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Some New Congruence Identities of General Partition for $p_r(n)$

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Abstract. In the present work, we deduce some new congruences modulo 3 and 5 for $p_r(n)$, where $r \in \{-(3\lambda+3), -(5\lambda+3) \mid \lambda$ is any non-negative integer$\}$. Our emphasis throughout this paper is to exhibit the use of q-identities to generate the congruences for $p_r(n)$.

Keywords: q-identity, partition congruence, Ramanujan's general partition function congruences.

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1. Introduction

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is given by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}.$$

By Jacobi's triple product identity [5, p. 35], we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,$$

where here and throughout the paper, we utilize the following q-shifted factorial:

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1.$$

One of the special case of $f(a,b)$ as defined by S. Ramanujan [5, p. 36] is as follows

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

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For convenience, we write $f(-q^n) = f_n$. Due to Euler, we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where $p(n)$ is the number of partitions of n. S. Ramanujan initiated the general partition function $p_r(n)$ as

$$\sum_{n=0}^{\infty} p_r(n)q^n = \frac{1}{f_1^r}, \tag{1}$$

for non-zero integer r. For partition function $p(n)$, Ramanujan's so called "most beautiful identity" is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

which readily implies

$$p(5n+4) \equiv 0 \pmod{5}.$$

Further he also recorded two more congruences as follows:

$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

The generalization of the congruences modulo powers of 5 and 7 for all $p_r(n)$ was proved by K. G. Ramanathan [16]. Later A. O. L. Atkin [2] found that Ramanathan's proof is not correct. M. Newmann [13–15], studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. The functions $p_r(n)$ have been studied by many mathematicians. For the wonderful work one can see [1–4, 6, 8–10, 12, 17–22]. For $r = -2$, R. Hammond and R. Lewis [11] proved that

$$p_{-2}(5n+\ell) \equiv 0 \pmod{5},$$

where $\ell \in \{2, 3, 4\}$. Also in [7], W. Y. C. Chen et. al. proved

$$p_{-2}(25n+23) \equiv 0 \pmod{25}$$

by using modular forms. More recently D. Tang [24] for $p_r(n)$ proved some new congruences for $p_r(n)$, where $r \in \{-2, -6, -7\}$. For example,

$$p_{-2}\left(5^{2\delta-1}n + \frac{7\times 5^{2\delta-1}+1}{12}\right) \equiv 0 \pmod{5^\delta},$$
$$p_{-6}\left(5^{2\delta}n + \frac{3\times 5^{\delta}+1}{4}\right) \equiv 0 \pmod{5^\delta}$$

and

$$p_{-7}\left(5^{2\delta-1}n + \frac{13\times 5^{2\delta-1}+7}{24}\right) \equiv 0 \pmod{5^\delta}.$$

Motivated by the above work in this paper, we deduce some new congruences modulo 3 and 5 for $p_r(n)$, where $r \in \{-(3\lambda+3), -(5\lambda+3) \mid \lambda \text{ is any non-negative integer}\}$. By using the binomial theorem, one can easily deduce the below mentioned congruence, and it will be used again and again in our proof without mentioning precisely. For a prime p, we have

$$f_1^p \equiv f_p \pmod{p}.$$

Now we state our results which we are proving in this paper. The following congruences holds good for any non-negative integer λ:

Theorem 1.1. *We have*

$$p_{-(3\lambda+3)}(3n+1) \equiv 0 \pmod{3}, \tag{2}$$
$$p_{-(3\lambda+3)}(3n+2) \equiv 0 \pmod{3}. \tag{3}$$

Theorem 1.2. *We have*

$$p_{-(5\lambda+3)}(5n+2) \equiv 0 \pmod{5}, \tag{4}$$
$$p_{-(5\lambda+3)}(5n+3) \equiv 0 \pmod{5}, \tag{5}$$
$$p_{-(5\lambda+3)}(5n+4) \equiv 0 \pmod{5}. \tag{6}$$

2. Congruences of modulo 3 and 5

Proof of Theorem 1.1. Set $r = -(3\lambda+3)$ in (1), we observe that

$$\sum_{n=0}^{\infty} p_{-(3\lambda+3)}(n)q^n = f_1^{3\lambda+3} = f_1^{3\lambda} f_1^3. \tag{7}$$

From [5, p. 345 Chapter 20, Entry 1], we have

$$f_1^3 = f_9^3 \left(\frac{1}{u} - 3q + 4q^3u^2 \right), \tag{8}$$

where

$$u = \frac{f_3 f_{18}^3}{f_6 f_9^3}.$$

Substituting (8) in (7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(3\lambda+3)}(n)q^n &= f_1^{3\lambda} f_9^3 \left(\frac{1}{u} - 3q + 4q^3u^2 \right) \\ &\equiv f_3^{\lambda} f_{27} \left(\frac{1}{u} - 3q + 4q^3u^2 \right) \pmod{3}.\end{aligned}$$

On extracting the powers of q^{3n+1} and q^{3n+2} in the above congruence, we deduce (2) and (3). □

Proof of Theorem 1.2. From [20], we have

$$f_1 = f_{25} \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right), \tag{9}$$

where

$$R(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})}{(1-q^{5n-3})(1-q^{5n-2})}.$$

Set $r = -(5\lambda+3)$ in (1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(5\lambda+3)}(n)q^n &= f_1^{5\lambda+3} = f_1^{5\lambda} f_1^3 \\ &\equiv f_5^{\lambda} f_1^3 \pmod{5}.\end{aligned} \tag{10}$$

From (9), we observe that

$$f_1^3 = f_{25}^3 \left(\frac{1}{R^3(q^5)} - \frac{3q}{R^2(q^5)} + 5q^3 - 3q^5 R^2(q^5) - q^6 R^3(q^5) \right). \tag{11}$$

Utilizing (11) in (12), one can easily see that

$$\sum_{n=0}^{\infty} p_{-(5\lambda+3)}(n) q^n \equiv f_5^{\lambda+6} \left(\frac{1}{R^3(q^5)} + \frac{2q}{R^2(q^5)} + 2q^5 R^2(q^5) + 4q^6 R^3(q^5) \right) \pmod{5}. \tag{12}$$

On extracting the powers of q^{5n+2}, q^{5n+3} and q^{5n+4} in the above congruence, we obtain (4), (5) and (6). The above result is also due to B. R. Srivatsa Kumar et al. [23]. □

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Некоторые новые конгруэнтные тождества общего разбиения для $p_r(n)$

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Аннотация. В настоящей работе мы выводим некоторые новые сравнения по модулю 3 и 5 для $p_r(n)$, где $r \in \{-(3\lambda+3), -(5\lambda+3) \mid \lambda$ любое неотрицательное целое число$\}$. В этой статье мы делаем упор на демонстрацию использования q-тождеств для генерации сравнений для $p_r(n)$.

Ключевые слова: q-идентичность, конгруэнтность разбиений, общие конгруэнции статистической суммы Рамануджана.