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Remarks on: "Generalized Contractions to Coupled Fixed Point Theorems in Partially Ordered Metric Spaces"

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Abstract. Remarks on the paper [1] are given and corrected proofs are presented. Their results remain correct.

Keywords: partially ordered metric spaces, rational contractions, coupled fixed point, monotone property.

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1. The problem

Recently the paper [1] gave some interesting results for fixed point theorems of a self mapping obeying some rational type contractions. However even if their results are correct, their proofs suffered a fundamental flaw that we will address here.

The condition for the self mapping $f : X \times X \to X$ found in [1] formula (1) has the following form:

$$d(f(x,y), f(\mu,\nu)) \leqslant \alpha \frac{d(x, f(x,y))[1 + d(\mu, f(\mu,\nu))]}{1 + d(x,\mu)} + \beta \frac{d(x, f(x,y)) d(\mu, f(\mu,\nu))}{d(x,\mu)} + \\ + \gamma[d(x, f(x,y)) + d(\mu, f(\mu,\nu))] + \delta[d(x, f(\mu,\nu)) + d(\mu, f(x,y))] + \lambda d(x,\mu) \quad (1)$$

for all $x, y, \mu, \nu \in X$ with $x \geqslant \mu$ and $y \leqslant \nu$. Observing the term proportional to β in (1) one sees that whenever $x = \mu$ we have a division by zero, which has no meaning. This fact spoils the proofs of theorems present in [1].

In order to correct the problem, instead of the term of (1) proportional to β we will use

$$\beta \frac{d(x, f(x,y)) d(\mu, f(\mu,\nu))}{1 + d(x, f(x,y))} \quad (2)$$

in analogy to the work [2] that among other results corrects a similar problem. The inequality written in this new manner avoids divisions by zero when $x = \mu$.

We will now proceed rewriting parts of [1] in order to give correct proofs of theorems present in there.

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2. Main results

Let us rewrite Theorem 1 of [1]. We have

$$
\begin{aligned}
d(f(x,y),f(\mu,\nu)) \leqslant \alpha\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\nu))]}{1+d(x,\mu)} + \beta\frac{d(x,f(x,y))d(\mu,f(\mu,\nu))}{1+d(x,f(x,y))} + \\
+\gamma[d(x,f(x,y))+d(\mu,f(\mu,\nu))] + \delta[d(x,f(\mu,\nu))+d(\mu,f(x,y))] + \lambda d(x,\mu) \quad (3)
\end{aligned}
$$

for all $x,y,\mu,\nu \in X$ with $x \geqslant \mu$ and $y \leqslant \nu$ with $\alpha,\beta,\gamma,\delta,\lambda \in [0,1)$ with

$$
\frac{\beta+\gamma+\delta+\lambda}{1-\alpha-\delta-\gamma} < 1 \; . \quad (4)
$$

The sequences are

$$
x_{n+1} = f(x_n,y_n) \text{ and } y_{n+1} = f(y_n,x_n) \text{ for all } n \geqslant 0 \; . \quad (5)
$$

Following [1] we know that $x_n < x_{n+1}$ and $y_n > y_{n+1}$, and we get

$$
\begin{aligned}
d(x_{n+1},x_n) = d(f(x_n,y_n),f(x_{n-1},y_{n-1})) \leqslant \\
\leqslant \alpha\frac{d(x_n,f(x_n,y_n))[1+d(x_{n-1},f(x_{n-1},y_{n-1}))]}{1+d(x_n,x_{n-1})} + \\
+\beta\frac{d(x_n,f(x_n,y_n))d(x_{n-1},f(x_{n-1},y_{n-1}))}{1+d(x_n,f(x_n,y_n))} + \\
+\gamma[d(x_n,f(x_n,y_n))+d(x_{n-1},f(x_{n-1},y_{n-1}))] + \\
+\delta[d(x_n,f(x_{n-1},y_{n-1}))+d(x_{n-1},f(x_n,y_n))] + \lambda d(x_n,x_{n-1}) \, , \quad (6)
\end{aligned}
$$

it implies that

$$
\begin{aligned}
d(x_{n+1},x_n) \leqslant \alpha\frac{d(x_n,x_{n+1})[1+d(x_{n-1},x_n)]}{1+d(x_n,x_{n-1})} + \beta\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{1+d(x_n,x_{n+1})} + \\
+\gamma[d(x_n,x_{n+1})+d(x_{n-1},x_n)] + \delta[d(x_n,x_n)+d(x_{n-1},x_{n+1})] + \lambda d(x_n,x_{n-1}) \; . \quad (7)
\end{aligned}
$$

Using the triangular inequality, $d(x_{n-1},x_{n+1}) \leqslant d(x_{n-1},d_n) + d(x_n,x_{n+1})$, and solving for $d(x_{n+1},x_n)$ we obtain

$$
d(x_{n+1},x_n) \leqslant \left(\frac{\beta+\gamma+\delta+\lambda}{1-\alpha-\gamma-\delta}\right) d(x_{n-1},x_n) \; . \quad (8)
$$

Observe the difference with equation (9) of [1], in our case the β is at the numerator. Imposing the condition (4) we end up with a contraction, see [1] page 495.

For the last part of the proof, we have

$$
\begin{aligned}
d(x_{n+1},z_{n+1}) = d(f(x_n,y_n),f(z_n,y_n)) \leqslant \alpha\frac{d(x_n,f(x_n,y_n))(1+d(z_n,f(z_n,y_n))}{1+d(x_n,z_n)} + \\
+\beta\frac{d(x_n,f(x_n,y_n))d(z_n,f(z_n,y_n))}{1+d(x_n,f(x_n,y_n))} + \gamma[d(x_n,f(x_n,y_n))+d(z_n,f(z_n,y_n))] + \\
+\delta[d(x_n,f(z_n,y_n))+d(z_n,f(x_n,y_n))] + \lambda d(x_n,z_n). \quad (9)
\end{aligned}
$$

In the limit $n \to +\infty$ we get

$$
d(x,z) \leqslant (2\delta+\lambda)d(x,z) \, , \quad (10)
$$

in complete analogy to [1], as in the above limit the term of (3) proportional to β goes to zero anyway. This completes the proof of Theorem 1.

Theorem 2. According to the bound given in (3), the inequality of Theorem 2 of [1] becomes

$$d(x,u_{n+1}) = d(f(x,y),f(u_n,v_n)) \leqslant \alpha\frac{d(x,f(x,y))[1+d(u_n,f(u_n,v_n))]}{1+d(x,u_n)}+ \\ +\beta\frac{d(x,f(x,y))d(u_n,f(u_n,v_n))}{1+d(x,f(x,y))}+\gamma[d(x,f(x,y))+d(u_n,f(u_n,v_n))]+ \\ +\delta[d(x,f(u_n,v_n))+d(u_n,f(x,y))]+\lambda d(x,u_n)\ , \tag{11}$$

leading to

$$d(x,u_{n+1}) \leqslant \left(\frac{\gamma+\delta+\lambda}{1-\gamma-\delta}\right)d(x,u_n) \tag{12}$$

in complete analogy to [1], as $d(x,f(x,y)) = d(x,x) = 0$ independently from the vaules of α and β. The rest of the proof is the same.

Theorem 3. Like for Theorem 2, from (3) we have

$$d(x_{n+1},y_{n+1}) = d(f(x_n,y_n),f(y_n,x_n)) \leqslant \alpha\frac{d(x_n,f(x_n,y_n))[1+d(y_n,f(y_n,x_n))]}{1+d(x_n,y_n)}+ \\ +\beta\frac{d(x_n,f(x_n,y_n))d(y_n,f(y_n,x_n))}{1+d(x_n,f(x_n,y_n))}+\gamma[d(x_n,f(x_n,y_n))+d(y_n,f(y_n,x_n))]+ \\ +\delta[d(x_n,f(y_n,x_n))+d(y_n,f(x_n,y_n))]+\lambda d(x_n,y_n)\ , \tag{13}$$

thus obtaining in the limit $n\to+\infty$

$$d(x,y) \leqslant (2\lambda+\delta)\,d(x,y)\ , \tag{14}$$

ending with a contradiction as $2\delta+\lambda<1$. The rest of the proof is the same.

Remarks

1. $\alpha=\gamma=\delta=0$ Čirić et. al [3]: inapplicable, same problem of division by 0 of (1).
2. $\alpha=0$ Chandok et al. [4]: inapplicable, same problem of division by 0 of (1).
3. $\alpha=\beta=\gamma=\delta=0$ Banach [5]: no variations, remains the same for $\lambda<1$.
4. $\alpha=\beta=\delta=\lambda=0$ Kannan [6]: no variations, remains the same for $2\gamma<1$.
5. $\alpha=\beta=\gamma=\lambda=0$ Chatterjee [7]: no variations, remains the same for $2\delta<1$.
6. $\alpha=\gamma=0$ Singh and Chatterjee [8]: inapplicable, same problem of division by 0 of (1).

3. Applications

Theorem 4. The condition of [1] becomes:

$$\int_0^{d(f(x,y),f(\mu,\nu))}\varphi(t)dt \leqslant \alpha\int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\nu))]}{1+d(x,\mu)}}\varphi(t)dt+\beta\int_0^{\frac{d(x,f(x,y))d(\mu,f(\mu,\nu))}{1+d(x,f(x,y))}}\varphi(t)dt+ \\ +\gamma\int_0^{d(x,f(x,y))+d(\mu,f(\mu,\nu))}\varphi(t)dt+\delta\int_0^{d(x,f(\mu,\nu))+d(\mu,f(x,y))}\varphi(t)dt+ \\ +\lambda\int_0^{d(x,\mu)}\varphi(t)dt\,. \tag{15}$$

Theorem 5. The condition of [1] becomes:

$$\int_0^{d(f(x,y),f(\mu,\nu))} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\nu))]}{1+d(x,\mu)}} \varphi(t)dt + \beta \int_0^{\frac{d(x,f(x,y))d(\mu,f(\mu,\nu))}{1+d(x,f(x,y))}} \varphi(t)dt + \\ + \lambda \int_0^{d(x,\mu)} \varphi(t)dt\,. \qquad (16)$$

Theorem 6, **Theorem 7** and **Theorem 8**: as the conditions of those theorems do not depend on the parameter β, there are no variations.

For further references, consult [1] and references therein.

References

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Замечания по статье: "Обобщенные сжатия теорем о связанных неподвижных точках в частично упорядоченных метрических пространствах"

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Аннотация. Даны примечания к статье [1] и приведены исправленные доказательства. Ее результаты остаются верными.

Ключевые слова: частично упорядоченные метрические пространства, рациональные стягивания, связанная неподвижная точка, монотонность.