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# On a Spectral Problem for Convection Equations

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**Abstract.** Spectral problems for stationary unidirectional convective flows in vertical heat exchangers at various boundary temperature conditions are considered. The constant temperature gradient on the vertical walls is used as a spectral parameter. The heat exchanger cross-section can be of an arbitrary shape. The general properties of the spectral problem solutions are established. Solutions are obtained in an analytical form for rectangular and a circular cross sections. The critical values of temperature gradient at which convective flow arises are found. The corresponding vertical velocity profiles are constructed. The properties of solutions of a new transcendental equation for the spectral values are studied.

**Keywords:** convection, spectral problem, eigenfunctions, eigenvalues.

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## 1. Problem formulation

The system of equations for convective motion in the Oberbeck–Boussinesq approximation has the form [1]

$$\mathbf{u_t} + (\mathbf{u} \cdot \bigtriangledown)\mathbf{u} + \frac{1}{\rho} \bigtriangledown p = \nu \Delta \mathbf{u} + g\beta\theta\mathbf{e}, \tag{1.1}$$

$$div\mathbf{u} = 0, \tag{1.2}$$

$$\theta_t + \mathbf{u} \cdot \bigtriangledown\theta = \chi\Delta\theta. \tag{1.3}$$

Here $\mathbf{u}$=$(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ is the velocity vector, $p(x,y,z,t)$ is the modified pressure, $\theta(x,y,z,t)$ is temperature; $\rho, \nu, g, \beta, \chi$ are density, kinematic viscosity, gravity acceleration, the coefficients of thermal expansion and thermal diffusivity of the medium, respectively. $\mathbf{e} = (0,0,-1)$ is unit vector. Thus the gravity acceleration is directed in the opposite direction to the $z$ axis.

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System (1.1)–(1.3) admits operator $\partial_z - A(\partial_\theta + \rho g\beta z\partial_p)$ with the constant $A$. This operator has invariants $x, y, t, u_1, u_2, u_3$, $p+\rho g\beta z^2/2$, $\theta+Az$. Then invariant solutions of rank three should be sought in the form [1]

$$\begin{aligned} \mathbf{u} &= (u(x,y,t), v(x,y,t), w(x,y,t)), \\ p &= -\rho g\beta A\frac{z^2}{2} + q(x,y,t), \\ \theta &= -Az + T(x,y,t). \end{aligned} \tag{1.4}$$

Substitution of (1.4) into (1.1)–(1.3) results in the system that contains only the invariants

$$\begin{aligned} u_t + uu_x + vu_y + \frac{1}{\rho}q_x &= \nu(u_{xx}+u_{yy}), \\ v_t + uv_x + vv_y + \frac{1}{\rho}q_y &= \nu(v_{xx}+v_{yy}), \\ u_x + v_y &= 0; \end{aligned} \tag{1.5}$$

$$\begin{aligned} w_t + uw_x + vw_y &= \nu(w_{xx}+w_{yy}) + \rho g\beta T, \\ T_t + uT_x + vT_y &= Aw + \chi(T_{xx}+T_{yy}). \end{aligned} \tag{1.6}$$

Equations (1.5) are Navier–Stokes system for plane motion of purely viscous fluid. The exact solutions for the system were found [2, 3]. Therefore, our attention is focussed on system (1.6) (so far without considering the problems with interfaces).

Suppose that $u = v = 0$, $q = q(t)$, then (1.6) is converted into the system of linear parabolic equations. Its stationary solution satisfies the following equations

$$\begin{aligned} \nu\Delta w + \rho g\beta T &= 0, \\ \chi\Delta T + Aw &= 0, \end{aligned} \tag{1.7}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. Equations (1.7) are satisfied in a certain region $\Omega$ on the plane $x, y$ with boundary $\Gamma$. Here we consider two boundary conditions on $\Gamma$, namely, the boundary condition of the first kind

$$w = 0, \qquad T = 0 \tag{1.8}$$

and the boundary condition of the third kind

$$w = 0, \qquad k\frac{\partial T}{\partial n} + bT = 0, \tag{1.9}$$

where $k > 0$ is the constant thermal conductivity coefficient of fluid, $b \geqslant 0$ is the constant heat transfer coefficient.

Boundary value problems (1.7), (1.8) or (1.7), (1.9) are spectral problems. Here the temperature gradient $A$ is the spectral parameter. For all $A$ the trivial solution satisfies these problems. The question arises: at what values of $A$ there is a nontrivial solution of spectral problems (1.7), (1.8) or (1.7), (1.9)? We are also interested in the minimum value of spectral parameter $A$.

## 2. Some general properties of spectral problems

We note that for both problems $A \neq 0$, otherwise $w \equiv 0$ and $T \equiv 0$. Let us consider spectral problem (1.7), (1.8). The following lemma holds

**Lemma 1.** *The eigenfunctions of spectral problem (1.7), (1.8) are real functions, and the eigenvalues are always positive. The eigenfunctions corresponding to different eigenvalues are orthogonal functions in the sense of space $L_2(\Omega)$.*

*Proof.* Let us show that the parameter $A$ can take only real values. Let $w(x,y) = w_1(x,y)+$ $+iw_2(x,y)$, $T(x,y) = T_1(x,y) + iT_2(x,y)$. We multiply the first equation of system (1.7) by $\overline{w}(x,y)$ and the second equation by $\overline{T}(x,y)$ (here the bar denotes the complex conjugate), integrate them over $\Omega$ and combine the results. Then we obtain

$$\int_\Omega (T\overline{w} + \overline{T}w)dxdy = \frac{\nu}{\rho g\beta}\int_\Omega |\nabla w|^2\, dxdy + \frac{\chi}{A}\int_\Omega |\nabla T|^2\, dxdy \tag{2.1}$$

Relation (2.1) was obtained with the use of the vector analysis formula $a\,\mathrm{div}\mathbf{b} = \mathrm{div}(a\mathbf{b}) - \nabla a\cdot\mathbf{b}$, $a, \mathbf{b} \in C^1(\Omega)$, the Gauss–Ostrogradsky theorem and the fact that $A \neq 0$. It is clear that $T\overline{w} + \overline{T}w$ is a real function. It follows from (2.1) that $A \in R^1$. Further, if $w_0(x,y)$, $T_0(x,y)$ are eigenfunctions corresponding to the eigenvalue $A_0 \in R^1$ then separating the real and imaginary parts in (1.7), (1.8), we obtain that they are eigenfunctions of the same spectral problem. In other words, one can assume that $w = \overline{w}$, $T = \overline{T}$.

Similar reasoning results in the following relation

$$\frac{\nu}{\rho g\beta}\int_\Omega |\nabla w|^2\, dxdy - \frac{\chi}{A}\int_\Omega |\nabla T|^2\, dxdy = 0.$$

Then we have

$$A = \rho g\beta\chi\nu^{-1}\frac{\int\limits_\Omega |\nabla T|^2\, dxdy}{\int\limits_\Omega |\nabla w|^2\, dxdy} > 0. \tag{2.2}$$

Let us turn to the proof of the orthogonality of eigenfunctions. Let us reduce system (1.7) to one equation for $w(x,y)$

$$\Delta\Delta w = \lambda w, \quad (x,y) \in \Omega \tag{2.3}$$

with boundary conditions of the first kind (1.8). It means that $w = 0$, $\Delta w = 0$, $(x,y) \in \Gamma$. The following notation was introduced

$$\lambda = \frac{\rho g\beta}{\nu\chi}A. \tag{2.4}$$

Let us assume that $\lambda_1 \neq \lambda_2\, (A_1 \neq A_2)$ are eigenvalues and $w_1(x,y)$, $w_2(x,y)$ corresponding eigenfunctions. Then $\Delta\Delta w_{1,2} = \lambda_{1,2}w_{1,2}$, $(x,y) \in \Omega$ and $w_{1,2} = 0$, $\Delta w_{1,2} = 0$ on the boundary $\Gamma$. We have $w_2\Delta\Delta w_1 - w_1\Delta\Delta w_2 = \mathrm{div}(w_2\nabla\Delta w_1 - w_1\nabla\Delta w_2) + \nabla w_1\cdot\nabla\Delta w_2 - \nabla w_2\cdot\nabla\Delta w_1$. Then we have

$$\int_\Omega (w_2\Delta\Delta w_1 - w_1\Delta\Delta w_2)dxdy =$$
$$= \int_\Omega [\nabla w_1\cdot\Delta(\nabla w_2) - \nabla w_2\cdot\Delta(\nabla w_1)]dxdy = (\lambda_1 - \lambda_2)\int_\Omega w_1 w_2\, dxdy \tag{2.5}$$

because $w_{1,2} = 0$ on $\Gamma$ and $\nabla(\Delta u) = \Delta(\nabla u)$ for any $u \in C^2(\Omega)$. Let us write the expression in square brackets under the integral in the form

$$\frac{\partial w_1}{\partial x}\Delta\left(\frac{\partial w_2}{\partial x}\right) - \frac{\partial w_2}{\partial x}\Delta\left(\frac{\partial w_1}{\partial x}\right) + \frac{\partial w_1}{\partial y}\Delta\left(\frac{\partial w_2}{\partial y}\right) -$$
$$- \frac{\partial w_2}{\partial y}\Delta\left(\frac{\partial w_1}{\partial y}\right) + \frac{\partial w_1}{\partial z}\Delta\left(\frac{\partial w_2}{\partial z}\right) - \frac{\partial w_2}{\partial z}\Delta\left(\frac{\partial w_1}{\partial z}\right). \tag{2.6}$$

Using the vector analysis formula

$$\int_{\Omega}(\phi\Delta\psi-\psi\Delta\phi)d\Omega=\int_{\Gamma}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)d\Gamma,$$

where $\mathbf{n}$ is the outer normal to the boundary $\Gamma$, for the first two terms in (2.6) we have

$$\int_{\Omega}\left[\frac{\partial w_1}{\partial x}\Delta\left(\frac{\partial w_2}{\partial x}\right)-\frac{\partial w_2}{\partial x}\Delta\left(\frac{\partial w_1}{\partial x}\right)\right]dxdy=$$
$$=\int_{\Gamma}\left[\frac{\partial w_1}{\partial x}\frac{\partial}{\partial n}\left(\frac{\partial w_2}{\partial x}\right)-\frac{\partial w_2}{\partial x}\frac{\partial}{\partial n}\left(\frac{\partial w_1}{\partial x}\right)\right]d\Gamma. \qquad (2.7)$$

Let us show that the integral over $\Gamma$ in (2.7) is equal to zero if $w_{1,2}(x,y)=0$ on $\Gamma$. Let $(x_0,y_0)$ be an arbitrary point on the line $\Gamma$. Let us choose the local rectangular coordinate system $\xi,\eta$ with the origin at the point $(x_0,y_0)$, directing the $\eta$ axis along the normal $\mathbf{n}$ [4]. In the vicinity of the origin $\xi=0$, $\eta=0$ (point $(x_0,y_0)$) the line $\Gamma$ is defined by the equation $\eta=f(\xi)\in C^2$ and $f(0)=0$, $f'(0)=0$. The latter is because $f(\xi)$ is tangent to $\Gamma$ at the point $(0,0)$. Then we have

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial\xi}+f'(\xi)\frac{\partial}{\partial\eta},\quad \frac{\partial}{\partial n}=\frac{\partial}{\partial\eta}.$$

Therefore, the integrand on the right-hand side of (2.7) takes the form

$$\left(\frac{\partial w_1}{\partial\xi}+f'(\xi)\frac{\partial w_1}{\partial\eta}\right)\left(\frac{\partial^2 w_2}{\partial\eta\partial\xi}+f'(\xi)\frac{\partial^2 w_2}{\partial\eta^2}\right)-\left(\frac{\partial w_2}{\partial\xi}+f'(\xi)\frac{\partial w_2}{\partial\eta}\right)\left(\frac{\partial^2 w_1}{\partial\eta\partial\xi}+f'(\xi)\frac{\partial^2 w_1}{\partial\eta^2}\right)=$$
$$=\frac{\partial w_1}{\partial\xi}\frac{\partial^2 w_2}{\partial\eta\partial\xi}-\frac{\partial w_2}{\partial\xi}\frac{\partial^2 w_1}{\partial\eta\partial\xi}\equiv I(x_0,y_0) \qquad (2.8)$$

at the point $\xi=0$, $\eta=0$ or $x_0,y_0$. Since near the point $(x_0,y_0)\in\Gamma$ we have $w_{1,2}(\xi,f(\xi))=0$ then $\partial w_{1,2}/\partial\xi=0$ at this point and $I(x_0,y_0)\equiv 0$. One can prove in a similar way that integrals over $\Omega$ of the remaining two terms in (2.6) are equal to zero. Then it follows from (2.5) that eigenfunctions $w_1$ and $w_2$ are orthogonal functions in the sense of $L_2(\Omega)$. Orthogonality of $T_1(x,y)$, $T_2(x,y)$ is beyond doubt. Lemma 1 is proved. □

**Remark 1.** *To prove the orthogonality the boundary condition $w|_\Gamma=0$ (or $T|_\Gamma=0$) is only needed.*

**Lemma 2.** *The eigenfunctions of spectral problem (1.7), (1.9) are real, and eigenvalues are always positive. The eigenfunctions corresponding to different eigenvalues are orthogonal functions in the sense of the space $L_2(\Omega)$.*

Proof is similar to the proof of Lemma 1. Here, instead of (2.2) we obtain the following formula for the spectral parameter

$$A=\rho g\beta\chi\nu^{-1}\cdot\frac{\int\limits_{\Omega}|\nabla T|^2\,dxdy+bk^{-1}\int\limits_{\Gamma}T^2dxdy}{\int\limits_{\Omega}|\nabla w|^2\,dxdy}>0. \qquad (2.9)$$

So eigenvalues are positive functionals of $w$ and $T$ defined by expressions (2.2), (2.9): $A=A(w,T)$. One needs to find minimum values of $A$, that is, one needs to find $A_0=\min\limits_{w\neq 0,T\neq 0}A(w,T)$. In the following paragraphs, this problem is solved for two practically important cross sections, when $\Omega$ is a rectangle or a circle.

# 3. The solution of spectral problem in the case of a rectangular cross-section

Let us consider a rectangular domain $\Omega$ with boundary $\Gamma$:

$$\begin{aligned}\Omega &= \{0 < x < l_1, 0 < y < l_2\}, \\ \Gamma &= \{x = 0\} \cup \{x = l_1\} \cup \{y = 0\} \cup \{y = l_2\}.\end{aligned} \tag{3.1}$$

Let us solve system (1.7) in the region $\Omega$ with boundary condition (1.8). Expressing $w$ from the second equation in (1.7) and substituting it into the first equation in (1.7), we obtain

$$\Delta^2 T = \frac{A\rho g\beta}{\nu\chi} T = \lambda T. \tag{3.2}$$

Let us use the separation of variables. We seek a solution of equation (3.2) in the form $T(x, y) = P(x)F(y)$. Substituting the form into (3.2), we obtain

$$\begin{gathered}P''(x) = \mu_1 P(x), \\ F^{(4)}(y) + 2\mu_1 F''(y) = (\lambda - \mu_1^2)F(y).\end{gathered} \tag{3.3}$$

The original boundary conditions of the problem are reduced to

$$P(0) = P(l_1) = 0, \quad F(0) = F(l_2) = 0. \tag{3.4}$$

Solving system (3.3) with the boundary conditions (3.4), we find

$$P_n(x) = \sin\frac{\pi n}{l_1}x, \quad F_m(y) = \sin\frac{\pi m}{l_2}y; \quad n, m = 1, 2, \ldots \tag{3.5}$$

Hence, taking into account the orthonormalization, we obtain the following solutions

$$\begin{aligned}T_{nm} &= A_{nm}\sin\frac{\pi n}{l_1}x\sin\frac{\pi m}{l_2}y, \quad w_{nm} = B_{nm}\sin\frac{\pi n}{l_1}x\sin\frac{\pi m}{l_2}y, \\ A_{nm} &= \frac{2}{\sqrt{l_1 l_2}}, \qquad B_{nm} = \frac{2A}{\chi\sqrt{l_1 l_2}}\left(\left(\frac{\pi n}{l_1}\right)^2 + \left(\frac{\pi m}{l_2}\right)^2\right)^{-1}.\end{aligned} \tag{3.6}$$

In this case, the eigenvalue is $\lambda_{nm} = \pi\sqrt{n^2/l_1^2 + m^2/l_2^2}$. Then the smallest eigenvalue is $\lambda_{11} = \pi l_1^{-1}\sqrt{1 + l^2}$, where $l = l_1/l_2$. From here we determine the minimum value of the spectral parameter $A_0 = \pi\nu\chi(\rho g\beta l_1)^{-1}\sqrt{1 + l^2}$.

Fig. 1 shows the vertical velocity profile $w_{11}$ as a function of dimensionless coordinates $(\xi, \eta)$, where $\xi = x/l_1$, $\eta = y/l_2$. The fluid flow rate is not equal to zero, so fluid moves in one direction.

If boundary conditions (1.8) are replaced by conditions (1.9) and $b = 0$ (that is, the wall is thermally insulated), we have the following problem

$$\begin{aligned}\nu\Delta w + \rho g\beta T &= 0, \\ \chi\Delta T + Aw &= 0, \ (x, y) \in \Omega; \\ w = 0, \qquad \frac{\partial T}{\partial n} &= 0, \ (x, y) \in \Gamma.\end{aligned} \tag{3.7}$$

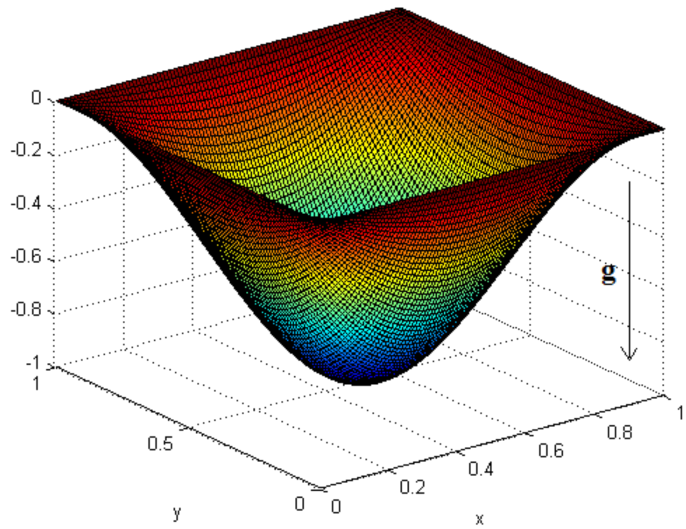


Fig. 1. The velocity profile $w_{11}$ for problem (1.7), (1.8)

Solving this problem, one can obtain the eigenfunctions

$$\begin{aligned} &T_{nm} = A_{nm}\cos\frac{\pi n}{l_1}x\cos\frac{\pi m}{l_2}y, \qquad w_{nm} = B_{nm}\cos\frac{\pi n}{l_1}x\cos\frac{\pi m}{l_2}y, \\ &A_{nm} = \frac{2}{\sqrt{l_1 l_2}}, \qquad B_{nm} = \frac{2A}{\chi\sqrt{l_1 l_2}}\left(\left(\frac{\pi n}{l_1}\right)^2 + \left(\frac{\pi m}{l_2}\right)^2\right)^{-1}. \end{aligned} \tag{3.8}$$

The eigenvalues of the problem are $\lambda_{nm} = \pi\sqrt{n^2/l_1^2 + m^2/l_2^2}$. The smallest eigenvalue is attained at $n = m = 1$. The minimum value of spectral parameter is $A_0 = \pi\nu\chi(\rho g\beta l_1)^{-1}\sqrt{1+l^2}$. Note that the minimum value of spectral parameters of both problems coincide.

Fig. 2 shows the vertical velocity profile $w_{11}$ as a function of dimensionless coordinates $(\xi, \eta)$, where $\xi = x/l_1$, $\eta = y/l_2$. In this case fluid moves in different directions since the flow rate is equal to zero.

## 4. The solution of spectral problem in the case of a circular cross-section

Let us consider domain $\Omega$ in the form of a circle with boundary $\Gamma$

$$\begin{aligned} \Omega &= \{x^2 + y^2 < a^2\}, \\ \Gamma &= \{x^2 + y^2 = a^2\}. \end{aligned} \tag{4.1}$$

Using the change of variables $x = r\cos\phi$, $y = r\sin\phi$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant r \leqslant a$, domain $\Omega$ and boundary $\Gamma$ take the form

$$\begin{aligned} &\Omega = \{r, \phi \mid r < a, \phi \in [0, 2\pi]\}, \\ &\Gamma = \{r, \phi \mid r = a, \phi \in [0, 2\pi]\}. \end{aligned} \tag{4.2}$$

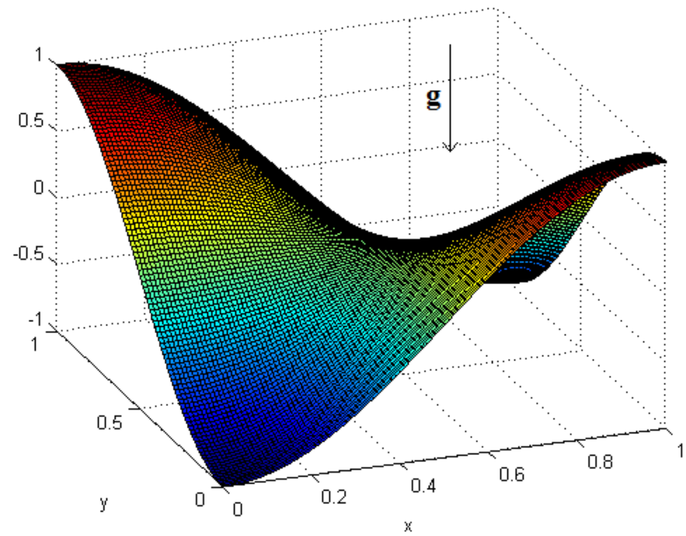


Fig. 2. The velocity profile $w_{11}$ for problem (1.7), (1.9)

Problem (1.7) in domain $\Omega$ with boundary condition (1.8) have the form

$$\begin{gathered}\nu\left(w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\phi\phi}\right)+\rho g\beta T=0,\\ \chi\left(T_{rr}+\frac{1}{r}T_r+\frac{1}{r^2}T_{\phi\phi}\right)+Aw=0,\ (r,\phi)\in\Omega;\\ w=0,\qquad T=0,\ (r,\phi)\in\Gamma.\end{gathered}\tag{4.3}$$

To solve problem (4.3) approach used to solve the problem for a rectangular cross-section is followed. Expressing $w$ from the second equation in (4.3) and substituting it into the first equation in (4.3), we obtain

$$\Delta^2 T=\lambda T;\quad \lambda=\frac{A\rho g\beta}{\nu\chi}.\tag{4.4}$$

Let us use the separation of variables. The solution of equation (4.4) is sought in the form $T(r,\phi)=P(r)F(\phi)$. Substituting this solution into (4.4), we obtain

$$F''+d^2F=0,\tag{4.5}$$

$$r^4P^{(4)}+2r^3P'''-r^2(1+2d^2)P''+r(1+2d^2)P'-(\lambda r^4-d^2(d^2-4))P=0.$$

Because $F$ must be a single-valued function ($F(0)=F(2\pi)$) coefficient $d$ must be an integer, that is, $d=n$. The solution of the first equation in (4.5) is $F_n(\phi)=D_1\cos n\phi+D_2\sin n\phi$. The solution of the second equation in (4.5) is $P_n(r)=A_1J_n(kr)+A_2Y_n(kr)+A_3I_n(kr)+A_4K_n(kr)$, where $k=(\lambda)^{\frac{1}{4}}$ [5]. Thus, the solution of problem (4.4) has the form $T_n(r,\phi)=(A_1J_n(kr)+$ $+A_2Y_n(kr)+A_3I_n(kr)+A_4K_n(kr))(D_1\cos n\phi+D_2\sin n\phi)$.

Let us find the unknown constants $A_i, i=\overline{1,4}$. The natural requirement is $|T(0,\phi)|<\infty$. Functions $Y_n$, $K_n$ are not bounded at the point 0 so we obtain $A_2=A_4=0$. Further, the boundary condition $T=0$ on $\Gamma$ of the original problem is written as

$$A_1J_n(ka)+A_3I_n(ka)=0.\tag{4.6}$$

The boundary condition $w = 0$ on $\Gamma$ results in the relation $\Delta T = 0$, $r = a$. The last expression can be represented as $P''F + r^{-1}P'F + r^{-2}PF'' = 0$. Because $F = D_1 \cos n\phi + D_2 \sin n\phi$ we have the following condition

$$P'' + \frac{1}{r}P' - \frac{n^2}{r^2}P = 0, \quad r = a.$$

Substituting $P(r)$ into the previous relation, we arrive at the equality

$$A_1(J_n''(kr) + \frac{1}{r}J_n'(kr) - \frac{n^2}{r^2}J_n(kr)) + A_3(I_n''(kr) + \frac{1}{r}I_n'(kr) - \frac{n^2}{r^2}I_n(kr)) = 0, \quad r = a.$$

Using recurrent formulas for the Bessel functions, the last condition is reduced to the following equality

$$-A_1 J_n(ka) + A_3 I_n(ka) = 0. \tag{4.7}$$

To find two unknown constants $A_1$ и $A_3$ one should solve the system of two homogeneous equations (4.6) and (4.7) so $A_3 = 0$ and $J_n(ka) = 0$. Then $k = \xi_n^{(i)}/a$, where $\xi_n^{(i)}-$ is the i-th zero of the Bessel function of order n. In our notation we obtain $k^4 = \lambda = \rho g \beta A_n^{(i)} \nu^{-1}\chi^{-1}$. It means that $A_n^{(i)} = \nu\chi\left(\xi_n^{(i)}\right)^4\left(\rho g\beta a^4\right)^{-1}$.

The spectral parameter takes the minimum value at the minimum value of $\xi_n^{(i)}$. It is achieved at $i = 1$, $n = 1$. This follows from the property of zeros of the Bessel functions $\xi_n^{(1)} < < \xi_{n+1}^{(1)} < \xi_n^{(2)} < \xi_{n+1}^{(2)} < \ldots$ [6]. Therefore $A_0 = \nu\chi\left(\xi_1^{(1)}\right)^4\left(\rho g\beta a^4\right)^{-1}$, $\xi_1^{(1)} = 3.83171$.

Finally, the solutions of problem (4.3) are

$$\begin{aligned} T_{ni} &= B_{ni} J_n\left(\frac{\xi_n^{(i)} r}{a}\right)\left\{\begin{array}{c}\cos n\phi \\ \sin n\phi\end{array}\right\}, \\ w_{ni} &= D_{ni}\frac{\chi}{A}\left(\frac{a}{\xi_n^{(i)}}\right)^2 J_n\left(\frac{\xi_n^{(i)} r}{a}\right)\left\{\begin{array}{c}\cos n\phi \\ \sin n\phi\end{array}\right\}. \end{aligned} \tag{4.8}$$

Fig. 3 shows the profile of vertical velocity $w_{11}$. Fluid flow rate is equal to zero so it moves in different directions .

Normalization of functions (4.8) gives the constants

$$B_{ni} = \frac{2}{a^2\left[J_{n+1}(\xi_n^{(i)})\right]^2}, \quad D_{ni} = \frac{2A(\xi_n^{(i)})^2}{\chi a^4\left[J_{n+1}(\xi_n^{(i)})\right]^2}. \tag{4.9}$$

In our case, it is necessary to take $n, i = 1$.

Let us consider system (4.3) in region (4.2) with boundary conditions (1.9) with $b = 0$

$$\begin{aligned} \nu\left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\phi\phi}\right) + \rho g\beta T &= 0, \\ \chi\left(T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\phi\phi}\right) + Aw &= 0, \quad r \in [0, a],\ \phi \in [0, 2\pi]; \\ w = 0, \qquad \frac{\partial T}{\partial r} &= 0, \quad r = a,\ \phi \in [0, 2\pi] \end{aligned} \tag{4.10}$$

Solving this problem in the same manner as for problem (4.3), we obtain that the eigenfunction $T_n$ has the form $T_n(r,\phi) = (C_1 J_n(kr) + C_3 I_n(kr))\left\{\begin{array}{c}\cos n\phi \\ \sin n\phi\end{array}\right\}$. The conditions for finding

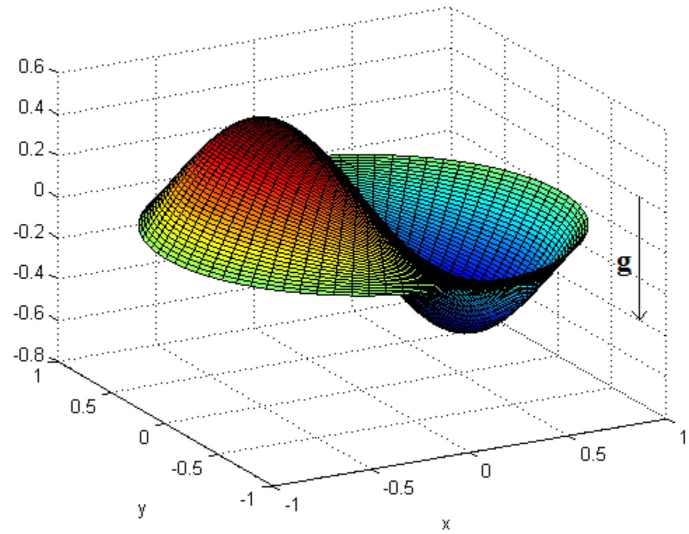


Fig. 3. The profile of velocity $w_{11}$ for problem (4.3)

unknown constants $C_1, C_3$ are

$$\begin{aligned} -C_1 J_n(ka) + C_3 I_n(ka) &= 0\ , \\ C_1 J_n'(ka) + C_3 I_n'(ka) &= 0\ . \end{aligned} \tag{4.11}$$

The solvability condition for system (4.11) is

$$J_n(ka) I_n'(ka) + J_n'(ka) I_n(ka) = 0. \tag{4.12}$$

Equation (4.12) is reduced to an expression without derivatives

$$J_n(ka) I_{n+1}(ka) - J_{n+1}(ka) I_n(ka) + \frac{2n}{ka} J_n(ka) I_n(ka) = 0. \tag{4.13}$$

The properties of solutions of this equation are presented in the next subsection. In particular, solutions are real and positive. Let us denote the i-th root of equation (4.13) by $\gamma_n^{(i)} = ka$. There are infinitely many roots. Hence, the spectral parameter $A$ is

$$A_n^i = \frac{\nu\chi}{\rho g \beta a^4} \left(\gamma_n^{(i)}\right)^4. \tag{4.14}$$

Returning to system (4.11), we find that $C_3 = A_1 J_n\left(\gamma_n^{(i)}\right) / I_n\left(\gamma_n^{(i)}\right)$. Finally, we obtain that eigenfunctions of problem (4.10) are

$$\begin{aligned} T_{ni} &= B_{ni} \frac{C_1}{I_n\left(\gamma_n^{(i)}\right)} \left[ I_n\left(\gamma_n^{(i)}\right) J_n\left(\frac{\gamma_n^{(i)} r}{a}\right) + J_n\left(\gamma_n^{(i)}\right) I_n\left(\frac{\gamma_n^{(i)} r}{a}\right)\right] \left\{ \begin{array}{c} \cos n\phi \\ \sin n\phi \end{array} \right\}, \qquad (4.15) \\ w_{ni} &= D_{ni} \frac{\chi}{A} \left(\frac{a}{\gamma_n^{(i)}}\right)^2 \frac{C_1}{I_n\left(\gamma_n^{(i)}\right)} \left[ I_n\left(\gamma_n^{(i)}\right) J_n\left(\frac{\gamma_n^{(i)} r}{a}\right) + J_n\left(\gamma_n^{(i)}\right) I_n\left(\frac{\gamma_n^{(i)} r}{a}\right)\right] \left\{ \begin{array}{c} \cos n\phi \\ \sin n\phi \end{array} \right\}. \end{aligned}$$

To find the minimum value of spectral parameter $A_0$, one need to choose the smallest root of equation (4.13). Unfortunately, in the well-known reference books [6, 7, 8] there is no solution of this equation so the roots were determined numerically. It was obtained that the magnitude of roots increases with their number and with an increase of the order of the Bessel function. Also the numerical results show that the equation has no complex roots, which is consistent with Lemma 2.

Thus, the root $\gamma_0^{(1)} = 4.6109$ should be chosen since it is the smallest root of equation (4.13). So we have $A_0 = \nu\chi\left(\gamma_0^{(1)}\right)^4\left(\rho g\beta a^4\right)^{-1}$.

As shown in Section 5, one can obtain asymptotic values of the roots of equation (4.13). Comparison of the asymptotic values of the roots with the numerically obtained roots for $n = 0$ is presented in Tab. 1. One can see that the difference between two adjacent roots tends to $\pi$ when the number of the root increases.

Table 1. The approximate values of roots of equation (4.13). First row presents results of numerical solution of equation (4.13). Second row presents asymptotic values (5.6)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Numerical calculation | 4.6109 | 7.7993 | 10.9518 | 14.1087 | 17.2557 | 20.401 | 23.5354 | 26.6889 | 29.8321 |
| Calculation by the formula (5.6) | 4.7124 | 7.854 | 10.9956 | 14.1372 | 17.2788 | 20.4204 | 23.5619 | 26.7035 | 29.8451 |

The profile of vertical velocity $w_{01}$ is shown in Fig. 4. The fluid moves in one direction since the flow rate is not equal to zero in this case.

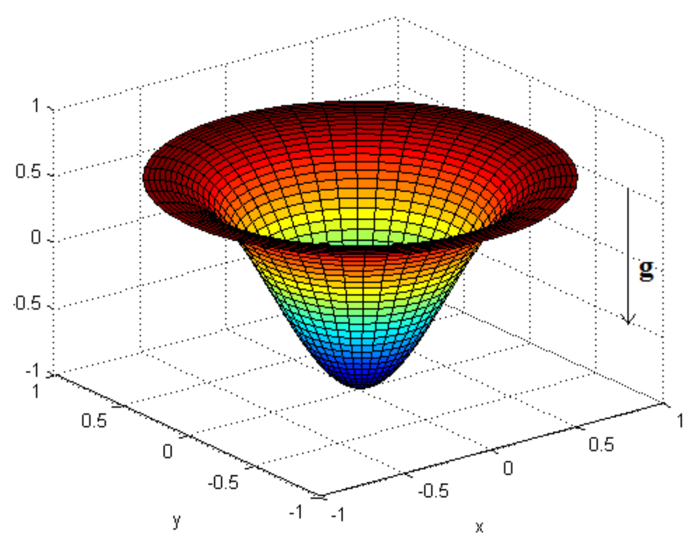


Fig. 4. The profile of velocity $w_{01}$ for problem (4.10)

## 5. Some properties of the roots of equation (4.13)

Let us assume that $z = ka$ is the complex root of equation (4.13).

**Lemma 3.** *The roots of equation (4.13) are real and isolated except for zero. Roots are symmetrically located with respect to the point* 0 *and they have no finite limit points.*

*Proof.* Using representation [6, 7]

$$\begin{aligned} J_n(z) &= \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!2^{2k}} z^{2k}, \\ I_n(z) &= \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!2^{2k}} z^{2k} \ , \end{aligned}$$

we rewrite equation (4.13) as

$$\begin{aligned} \left(\frac{z}{2}\right)^{2n} \Bigg\{ \left(\frac{z}{2}\right)^2 \Bigg[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!2^{2k}} z^{2k} \sum_{k=0}^{\infty} \frac{1}{k!(k+n+1)!2^{2k}} z^{2k} \ - \\ - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n+1)!2^{2k}} z^{2k} \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!2^{2k}} z^{2k} \Bigg] + \\ + n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!2^{2k}} z^{2k} \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!2^{2k}} z^{2k} \Bigg\} = 0. \end{aligned} \tag{5.1}$$

It is clear that the root $z = 0$ has at least multiplicity $2n$. We transform the expression in curly brackets of equation (5.1). If there are two converging power series $\sum\limits_{k=0}^{\infty} a_k z^{2k}$, $\sum\limits_{k=0}^{\infty} b_k z^{2k}$ then their product is the power series $\sum\limits_{k=0}^{\infty} c_k z^{2k}$ with coefficients $c_k = \sum\limits_{j=0}^{k} a_j b_{k-j}$ (by induction). Designating the coefficients for the products $c_k^1, c_k^2, c_k^3$, sequentially, we find

$$\begin{aligned} c_k^1 &= \sum_{j=0}^{k} \frac{(-1)^j}{j!(j+n)!(k-j)!(k-j+n+1)!2^{2k}} \ , \\ c_k^2 &= \sum_{j=0}^{k} \frac{(-1)^j}{j!(j+n+1)!(k-j)!(k-j+n)!2^{2k}} \ , \\ c_k^3 &= \sum_{j=0}^{k} \frac{(-1)^j}{j!(j+n)!(k-j)!(k-j+n)!2^{2k}} \ . \end{aligned} \tag{5.2}$$

Then we have

$$\begin{aligned} c_k^1 - c_k^2 &= \sum_{j=0}^{k} \frac{(-1)^j}{j!(j+n)!(k-j)!(k-j+n)!2^{2k}} \left[ \frac{1}{k-j+n+1} - \frac{1}{j+n+1} \right] = \\ &= \sum_{j=0}^{k} \frac{(-1)^j (2j-k)}{j!(j+n+1)!(k-j)!(k-j+n+1)!2^{2k}} \equiv d_k. \end{aligned} \tag{5.3}$$

Now we obtain from (5.1) the following equation

$$\sum_{k=1}^{\infty} \frac{d_k}{4} z^{2k+2} - \sum_{k=0}^{\infty} n c_k^3 z^{2k} = 0, \tag{5.4}$$

because $d_0 = 0$. The last equation is transformed into (the replacement $k+1 \leftrightarrow k$ replacement for the first row)

$$-nc_0^3 - nc_1^3 z^2 + \sum_{k=2}^{\infty} \left( \frac{d_{k-1}}{4} - nc_k^3 \right) z^{2k} = 0,$$

or

$$\sum_{k=0}^{\infty} d_k^1 z^{2k} = 0, \quad d_0^1 = -nc_0^3, \quad d_1^1 = -nc_1^3, \quad d_k^1 = \frac{d_{k-1}}{4} - nc_k^3, \quad k \geqslant 2. \tag{5.5}$$

Thus $d_0^1 = -n/(n!)^2 \neq 0$ when $n \neq 0$, $d_1^1 = 0$ then multiplicity of the root $z = 0$ is $2n$. For $n = 0$ it follows from (5.4) that multiplicity of the root $z = 0$ is two, that is, $d_1 \neq 0$, according to (5.3).

In (5.5) all the numbers $d_k^1, k = 0, 1, 2, \ldots$ are real, therefore $z$ and $\bar{z}$ are solutions. Let $z = |z|e^{i\phi}$ and $\bar{z} = |z|e^{-i\phi}$ are roots. Then we obtain from (5.5)

$$\sum_{k=0}^{\infty} d_k^1 |z|^{2k} (\cos 2k\phi \pm i \sin 2k\phi) = 0.$$

Then $\sin 2k\phi = 0$ at $\phi = 0, \pi$; $\cos 2k\phi = 1$, $k = 0, 1, 2, \ldots$ and the imaginary part of the root $z$ is zero.

So, the non-zero roots of equation (4.13) are real, $z = \gamma \in R^1$. If $\gamma$ is a root, $-\gamma$ is also a root. So it is sufficient to find positive roots. Further, function $f(\gamma) = \sum\limits_{k=0}^{\infty} d_k^1 \gamma^{2k}$ is a whole function. Zeros of the function are isolated and they cannot have finite limit points [9]. The Lemma is proved. □

**Remark 2.** For $\gamma \gg 1$ [8]

$$\begin{aligned} J_n(\gamma) &= \sqrt{\frac{2}{\pi\gamma}} \cos\left(\gamma - \frac{\pi n}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{\sqrt{\gamma}}\right), \\ I_n(\gamma) &= \sqrt{\frac{1}{2\pi\gamma}} e^{\gamma} \left[1 + O\left(\frac{1}{\gamma}\right)\right], \end{aligned}$$

and the roots of equation (4.13) have asymptotic representation

$$\gamma_n^{(m)} \approx \frac{(n+1)\pi}{2} + m\pi, \ m \gg 1. \tag{5.6}$$

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## Об одной спектральной задаче для уравнений конвекции

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**Аннотация.** Рассматриваются спектральные задачи, возникающие при моделировании стационарных однонаправленных конвективных течений в вертикальных теплообменниках при различных температурных режимах на их границах. Роль спектрального параметра играет постоянный температурный градиент на вертикальных стенках. При этом поперечное сечение теплообменника может быть произвольной формы. Установлены общие свойства решений спектральных задач. Для практически важных сечений — прямоугольника и круга — решения получены в аналитическом виде. Найдены критические значения градиента температуры, при которых возникает конвективное течение, и построены соответствующие профили вертикальной скорости. Изучены свойства решений нового трансцендентного уравнения, определяющего спектральные значения.

**Ключевые слова:** конвекция, спектральная задача, собственные функции, собственные значения.