and it differs from the double-layer potential in having a somewhat worse boundary behavior. At the same time, it establishes a relation between harmonic and holomorphic functions in $\mathbb{C}^n$ when $n > 1$.

Formula (2.2) implies the jump theorem for the Bochner–Martinelli integral.

Let D be a bounded domain in $\mathbb{C}^n$ with piecewise-smooth boundary, and let f be a function in $\mathcal{C}^1(\overline{D})$. We consider the Bochner–Martinelli integral

$$Mf(z) = \int_{\partial D} f(\zeta)U(\zeta,z), \quad z \notin \partial D. \tag{2.4}$$

We will write $M^+f(z)$ for $z \in D$ and $M^-f(z)$ for $z \notin \overline{D}$. The function $Mf(z)$ is a harmonic function for $z \notin \partial D$ and $Mf(z) = O(|z|^{1-2n})$ as $|z| \to \infty$.

Corollary 2.4. *Under these conditions the function M^+f has a continuous extension on $\overline{D}$, the function M^-f has a continuous extension on $\mathbb{C}^n \setminus D$ and*

$$M^+f(z) - M^-f(z) = f(z), \quad z \in \partial D. \tag{2.5}$$

Formula (2.5) is the simplest jump formula for the Bochner–Martinelli integral. There exist many jump theorems for different classes of functions: Hölder functions [12], continuous functions [13, 14], integrable functions [15, 16], distributions, hyperfunctions [17].

3. Functions with the Morera property along complex and real planes

Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected smooth boundary ∂D of class $\mathcal{C}^2$.

Definition 3.1. *We say that a continuous function f on ∂D ($f \in \mathcal{C}(\partial D)$) satisfies the Morera property (condition) along a complex plane l of dimension k, $1 \leqslant k \leqslant n-1$, if*

$$\int\limits_{\partial D \cap l} f(\zeta)\beta(\zeta) = 0 \tag{3.1}$$

for any differential form β of type $(k, k-1)$ with constant coefficients.

It is assumed that the plane l transversally intersects the boundary of the domain D. If l is a complex line intersecting ∂D transversally, then the Morera property along l consists of the equality

$$\int\limits_{\partial D \cap l} f(z+bt)dt = \int\limits_{\partial D \cap l} f(z_1 + b_1 t, \ldots, z_n + b_n t)dt = 0$$

for the given parametrization $\zeta = z + bt$ ($z, b \in \mathbb{C}^n$, $t \in \mathbb{C}$) of the complex line l.

Clearly, the boundary values of functions $F \in \mathcal{A}(D)$ (i.e., functions holomorphic in D and continuous in the closure of the domain $\overline{D}$) satisfy this property. Moreover, the same is true for continuous CR-functions f on ∂D. Recall that

Definition 3.2. *A function $f \in \mathcal{C}(\partial D)$ is called a CR-function on ∂D if*

$$\int_{\partial D} f(\zeta)\, \bar{\partial}\alpha(\zeta) = 0 \tag{3.2}$$

for all exterior differential forms α of type $(n, n-2)$ with coefficients of class $\mathcal{C}^\infty$ in the $\overline{D}$.

Conditions (3.2) are called the tangential Cauchy–Riemann equations.

The Hartogs–Bochner theorem, which is now classical, tells us that *any continuous function f on ∂D is a CR-function if and only if it is holomorphically extended to D up to a certain function $F \in \mathcal{A}(D)$ (the boundary of D is connected).*

Globevnik and Stout [18] considered the following inverse problem: let a function $f \in \mathcal{C}(\partial D)$ satisfy the Morera property (3.1) along any complex k-plane l intersecting ∂D transversally. Is it true that f is a CR-function on ∂D?

Obviously, the greater the dimension k of the complex plane, the weaker the Morera property along complex k-planes. Therefore, if the Morera property holds along all complex lines, so it does along all complex hyperplanes. The following theorem is the first sufficiently general assertion on the solution of this problem.

Theorem 3.1 (Globevnik, Stout [18]). *Let $1 \leqslant k \leqslant n-1$, and let a function $f \in \mathcal{C}(\partial D)$ satisfy the Morera property (3.1) along any complex k-plane l intersecting ∂D transversally, then f is a CR-function on ∂D (and, therefore, it is holomorphically continued to D by the Hartogs–Bochner theorem).*

A more precise analysis shows that Theorem 3.1 holds for real planes. By definition, the CR-dimension of a real plane l in $\mathbb{C}^n$ is the dimension of the maximal complex plane belonging to l. Denote by $\dim_R l$ and $\dim_{CR} l$ the real dimension of the plane l and the CR-dimension of l, respectively. Then, obviously,

$$\max(0, \dim_R l - n) \leqslant \dim_{CR} l \leqslant \left[\frac{\dim_R l}{2}\right].$$

A continuous function f on ∂D satisfies the Morera condition along a real k-dimensional plane l of CR-dimension p that transversally intersects the boundary ∂D if

$$\int\limits_{\partial D \cap l} f\beta = 0$$

for all $(k-p, p-1)$-differential forms β with constant coefficients.

Theorem 3.2 (Govekar [19]). *Let $2 \leqslant k \leqslant 2n-1$ and $\max(1, k-n) \leqslant p \leqslant [k/2]$. A continuous function f on ∂D is a CR-function if and only if f satisfies the Morera property along any k-dimensional plane l of CR-dimension p that intersects ∂D transversally.*

In particular, for real hypersurfaces, the previous theorem yields the following assertion

Theorem 3.3 (Govekar [19]). *A function $f \in (\partial D)$ is a CR-function on ∂D if and only if*

$$\int\limits_{\partial D \cap l} f\beta = 0$$

for all real hyperplanes l intersecting ∂D transversally and for all differential forms β of type $(n, n-2)$ with constant coefficients.

For complex k-planes l, we have $\dim_R l = 2k$ and $p = k$, therefore, Theorem 3.2 transforms into Theorem 3.1.

4. Functions with the Morera property and with the one-dimensional holomorphic extension property along complex lines

Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected smooth boundary ∂D of class $\mathcal{C}^2$. The classical Hartogs theorem asserts that *any function f is holomorphic in the domain D if its restriction to any complex line parallel to one of the coordinate complex lines is holomorphic.*

The following natural question arises: for which sets of complex one-dimensional cross sections of the domain does the existence of holomorphic continuations along the cross sections imply the existence of a holomorphic continuation to the whole domain?

Consider one-dimensional complex lines l of the form

$$l = \{\zeta \in \mathbb{C}^n : \ \zeta_j = z_j + b_j t, \ j = 1, \ldots, n, \ t \in \mathbb{C}\} \tag{4.1}$$

passing through a point $z \in \mathbb{C}^n$ in the direction of a vector $b \in \mathbb{CP}^{n-1}$ (the direction of b is determined with an accuracy of up to multiplication by a complex number $\lambda \neq 0$).

By Sard's theorem, for almost all $z \in \mathbb{C}^n$ and almost all $b \in \mathbb{CP}^{n-1}$, the intersection $l \cap \partial D$ is a finite set of piecewise-smooth curves (except for the degenerate case where $\partial D \cap l = \varnothing$).

Let us give the following definition.

Definition 4.1. *The function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along complex lines l of the form (4.1) if for any line l such that $\partial D \cap l \neq \varnothing$, there exists a function F having the following properties:*

1. *$F \in \mathcal{C}(\overline{D} \cap l)$,*
2. *$F = f$ on the set $\partial D \cap l$,*
3. *the function F is holomorphic at interior (with respect to the topology of l) points of the set $\overline{D} \cap l$.*

An analogous definition can be made for complex k-planes. Clearly, if a function f satisfies the holomorphic extension property along all complex k-planes, then it satisfies this property along complex lines. Therefore, in what follows, we restrict ourselves to consideration of this case. This property is a stronger property than the Morera property.

Theorem 4.1 (Stout [20]). *If a function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along complex lines of the form (4.1), then f is holomorphically extended into D.*

A narrower set of complex lines sufficient for the holomorphic continuation was considered by M. A. Agranovsky and A. M. Semenov [21].

Consider an open set $V \subset D$ and a set $\mathfrak{L}_V$ of complex lines intersecting this set.

Theorem 4.2 (Agranovsky, Semenov [21]). *If a function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along lines from the set $\mathfrak{L}_V$ for a certain open set $V \subset D$, then the function f is holomorphically extended into D.*

The strengthening of the previous results consists of assertions dealing with the boundary analogs of the Morera theorem, since they are completely implied by them. We now formulate the assertion belonging to J. Globevnik and E. L. Stout (a special case of Theorem 3.1).

Theorem 4.3 (Globevnik, Stout [18]). *Let a function $f \in \mathcal{C}(\partial D)$, and for almost all $z \in \mathbb{C}^n$ and almost all $b \in \mathbb{CP}^{n-1}$, let*

$$\int\limits_{\partial D \cap l} f(z + bt)\, dt = \int\limits_{\partial D \cap l} f(z_1 + b_1 t, \ldots, z_n + b_n t)\, dt = 0. \tag{4.2}$$

Then the function f is holomorphically extended into D to a function $F \in \mathcal{C}(\overline{D})$. (If $\partial D \cap l = \varnothing$, then the integral in (4.2) is assumed to be equal to zero.)

We note that without the connectedness condition of the boundary of the domain Theorem 4.3 is obviously false.

The problem of finding *sufficient* sets of complex lines $\mathfrak{L} = \{l\}$ for which condition (4.2) for $l \in \mathfrak{L}$ implies a holomorphic extension of the function f to D was posed by Globevnik, Stout [18]. For example, is a set $\mathfrak{L}_V$ of lines l intersecting a certain open set $V \subset D$ such a sufficient set?

Theorem 4.4 (Kytmanov, Myslivets [22, 23]). *Let k be a fixed nonnegative integer and let a function $f \in \mathcal{C}(\partial D)$. If, for almost all $z \in \mathbb{C}^n$ and almost all $b \in \mathbb{CP}^{n-1}$, the condition*

$$\int\limits_{\partial D \cap l} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^k\, dt = 0 \tag{4.3}$$

holds, then f is holomorphically extended to D.

For $k = 0$, we obtain Theorem 4.3.

Theorem 4.5 (Kytmanov, Myslivets [22, 23]). *For a fixed k and a function $f \in \mathcal{C}(\partial D)$, let condition (4.3) hold for almost all lines l (of the form (4.1)) intersecting an open set $V \subset D$ (or an open set $V \subset \mathbb{C}^n \setminus \overline{D}$), then the function f is holomorphically extended into D.*

Corollary 4.1. *Let A be an algebraic hypersurface in $\mathbb{C}^n$. If condition (4.3) for a function f holds for almost all complex lines l intersecting A, then the function f is holomorphically extended into D.*

5. Holomorphic extension along complex curves

Consider classes of complex curves $l_{z,b}$ of the following types:

Type 1: algebraic curves

$$l_{z,b} = \left\{ \zeta \in \mathbb{C}^n : \ \zeta_1 = z_1 + t^{k_1},\ \zeta_j = z_j + b_j t^{k_j}, j = 2, \ldots, n,\ t \in \mathbb{C} \right\},$$

where the constants $k_j \in \mathbb{N}$ are fixed, $j = 1, \dots, n$, and the vector $b = (1, b_2, \dots, b_n) \in \mathbb{C}^n$;
Type 2: complex curves of the form

$$l_{z,b} = \left\{\zeta \in \mathbb{C}^n : \ \zeta_1 = z_1 + t, \ \zeta_j = z_j + b_j t^{k_j} \chi_j(t), j = 2, \dots, n, \ t \in \mathbb{C}\right\},$$

where $\chi_j(t)$ are the entire holomorphic functions of the variable t, and, moreover these functions do not vanish at any point, $j = 2, \dots, n$;
Type 3: complex curves of the form

$$l_{z,b} = \left\{\zeta \in \mathbb{C}^n : \ \zeta_1 = z_1 + t^{k_1}, \ \zeta_j = z_j + b_j t^{k_j} \chi_j(t^{k_1}), j = 2, \dots, n, \ t \in \mathbb{C}\right\}, \tag{5.1}$$

where $\chi_j(\tau)$ are the entire complex functions of the variable τ that do not vanish at any point, $j = 2, \dots, n$.

The third class of curves also contains curves of the form

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \ \zeta_1 = z_1 + \varphi_1(t), \zeta_j = z_j + b_j \varphi_j(t), j = 2, \dots, n, \ t \in \mathbb{C}\},$$

where $\varphi_j(t)$ are the entire functions of the variable t having one zero of the first order at the point $t = 0$. Indeed, in this case, we can introduce a different parametrization taking the first function φ_1 as the parameter t.

Definition 5.1. *A function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along complex curves of the form $l_{z,b}$ if for any curve $l_{z,b}$ such that $\partial D \cap l_{z,b} \neq \varnothing$, there exists a function $F_{z,b}(t)$ having the following properties:*

1. *$F_{z,b} \in \mathcal{C}(\overline{D} \cap l_{z,b})$,*
2. *$F_{z,b} = f$ on the set $\partial D \cap l_{z,b}$,*
3. *the function $F_{z,b}$ is holomorphic with respect to t in interior (in the topology of $l_{z,b}$) points of the set $\overline{D} \cap l_{z,b}$.*

Therefore, this definition is completely analogous to that of functions with the one-dimensional holomorphic extension property along lines.

Theorem 5.1 (Kytmanov, Myslivets [24])**.** *Let $\partial D \in \mathcal{C}^2$, and a function $f \in \mathcal{C}(\partial D)$ has the one-dimensional holomorphic extension property along complex curves $l_{z,b}$, then f is holomorphically extended into D.*

This assertion generalizes Stout's Theorem 4.1 (see Theorem 2.1) on functions with the one-dimensional holomorphic extension property along complex lines.

Let us dwell on sufficient families of curves $l_{z,b}$ the holomorphic extension along which can ensure the holomorphic extension to the domain D. The first such family comprises the curves $l_{z,b}$ with the point z belonging to a certain open set $V \subset \mathbb{C}^n \setminus \overline{D}$, and b being any vector.

6. Morera theorem in classic domains

In this section, we consider the boundary variant of the Morera theorem for classic domains. The starting point of this theorem is the result of Nagel and Rudin [25], which says that if a function f is continuous on the boundary of a ball in $\mathbb{C}^N$ and the integral

$$\int_0^{2\pi} f(\psi(e^{i\varphi}, 0 \dots, 0))\, e^{i\varphi}\, d\varphi = 0$$

for all (holomorphic) automorphisms ψ of the ball, then the function f is holomorphically extended to the ball.

6.1. Classic domains

We recall definitions and introduce notation needed for further discussion. By a classic domain $D \subset \mathbb{C}^N$, we understand an irreducible bounded symmetric domain of several complex variables of one of the following four types:

1. The domain D_I is formed by matrices Z consisting of m rows and n columns (entries of matrices are complex numbers) and satisfying the condition
$$I^{(m)} - ZZ^* > 0.$$
Here, $I^{(m)}$ is the identity matrix of order m, $Z^* = \overline{Z}'$ is the matrix complex-conjugate to the transposed matrix Z', and, as usual, the inequality $H > 0$ for an Hermitian matrix H means that this matrix is positive definite.

2. The domain D_{II} is formed by symmetric (square) matrices Z of order n satisfying the condition
$$I^{(n)} - Z\overline{Z} > 0.$$

3. The domain D_{III} is formed by skew-symmetric matrices Z of order n satisfying the condition
$$I^{(n)} + Z\overline{Z} > 0.$$

4. The domain D_{IV} is formed by n-dimensional vectors $z = (z_1, \dots, z_n)$ satisfying the condition
$$|zz'|^2 + 1 - 2\bar{z}z' > 0, \qquad |zz'| < 1.$$

The complex dimension of these four types of domains is equal to mn, $\dfrac{n(n+1)}{2}$, $\dfrac{n(n-1)}{2}$, and n, respectively. These domains are complete circular convex domains. In our case, the domain D means a domain of one of the types presented above.

6.2. Morera theorem in classic domains

We define the class $\mathcal{H}^1(D)$ as a class of all functions f holomorphic in D such that
$$\sup_{0<r<1} \int_S |f(r\zeta)| d\mu < +\infty,$$
here, $r\zeta = (r\zeta_1, \dots, r\zeta_N)$, and $d\mu$ is the normalized Lebesgue measure on the manifold S, which is a Haar measure, and, therefore, it is invariant with respect to rotations.

For any function f in D and any $\zeta \in S$ consider a cut-function f_ζ in $\triangle = \{t \in \mathbb{C} : \ |t| < 1\}$ of the following form: $f_\zeta(t) = f(t\zeta)$. This cut-function allows us to relate certain N-dimensional properties of the function f to one-dimensional properties of f_ζ.

Fix a point $\lambda_0 \in S$ ($\lambda_0 = (\lambda_1^0, \dots, \lambda_N^0)$) and consider the following embedding of a disk $\triangle$ in the domain D:
$$\{\zeta \in \mathbb{C}^N : \ \zeta_j = t\lambda_j^0, \ j = 1, \dots, N, \ |t| < 1\}. \tag{6.1}$$

Under this embedding, the boundary T of the disk $\triangle$ moves to a circle lying on S. If ψ is an arbitrary (holomorphic) automorphism of the domain D (i.e., a biholomorphic self-map of the domain D), then the set of the form (6.1) passes to a certain analytic disk with the boundary on S under the action of this automorphism.

Theorem 6.1 (Kytmanov, Kosbergenov, Myslivets [26]). *If a function $f \in \mathcal{C}(S)$ satisfies the condition*

$$\int_T f(\psi(t\lambda_0))\, dt = 0 \tag{6.2}$$

for all automorphisms ψ of the domain D, then the function f is holomorphically extended into D to a function F of class $\mathcal{C}(\overline{D})$.

7. Multidimensional analogue of Morera theorem for continuous functions

This section contains some results related to the holomorphic extension of continuous functions given on the boundary of a bounded domain to this domain. We consider functions that satisfy the Morera property (Definition 3.1). So let us consider a set of complex lines intersecting the germ of a real-analytic manifold of real dimension $(2n-2)$ to be a sufficient set.

Let $D \subset \mathbb{C}^n$ $(n > 1)$ be a bounded domain with a connected smooth boundary of the form

$$D = \{z \in \mathbb{C}^n : \ \rho(z) < 0\},$$

where $\rho(z)$ is smooth real-valued function in a neighbourhood of the set $\overline{D}$ such that $d\rho\big|_{\partial D} \neq 0$. Consider complex lines $l_{z,b}$ of the form (4.1)

$$l_{z,b} = \{\zeta \in \mathbb{C}^n : \ \zeta_j = z_j + b_j t, \ j = 1, \ldots, n, \ t \in \mathbb{C}\},$$

passing through the point $z \in \mathbb{C}^n$ in the direction of the vector $b = \{b_1, \ldots, b_n\} \in \mathbb{CP}^{n-1}$ (the direction b is determined up to multiplication by a complex number $\lambda \neq 0$).

Let Γ be the germ of a real-analytic manifold of real dimension $(2n-2)$.

Theorem 7.1 (Kytmanov, Myslivets [27]). *Let a domain $D \subset \mathbb{C}^n$ satisfy a conditions*

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}\, b_j \neq 0,$$

for the points z, lying in the neighbourhood of a manifold Γ such that $\partial D \cap \Gamma = \varnothing$. Let a function $f \in \mathcal{C}(\partial D)$ satisfy the generalized Morera property, i.e.,

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t) t^k dt = 0 \tag{7.1}$$

for all $z \in \Gamma$, $b \in \mathbb{CP}^{n-1}$ and for a fixed integral non-negative number k. Then the function f has the holomorphically extension into the domain D.

For $k = 0$ condition (7.1) takes us to the boundary Morera property

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \ldots, z_n + b_n t)\, dt = 0. \tag{7.2}$$

Corollary 7.1. *Let a domain D satisfy the conditions of Theorem 7.1, and a function $f \in \mathcal{C}(\partial D)$ satisfy condition (7.2) for all $z \in \Gamma$ and $b \in \mathbb{CP}^{n-1}$, then f is holomorphically extended into the domain D.*

8. Functions with the one-dimensional holomorphic extension property in a ball

Historically, the first statements about the functions with the one-dimensional holomorphic extension property along the complex lines were obtained in a ball by M. L. Agranovskiy and R. E. Valskiy [31]. In the proof of their assertion, they used only the Morera property along complex lines intersecting the ball. So, in fact, they got a boundary Morera theorem. The ball thus became a model example, to obtain a series of statements, which were then extended to the case of domains of a more general form.

A number of papers dealt with classes of complex lines (or curves), sufficient for holomorphic extension into a ball. Thus, in the monograph by W. Rudin [6, Th. 12.3.11] it is shown that if a function $f \in \mathcal{C}(\partial B)$ (B is a unit ball in $\mathbb{C}^n$ centered at the origin) has the one-dimensional holomorphic extension property along all complex lines that are lying at a distance r from the center of the ball for $0 < r < 1$, then this is a CR-function on ∂B. The proof is based on the description of $\mathcal{U}$-invariant subspaces of functions in the ball. This statement was generalized to strictly convex domains with a real-analytic boundary by M. L. Agranovsky [32]. Finer families of complex lines sufficient for holomorphic continuation, were studied in [33, 34]. J. Globevnik [35] shows that a two-dimensional compact manifold of complex lines is a sufficient family for holomorphic extension into $\mathbb{C}^2$.

M. L. Agranovsky and A. M. Semenov in [21] prove the following result. Let R be a smooth analytic disc in $\mathbb{C}^n$, ie, $R = \varphi(\triangle)$, where $\triangle$ is an open unit disc in the complex plane $\mathbb{C}$, and $\varphi : \triangle \to \mathbb{C}^n$ is a holomorphic map of class $\mathcal{C}^1(\overline{\triangle})$. Denote the Shilov boundary of R by γ, i.e., we put

$$\Omega = \bigcup_{u \in \mathcal{U}(n)} u(\gamma),$$

where $\mathcal{U}(n)$ is the group of unitary transformations in $\mathbb{C}^n$. The set Ω is a spherical layer

$$\Omega = \left\{ \zeta : \min_{z \in \gamma} |z| \leqslant |\zeta| \leqslant \max_{z \in \gamma} |z| \right\}.$$

Theorem 8.1 (Agranovsky, Semenov [21]). *Assume the following conditions to be fulfilled:*

1. $0 \notin R \cup \gamma$;
2. *γ is not contained in any complex line in $\mathbb{C}^n$, passing through 0.*

Let $f \in \mathcal{C}^1(\Omega)$ and for any $u \in \mathcal{U}(n)$ the restriction f on $u(\gamma)$ admit a holomorphic extension to $u(R)$, which is smooth on $\overline{u(R)}$. Then f is holomorphic into Ω (and therefore extends holomorphically in the corresponding ball).

As already noted, E. Grinberg [36] was the first to formulate the boundary Morera theorem for a ball (in the case of complex lines). Although one of the assertions by Nagel and Rudin [44] can also be treated as a boundary Morera theorem.

We present one of the theorems, in which the class of complex lines is significantly narrowed.

Theorem 8.2 (Globevnik, Stout [18]). *Consider a unit ball $B \subset \mathbb{C}^2$. Suppose that the given number r, $0 < r < 1$, is such that the expression $r^{-1}(1 - r^2)^{1/2}$ is not a root of any polynomial with integer coefficients. Assume that the function $f \in \mathcal{C}(\partial B)$ and satisfies the Morera property along all complex lines lying at a distance r from the center of the ball, then f extends to B as a function from $\mathcal{A}(B)$.*

Example 8.1. Consider the example from [18], showing that the condition in Theorem 8.2 is essential. Let $r = \left(\frac{3}{5}\right)^{1/2}$. Then the function $g(z, \bar{z}) = z_1^3 \bar{z}_2^2$ has the Morera property along any complex line lying at a distance r from the center of the ball, but obviously, g does not extend holomorphically in B from the boundary ∂B.

Indeed, the calculations in Theorem 8.2, show that

$$\int_{-\pi}^{\pi}(e^{i\alpha}\rho+e^{i\theta}t(r^2-\rho^2)^{1/2}e^{-i\omega})^3(e^{-i\alpha}(r^2-\rho^2)^{1/2}e^{-i\omega}-e^{-i\theta}t\rho)^2e^{i\theta}d\theta=$$

$$=2\pi(r^2-\rho^2)^{1/2}(e^{i\alpha})^{1/2}(e^{-i\omega})\rho^3\cdot\left[\binom{3}{0}\binom{2}{1}t(-1)^1+\binom{3}{1}\binom{2}{2}t^3(-1)^2\right]=$$

$$=2\pi(r^2-\rho^2)^{1/2}e^{2i\alpha}e^{-i\omega}\rho^3(-2t+3t^3)=0,$$

since $-2t+3t^3=t\left(3\left(1-\frac{3}{5}\right)\frac{5}{3}-2\right)=0$.

Recently M. L. Agranovsky [37] and J. Globevnik [29] have shown that a family of complex lines passing through two fixed points in $\overline{D}$ is sufficient for holomorphic extension for real-analytic functions on the boundary of a ball. A family of complex lines passing through one point on the boundary of a ball was proved to be sufficient for holomorphic extension by L. Barakko [38].

Theorem 8.3 (Baracco [38]). *Let the point $z_0\in\partial B$, and the function f be of class $\mathcal{C}^\omega(\partial B)$, and suppose that f extends holomorphically from ∂B along each line passing through z_0. Then f extends holomorphically to B.*

9. On a boundary analogue of Hartogs' theorem in a ball

We emphasize here the papers which show that a family of complex lines passing through a finite number of points arranged in some way is sufficient for holomorphic extension. However this is only asserted for real-analytic or infinitely differentiable functions defined on the boundary of a ball. So, Agranovsky and Globevnik showed that, in $\mathbb{C}^2$, for real-analytic functions defined on the boundary of a ball just two points lying in the closure of the ball are enough.

9.1. Main results

Let $B=\{z\in\mathbb{C}^n:|z|<1\}$ be a unit ball in $\mathbb{C}^n$ centered at the origin and let $S=\partial B$ be the boundary of the ball. We will further say that the function $f\in\mathcal{C}(S)$ has *the one-dimensional holomorphic extension property along the family* $\mathfrak{L}_\Gamma$, if it has the one-dimensional holomorphic extension property along any complex line $l_{z,b}\in\mathfrak{L}_\Gamma$, where $l_{z,b}$ be the complex line of the form (4.1).

We will also say that *the set $\mathfrak{L}_\Gamma$ is sufficient for holomorphic extension*, if the function $f\in\mathcal{C}(S)$ has the one-dimensional holomorphic extension property along all complex lines in the family $\mathfrak{L}_\Gamma$, and the function f holomorphically extends to B (i.e., f is a CR-function on S). In [23, 39–41] it is shown that for a class of continuous functions given on the boundary of a ball a family of complex lines passing through finite points in the ball will be a sufficient family. Baracco was the first to prove this result, which was earlier explicitly conjectured by Agranovsky [37]. Globevnik [40] suggested an alternative proof, even for the case when the vertices lie outside the ball. Those results were obtained by completely different methods.

Theorem 9.1 (Kytmanov, Myslivets [41]). *Suppose $n=2$ and the function $f(\zeta)\in\mathcal{C}(S)$ has the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\{a,c,d\}}$, and the points a, c, $d\in B$ do not lie on one complex line in $\mathbb{C}^2$, then $f(\zeta)$ extends holomorphically into B.*

We denote by $\mathcal{A}$ a set of points $a_k\in B\subset\mathbb{C}^n$, $k=1,\dots,n+1$, lying outside the complex hyperplane $\mathbb{C}^n$.

Theorem 9.2 (Kytmanov, Myslivets [41]). *Let a function $f(\zeta) \in \mathcal{C}(S)$ has the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathcal{A}}$, then $f(\zeta)$ extends holomorphically into B.*

9.2. The example

Now we give an example based on the Globevnik's example which shows that for continuous functions on the boundary S of the family $\mathfrak{L}_{\mathcal{A}}$, where $\mathcal{A}$ is a set of points $a_k \in B \subset \mathbb{C}^n$, $k = 1, \dots, n$ is not enough for holomorphic extension.

Consider a part of a complex hyperplane

$$\Gamma = \{(z', w) \in B : \ w = 0\},$$

in the ball $B = \{(z', w) \in \mathbb{C}^n : \ |z'|^2 + |w|^2 < 1\}$, where $z' = (z_1, \dots, z_{n-1})$, $w \in \mathbb{C}$ and $|z'|^2 = |z_1|^2 + \dots + |z_{n-1}|^2$. Then the function $f = \dfrac{w^{k+2}}{\bar{w}}$ $(k \in \mathbb{Z},\ k \geqslant 0)$ has the one-dimensional property of holomorphic extension from ∂B along the complex line of the family $\mathfrak{L}_\Gamma$, which is smooth on ∂B, but does not extend holomorphically to B.

Consider complex lines intersecting Γ:

$$l_{a'} = \{(z', w) \in \mathbb{C}^n : \ z' = a' + b't,\ w = ct,\ t \in \mathbb{C}\}. \tag{9.1}$$

These lines pass through the point $(a', 0) \in \Gamma$. When $|a'| < 1$ the point $(a', 0) \in B$, while for $|a'| > 1$ the point $(a', 0) \notin \overline{B}$. Without loss of generality, we suppose that $|b'|^2 + |c|^2 = 1$. The intersection $l_{a'} \cap \partial B$ forms a circle

$$|t|^2 + \langle a', \bar{b}' \rangle \bar{t} + \langle \bar{a}', b' \rangle t = 1 - |a'|^2, \quad \text{or} \quad |t + a'\bar{b}'|^2 = 1 - |c|^2|a'|^2, \tag{9.2}$$

where $\langle a', b' \rangle = a_1 b_1 + \dots + a_{n-1} b_{n-1}$.

Indeed, since for complex lines of the form (9.1) on ∂B

$$\bar{t} = \frac{1 - |a'|^2 - \langle \bar{a}', b' \rangle t}{t + \langle a', \bar{b}' \rangle},$$

the function f on ∂B becomes

$$f = \frac{(t + \langle a', \bar{b}' \rangle)}{1 - |a'|^2 - \langle \bar{a}', b' \rangle t} \cdot (ct)^{k+2}.$$

The denominator of the fraction is equal to 0 at $t_0 = \dfrac{1 - |a'|^2}{\langle \bar{a}', b' \rangle}$. Substituting this point into the expression (9.2), we obtain

$$\frac{(1 - |a'|^2)^2}{|\langle a', b \rangle|^2} + 1 - |a'|^2 > 0, \quad \text{if} \quad |a'| < 1.$$

Therefore the point of the line $l_{a'}$, corresponding to t_0, lies outside the ball B. So the function f holomorphically extends to $l_{a'} \cap B$. Consider the finite set $\mathcal{A} = \{a_1, \dots, a_{n-1}, 0\} \in B$, then there exists a complex hyperplane containing $\mathcal{A}$. We can suppose that this is the hyperplane Γ.

10. On the functions with one-dimensional holomorphic extension property in circular domains

Recently, we obtained similar results for circular domains with the Nevanlinna property.

The above definition and statement will be applied to bounded domains G with a boundary of class $\mathcal{C}^2$, therefore (due to the principle of correspondence of boundaries) the function $k(\tau)$ extends to $\overline{\Delta}$ as a function of class $\mathcal{C}^1(\overline{\Delta})$ and $\tilde{k}(\tau)$ extends to $\mathbb{C}\setminus\overline{\Delta}$ as a function of class $\mathcal{C}(\mathbb{C}\setminus\overline{\Delta})$. Therefore, in whole, the function $\bar{t} = \dfrac{u(\tau)}{v(\tau)}$ is a meromorphic function in $\mathbb{C}$. There are various example of domains with the Nevanlinna property. For example, if ∂G is a real-analytic, then $k(\tau)$ is a rational function with no poles on the closure Δ.

In our further consideration we will need the domain G to possess *the strengthened Nevanlinna property*, that is the function $u_1(\tau) \neq 0$ in $\mathbb{C} \setminus \Delta$ and $\tilde{k}$ has at infinity zero of no more than first order. If $G = \Delta$ then $\overline{\tau} = \dfrac{1}{\tau}$ on $\partial\Delta$. Therefore, the meromorphic function $\dfrac{1}{\tau}$ has a zero of the first order at ∞.

For example, such domains include domains for which $k(\tau)$ is a rational function with no poles on $\overline{\Delta}$ and no zeros in $\mathbb{C} \setminus \Delta$.

We will say that a domain $D \subset \mathbb{C}^n$ possess *the strengthened Nevanlinna property in the point* $z \in D$ if the section $D \cap l_{z,b}$ possess the strengthened Nevanlinna property for any $b \in \mathbb{CP}^{n-1}$.

Theorem 10.1 (Kytmanov, Myslivets [42]). *Let $n = 2$ and D be a bounded strictly convex circular domain with twice smooth boundary and possess the strengthened Nevanlinna property in the points $a, c, d \in D$ and the function $f(\zeta) \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\{a,c,d\}}$, and the points a, c, d do not lie on one complex line in $\mathbb{C}^2$, then the function $f(\zeta)$ extends holomorphically into D.*

We denote by $\mathfrak{A}$ the set of points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \dots, n+1$, which do not lie on the complex hyperplane in $\mathbb{C}^n$.

Theorem 10.2 (Kytmanov, Myslivets [42]). *Let D be a bounded strictly convex circular domain with twice smooth boundary in $\mathbb{C}^n$ and possess the strengthened Nevanlinna property in the points from the set $\mathfrak{A}$ and the function $f(\zeta) \in \mathcal{C}(\partial D)$ have the one-dimensional holomorphic extension property along the family $\mathfrak{L}_{\mathfrak{A}}$, then the function $f(\zeta)$ extends holomorphically into D.*

11. The functions with the boundary Morera properties in domains with piecewise-smooth boundary

We also obtained an analog of Theorem 7.1 for domains with a piecewise-smooth boundary. Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected piecewise-smooth boundary ∂D.

Theorem 11.1 (Kytmanov, Myslivets [43]). *Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a connected piecewise-smooth boundary and let k be a fixed nonnegative integer and let a function $f \in \mathcal{C}(\partial D)$. If for almost all $z \in \mathbb{C}^n$ and almost all $b \in \mathbb{CP}^{n-1}$ the condition*

$$\int\limits_{\partial D \cap l_{z,b}} f(z_1 + b_1 t, \dots, z_n + b_n t) t^k \, dt = 0, \tag{11.1}$$

holds, then f is holomorphically extended into D.

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О многомерных граничных аналогах теоремы Морера

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Аннотация. Мы обсуждаем функции со свойством одномерного голоморфного продолжения вдоль прямых и кривых, а также граничные многомерные варианты теоремы Мореры. Мы хотим показать, как интегральные представления могут быть применены к изучению аналитического продолжения функций, в частности к многомерным граничным аналогам теорем Мореры.

Ключевые слова: свойство одномерного голоморфного продолжения, многомерные варианты теоремы Морера.