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Connecting Homomorphism and Separating Cycles

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Abstract. We discuss the construction of a long semi-exact Mayer–Vietoris sequence for the homology of any finite union of open subspaces. This sequence is used to obtain topological conditions of representation of the integral of a meromorphic n-form on an n-dimensional complex manifold in terms of Grothendieck residues. For such a representation of the integral to exist, it is necessary that the cycle of integration separates the set of polar hypersurfaces of the form. The separation condition in a number of cases turns out to be a sufficient condition for representing the integral as a sum of residues. Earlier, when describing such cases (in the works of Tsikh, Yuzhakov, Ulvert, etc.), the key was the condition that the manifold be Stein. The main result of this article is the relaxation of this condition.

Keywords: Mayer–Vietoris sequence, Grothendieck residue, separating cycle.

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Introduction

In the theory of functions of one complex variable the Cauchy residue of a function f at an isolated singular point a is represented in a local coordinate z by the integral over the cycle $\gamma^{(a)} = \{|z-a| = \varepsilon\}$ in a sufficiently small punctured neighborhood $U_a \setminus \{a\}$. The cycle $\gamma^{(a)}$ is called *local* cycle at a. By Cauchy's theorem the definition of the residue does not depend on the choice of a local cycle (choice of the local coordinate and the radius ε). It's usually not difficult to represent the integral

$$\int_\gamma f\,dz$$

of a meromorphic function f over a cycle γ lying outside the polar set of the function as a sum of residues: it suffices to know the homological expansion of the cycle γ in terms of local cycles.

The multidimensional analogue of the Cauchy residue is the *Grothendieck residue* of a meromorphic differential n-forms ω given on an n-dimensional complex-analytic manifold. This residue in turn is represented by the integral over a local n-cycle in a neighborhood of an isolated intersection point of polar hypersurfaces of ω. In this case, it is possible to show that in order for the integral of a meromorphic form to be represented in terms of residues, it is necessary that the cycle of integration in a certain sense *separates* the set of polar hypersurfaces of the form. The most complete results on the characterization of such separating cycles and their relationship to

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local cycles in Stein manifolds are presented by Tsikh and Yuzhakov (see [5, 9]) (these results have recently been complemented in [6, 7]). In this article, we develop a method for studying separating cycles, which makes it possible to weaken the homological conditions for the manifold and a family of polar hypersurfaces of integrating form, abandoning the Stain property of the manifold.

Our main tool is a generalization of the well-known Mayer–Vietoris long exact sequence. All the necessary information about the homology of the union of open subspaces (the homology of the topological space with open cover $\mathfrak{U}$) is discussed in Section 1. The case of the union of more than two subspaces leads to the study of the double complex of a cover and the *Mayer–Vietoris spectral sequence* (see [3]).

In Section 2 we construct a connecting homomorphism that allows us to get the long Mayer–Vietoris sequence for any finite cover. This sequence is not exact in the general case, but it is semi-exact. In obtaining our results, ideas from Gleason's article [4] are essentially used. The notion of the *resolution for a cycle* associated with an open covering of the space (the $\mathfrak{U}$-resolution) in [4] is not standard, but it is very appropriate in our opinion.

The main results are presented in Section 3. Note that our results are formulated under the assumption that the $(2n-1)$-dimensional homology of the manifold is trivial and that the intersection of the set of n polar hypersurfaces of the integrating form ω is discrete. In terms of the corresponding long Mayer–Vietoris sequence, we have obtained (Theorem 3.1) a necessary and sufficient condition under which any separating cycle is represented in terms of local cycles (and therefore the integral is calculated in terms of the residues). However, this is only a reformulation of the problem in the language of homological algebra. Theorem 3.2 gives a more practical sufficient condition (in terms of homology of complements of polar hypersurfaces of the form ω) under which any separating cycle is represented in terms of local cycles. This condition, in particular, is satisfied for Stein manifolds and arbitrary set of n polar hypersurfaces, which allows us to obtain another proof of the Tsikh theorem on separating cycles in Stein manifolds (Theorem 3.3). Therefore, the condition from Theorem 3.2 gives the desired relaxation of the condition of Steinnes of the manifold.

1. Homology of the union of open subspaces

Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of topological space X, where I is an ordered index set. We will use the standard notation $S_q(X)$ for the group of singular chains of dimension q (with coefficients in $\mathbb{C}$) in X. Also, by $S_* = S_*(X)$ we will denote the corresponding chain complex with the boundary operator ∂.

Definition 1.1. *A $\mathfrak{U}$-chain in X of multiplicity p and dimension q is an alternating function σ on I^{p+1} with values*

$$\sigma(i_0, i_1, \ldots, i_p) \in S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}),$$

which vanishes except at a finite number of a points of I^{p+1}.

Note that $\mathfrak{U}$-chains can be identified with elements of the bigraded group

$$C_{p,q} = \bigoplus_{i_0<i_1<\ldots<i_p} S_q(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p}), \quad p, q = 0, 1, \ldots$$

We denote by $S_q^{\mathfrak{U}} = S_q^{\mathfrak{U}}(X)$ the subgroup in $S_q = S_q(X)$ generated by singular q-simplices Δ, such that $\operatorname{supp}\Delta \subset U_i$ for some $U_i \in \mathfrak{U}$. The natural inclusion $\iota\colon S_*^{\mathfrak{U}} \to S_*(X)$ obviously is a chain map.

For the chain complex C_* with the boundary operator d we use the following standard notations for subgroups of cycles, subgroups of boundaries and homology groups:

$$Z_q(C_*) = \ker(d\colon C_q \to C_{q-1}),$$
$$B_q(C_*) = \operatorname{im}(d\colon C_{q+1} \to C_q),$$
$$H_q(C_*) = Z_q(C_*)/B_q(C_*).$$

The following fact shows that when calculating the homology of the space X it is sufficient to use the complex $S_*^{\mathfrak{U}}$.

Theorem 1.1 (see [8]). *The homomorphism $\iota_*\colon H(S_*^{\mathfrak{U}}) \to H(X)$ induced by the chain map $\iota\colon S_*^{\mathfrak{U}} \to S_*(X)$ is the isomorphism.*

Consider first the case $I = \{1, 2\}$. In this situation, there is the well-known long exact Mayer–Vietoris sequence for homology of the union of two open subsets. This sequence is obtained from short exact sequences

$$S_q^{\mathfrak{U}}(X) \xleftarrow{\varepsilon} S_q(U_1) \oplus S_q(U_2) \xleftarrow{\delta} S_q(U_1 \cap U_2), \tag{1}$$

where $\varepsilon\colon (\sigma_1, \sigma_2) \mapsto \sigma_1 + \sigma_2$ is an epimorphism and $\delta\colon \sigma \mapsto (\sigma, -\sigma)$ is a monomorphism, at that ε and δ are chain maps. Passing to homology, we get sequences

$$H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(U_1) \oplus H_q(U_2) \xleftarrow{\delta_*} H_q(U_1 \cap U_2),$$

in which in general ε_* is not an epimorphism and δ_* is not a monomorphism. The following properties hold: 1) $\operatorname{im}\delta_* = \ker\varepsilon_*$; 2) the *connecting homomorphism* $\varphi\colon H_q(S_*^{\mathfrak{U}}) \to H_{q-1}(U_1 \cap U_2)$ induced by the multivalued map $\delta^{-1}\partial\varepsilon^{-1}$ is correctly defined, at that $\operatorname{im}\varphi = \ker\delta_*$ and $\ker\varphi = \operatorname{im}\varepsilon_*$. We get the required long exact sequence:

$$\ldots \longleftarrow H_{q-1}(U_1 \cap U_2) \xleftarrow{\varphi} H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(U_1) \oplus H_q(U_2) \xleftarrow{\delta_*} H_q(U_1 \cap U_2) \longleftarrow \ldots,$$

in which groups $H_q(S_*^{\mathfrak{U}})$ are replaced by isomorphic groups $H_q(X)$.

Let now $\operatorname{card}(I) \geqslant 2$. We will describe a sequence of chain groups generalizing the short exact sequence (1). Inclusions $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_p} \hookrightarrow U_{i_0} \cap U_{i_1} \cap \ldots [i_k] \ldots \cap U_{i_p}$, $k = 0, \ldots, p$, induce the Čech boundary operator $\delta\colon C_{p,q} \to C_{p-1,q}$ defined by the formula "alternating sum":

$$(\delta\sigma)(i_0, i_1, \ldots, i_{p-1}) = \sum_{i \in I} \sigma(i, i_0, \ldots, i_{p-1}).$$

In turn, the inclusions $U_i \subset X$ induce the operator $\varepsilon\colon C_{0,q} \to S_q^{\mathfrak{U}}$, which acts according to the same "alternating sum" formula as follows:

$$\varepsilon\sigma = \sum_{i \in I} \sigma(i),$$

at that $\varepsilon\delta = 0$. We obtain the following Mayer–Vietoris sequence for the groups of singular chains of the union

$$0 \longleftarrow S_q^{\mathfrak{U}} \xleftarrow{\varepsilon} C_{0,q} \xleftarrow{\delta} C_{1,q} \xleftarrow{\delta} C_{2,q} \xleftarrow{\delta} \ldots. \tag{2}$$

Theorem 1.2 (see [1]). *The sequence* (2) *is exact for all* $q = 0, 1, \ldots$.

If $\mathrm{card}(I) = 2$ then this sequence coincides with the sequence (1). In what follows, we will assume that $\mathrm{card}(I) > 2$. Maps $\delta\colon C_{p,*} \to C_{p-1,*}$ and $\varepsilon\colon C_{0,*} \to S^{\mathfrak{U}}_*$ are, as above, chain maps of the corresponding complexes with the boundary operator ∂. In passing to homology, the exact sequence (2) goes over into the sequence

$$0 \longleftarrow H_q(S^{\mathfrak{U}}_*) \xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} H_q(C_{1,*}) \xleftarrow{\delta_*} H_q(C_{2,*}) \xleftarrow{\delta_*} \dots, \tag{3}$$

about which, in general, we can say that it is only semi-exact. In this case, the measure of "inexactness" is the homology groups of the sequence (3), considered as a chain complex with a boundary operator δ_* (or ε_*).

More abstract point of view on the generalization of the Mayer–Vietoris sequence for homology of the union relates to consideration of two spectral sequences of the double complex $C = (C_{p,q}; \delta, \partial)$ (see [2, 3]). This double complex is a *first quarter complex* ($C_{p,q} = 0$ for $p < 0$ or $q < 0$). By adding to C a column $(S^{\mathfrak{U}}_q; \partial)$ and a chain map $\varepsilon\colon C_{0,*} \to S^{\mathfrak{U}}_*$ we get the *extended double complex*

$$\begin{tikzcd}
 & \downarrow & \downarrow & \downarrow & \\
0 & S^{\mathfrak{U}}_q \arrow[l] \arrow[d, "\partial"'] & C_{0,q} \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] & C_{1,q} \arrow[l, "\delta"'] \arrow[d, "\partial"'] & \dots \arrow[l, "\delta"'] \\
0 & S^{\mathfrak{U}}_{q-1} \arrow[l] \arrow[d, "\partial"'] & C_{0,q-1} \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] & C_{1,q-1} \arrow[l, "\delta"'] \arrow[d, "\partial"'] & \dots \arrow[l, "\delta"'] \\
 & \dots \arrow[d, "\partial"'] & \dots \arrow[d, "\partial"'] & \dots \arrow[d, "\partial"'] & \\
0 & S^{\mathfrak{U}}_0 \arrow[l] \arrow[d] & C_{0,0} \arrow[l, "\varepsilon"'] \arrow[d] & C_{1,0} \arrow[l, "\delta"'] \arrow[d] & \dots \arrow[l, "\delta"'] \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4}$$

of singular chains which is dual to the familiar Čech–de Rham double complex for differential forms. Based on a double complex C we build a *total complex* TC, formed by a graded group

$$(TC)_n = \bigoplus_{p+q=n} C_{p,q}$$

and a boundary operator $D\colon (TC)_n \to (TC)_{n-1}$ such that $D\big|_{C_{p,q}} = \delta + (-1)^p\partial$.

The first of the spectral sequences $\{(E^r_{p,q}; d^r)\}$ of the complex C corresponds to filtration for TC determined by the formula $F_p(TC)_n = \bigoplus_{i\leqslant p} C_{i,n-i}$. We have $E^0_{p,q} = C_{p,q}$ and $d^0 = \pm\partial$, so $E^1_{p,q} = H_q(C_{p,*})$ (the vertical homology of the complex C) and the differential $d^1\colon E^1_{p,q} \to E^1_{p-1,q}$ coincides with the map induced by the chain map $\delta\colon C_{p,*} \to C_{p-1,*}$, i.e. $d^1 = \delta_*$. Further, the term $E^2_{p,q}$ (the horizontal homology of the vertical homology of the complex C) describes the homology of the sequence (3). Therefore, this spectral sequence (called the Mayer–Vietoris spectral sequence) is a generalization of the long exact Mayer–Vietoris sequence.

The second spectral sequence for C is determined by another filtration of the total complex: $F_p(TC)_n = \bigoplus_{j\leqslant p} C_{n-j,j}$. In this case $E^0_{p,q} = C_{q,p}$, so E^1 is determined by the horizontal homology of the complex C. Since the strings of the extended complex (4) are exact, then $E^1_{p,0} = S^{\mathfrak{U}}_p$ and $E^1_{p,q} = 0$ for $q > 0$. Considering the vertical homology, we obtain $E^2_{p,0} = H_p(X)$ and $E^2_{p,q} = 0$ for $q > 0$. This means that the second spectral sequence "degenerates" and gives the isomorphism $H_q(TC) \cong H_q(S^{\mathfrak{U}}_*)$. Thus, the double complex C "calculates" the homology of the space X.

2. Connecting homomorphism

Pursuing the goal of finding a generalization of the long exact Mayer–Vietoris sequence in the case of any finite open cover of the space X, we construct a homomorphism, which is a connecting homomorphism for semi-exact sequences (3).

We will use the following notation:

$$C^{\partial}_{p,q} = Z_q(C_{p,*}) = \ker(\partial\colon C_{p,q} \to C_{p,q-1}).$$

For the chain complex (2) consider the subcomplex

$$0 \longleftarrow Z_q(S^{\mathfrak{U}}_*) \xleftarrow{\varepsilon} C^{\partial}_{0,q} \xleftarrow{\delta} C^{\partial}_{1,q} \xleftarrow{\delta} C^{\partial}_{2,q} \xleftarrow{\delta} \dots.$$

In general, there are nontrivial homology groups $H_p(C^{\partial}_{*,q}) = Z_p(C^{\partial}_{*,q})/B_p(C^{\partial}_{*,q})$, $p = 0, 1, \dots$

Lemma 2.1. *Let ξ be the cycle belongs to $Z_p(C^{\partial}_{*,q})$, where $p \geqslant 0$, $q \geqslant 1$, and $\xi = \delta\xi_{p+1}$ for some $\mathfrak{U}$-chain $\xi_{p+1} \in C_{p+1,q}$. Then $\partial\xi_{p+1}$ belongs to $Z_{p+1}(C^{\partial}_{*,q-1})$ and the homology class $[\partial\xi_{p+1}] \in H_{p+1}(C^{\partial}_{*,q-1})$ depends only on the class $[\xi] \in H_p(C^{\partial}_{*,q})$. The correspondence $[\xi] \to [\partial\xi_{p+1}]$ defines a homomorphism*

$$\varphi_{p+1} = (\partial\delta^{-1})_*\colon H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1}).$$

Proof. Note that the existence of a $\mathfrak{U}$-chain ξ_{p+1} such that $\xi = \delta\xi_{p+1}$ follows from the fact that strings of the complex (4) are exact, due to condition $\delta\xi = 0$ ($\varepsilon\xi = 0$ for $p = 0$). We have $\partial(\partial\xi_{p+1}) = 0$, $\delta(\partial\xi_{p+1}) = \partial(\delta\xi_{p+1}) = \partial\xi = 0$ (see the diagram (5)), so $\partial\xi_{p+1} \in Z_{p+1}(C^{\partial}_{*,q-1})$.

$$\begin{tikzcd} 0 & \xi \arrow[l, "\delta"'] \arrow[d, "\partial"'] \arrow[dr, dashed, "\varphi_{p+1}"] & \xi_{p+1} \arrow[l, "\delta"'] \arrow[d, "\partial"] \\ & 0 & \partial\xi_{p+1} \arrow[l, "\delta"] \end{tikzcd} \qquad \begin{tikzcd} 0 & \xi \arrow[l, "\varepsilon"'] \arrow[d, "\partial"'] \arrow[dr, dashed, "\varphi_1"] & \xi_1 \arrow[l, "\delta"'] \arrow[d, "\partial"] \\ & 0 & \partial\xi_1 \arrow[l, "\delta"] \end{tikzcd} \tag{5}$$

If also $\xi = \delta\xi'_{p+1}$, then $\delta(\xi_{p+1} - \xi'_{p+1}) = 0$. So there exists a $\mathfrak{U}$-chain $\tau \in C_{p+2,q}$ such that $\xi_{p+1} - \xi'_{p+1} = \delta\tau$. This implies $\xi'_{p+1} = \xi_{p+1} - \delta\tau$. We have $\partial\xi'_{p+1} = \partial(\xi_{p+1} - \delta\tau) = \partial\xi_{p+1} - \partial(\delta\tau)$, at that $\partial(\delta\tau) \in B_{p+1}(C^{\partial}_{*,q-1})$, because $\partial(\delta\tau) = \delta(\partial\tau)$, where $\partial(\partial\tau) = 0$. Hence it follows $[\partial\xi_{p+1}] = [\partial\xi'_{p+1}] \in H_{p+1}(C^{\partial}_{*,q-1})$.

Finally, let ξ' be an arbitrary representative of the class $[\xi] \in H_p(C^{\partial}_{*,q})$. Then there exist a $\mathfrak{U}$-chain $\sigma \in C_{p+1,q}$, for which $\xi - \xi' = \delta\sigma$ and $\partial\sigma = 0$. If $\delta\xi_{p+1} = \xi$, then $\delta(\xi_{p+1} - \sigma) = \xi - \delta\sigma = \xi'$. The class $[\xi']$ maps to the class of the $\mathfrak{U}$-chain $\partial(\xi_{p+1} - \sigma) = \partial\xi_{p+1} - \partial\sigma = \partial\xi_{p+1}$, which was required.

It remains to note that for the map $[\xi] \mapsto [\partial\xi_{p+1}]$ the image of the sum of classes is obviously equal to the sum of the images, so this map is a homomorphism. □

Remark 2.1. We have $C^{\partial}_{p,0} = C_{p,0}$ for $p \geqslant 0$, so $H_p(C^{\partial}_{*,0}) = H_p(C_{*,0}) = 0$. Hence for $q = 1$ the homomorphism described in the Lemma 2.1 is trivial.

Lemma 2.2. *Let be the cycle belongs to ξ $Z_q(S^{\mathfrak{U}}_*)$, where $q \geqslant 1$, and $\xi = \varepsilon\xi_0$ for some $\mathfrak{U}$-chain $\xi_0 \in C_{0,q}$. Then $\partial\xi_0$ belongs to $Z_0(C^{\partial}_{*,q-1})$ and the homology class $[\partial\xi_0] \in H_0(C^{\partial}_{*,q-1})$ depends only on the class $[\xi] \in H_q(S^{\mathfrak{U}}_*)$. The correspondence $[\xi] \to [\partial\xi_0]$ defines a homomorphism*

$$\varphi_0 = (\partial\varepsilon^{-1})_*\colon H_q(S^{\mathfrak{U}}_*) \to H_0(C^{\partial}_{*,q-1}).$$

Proof. Existence of a $\mathfrak{U}$-chain ξ_0, such that $\xi = \varepsilon\xi_0$, follows from the surjectivity of ε. The fact that $\partial\xi_0 \in Z_0(C^{\partial}_{*,q-1})$ and independence of the class $[\partial\xi_0] \in H_0(C^{\partial}_{*,q-1})$ from choice of ξ_0 can be proved in the same way as in Lemma 2.1.

Let ξ' represents a class $[\xi] \in H_q(S^{\mathfrak{U}}_*)$. So there exists a chain $\sigma \in S^{\mathfrak{U}}_{q+1}$, such that $\xi - \xi' = \partial\sigma$. Since ε is an epimorphism, there exists $\tau \in C_{0,q+1}$, such that $\sigma = \varepsilon\tau$. We have $\varepsilon(\xi_0 - \partial\tau) = \xi - \varepsilon(\partial\tau) = \xi - \partial(\varepsilon\tau) = \xi - \partial\sigma = \xi'$, therefore ∂-cycle ξ' corresponds to the class $[\partial(\xi_0 - \partial\tau)] = [\partial\xi_0 - \partial\partial\tau] = [\partial\xi_0]$. In this way the map $[\xi] \mapsto [\partial\xi_0]$ is correctly defined. Obviously, this is a homomorphism. □

Remark 2.2. For $q = 1$ the homomorphism φ_0 is trivial.

The last two lemmas allow us to write the following "diagonal" sequence of maps:

$$H_r(S^{\mathfrak{U}}_*) \xrightarrow{\varphi_0} H_0(C^{\partial}_{*,r-1}) \xrightarrow{\varphi_1} H_1(C^{\partial}_{*,r-2}) \to \ldots \to H_{r-2}(C^{\partial}_{*,1}) \xrightarrow{\varphi_{r-1}} H_{r-1}(C^{\partial}_{*,0}) \cong 0. \qquad (6)$$

The sequence of homomorphisms (6) is related to the following notion of a *resolution for a cycle* by Gleason (see [4]).

Definition 2.1. *A $\mathfrak{U}$-resolution for the cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$ is a sequence $\{\xi_p\}_{p=0}^r$ of $\mathfrak{U}$-chains, $\xi_p \in C_{p,r-p}$, such that:*

1) $\varepsilon\xi_0 = \xi$;

2) $\delta\xi_p = \partial\xi_{p-1}$, $p = 1, \ldots, r$.

Remark 2.3. The existence of a $\mathfrak{U}$-resolution $\{\xi_p\}$ for any cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$ follows from the fact that the strings of the complex (4) are exact. Gleason's definition of a resolution suggests that $\xi \in Z_r(S_*)$. So the condition $\xi \in Z_r(S^{\mathfrak{U}}_*)$ is given as a criterion for the existence of the resolution.

Remark 2.4. The $\mathfrak{U}$-resolution for a cycle ξ (accurate to sign $\pm$ in front of its term) is, in fact, such D-cycle, which represents the image of the class $[\xi]$ under the isomorphism $H_q(S^{\mathfrak{U}}_*) \to H_q(TC)$. It is a "zig-zag" of the double complex C.

Comparing the definitions of homomorphisms φ_{p+1} and φ_0 from Lemmas 2.1, 2.2 with Definition 2.1, we obtain the following statement.

Proposition 2.1. *Let $\{\xi_p\}$ be the $\mathfrak{U}$-resolution of a cycle $\xi \in Z_r(S^{\mathfrak{U}}_*)$. Then the sequence of images of the cycle $[\xi] \in H_r(S^{\mathfrak{U}}_*)$ under homomorphisms* (6) *has the form*

$$[\xi] \longmapsto [\partial\xi_0] \longmapsto [\partial\xi_1] \longmapsto \ldots \longmapsto [\partial\xi_{r-2}] \longmapsto [\partial\xi_{r-1}] = [0].$$

In a similar way, one can consider a part of the sequence of homomorphisms (6) starting from the group $H_p(C^{\partial}_{*,q})$ for $p \geqslant 0$ and $q \geqslant 1$. In this case, we have the following sequence of homomorphisms

$$H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1}) \to \ldots \to H_{p+q-1}(C^{\partial}_{*,1}) \to H_{p+q}(C^{\partial}_{*,0}) \simeq 0,$$

which is naturally leads to following version of the notion of the $\mathfrak{U}$-resolution.

Definition 2.2. *Let ξ by a $\mathfrak{U}$-chain of the multiplicity $p \geqslant 0$ and the dimension $q \geqslant 1$ such that $\partial\xi = 0$ and $\delta\xi = 0$ ($\varepsilon\xi = 0$ for $p = 0$). A sequence $\{\xi_k\}$ of $\mathfrak{U}$-chains $\xi_k \in C_{k,p+q-k+1}$, is said to be the $\mathfrak{U}$-resolution of the ξ, if the following conditions hold:*

1) $\delta\xi_{p+1} = \xi$;

2) $\delta\xi_k = \partial\xi_{k-1}$, $k = p+2, \ldots, p+q+1$.

Remark 2.5. As in the case $\xi \in Z_r(S_*^{\mathfrak{U}})$, the resolution exists for any cycle $\xi \in Z_p(C_{*,q}^{\partial})$.

The existence of the connecting homomorphism assumes that the open cover $\mathfrak{U}$ of topological space X is finite ($\mathrm{card}(I) < \infty$). In what follows, we will assume that this covering consists of m ($m \geqslant 2$) elements.

In this case, for the double complex (4), we have $C_{p,q} \cong 0$ for $p \geqslant m$. Since the strings of the complex are exact, we see that $\delta\colon C_{m-1,*} \to C_{m-2,*}$ is monomorphism. Hence

$$Z_{m-1}(C_{*,q}) \cong 0, \quad H_{m-1}(C_{*,q}^{\partial}) = H_{m-1}(C_{*,q}) \cong 0.$$

For any resolution $\{\xi_k\}$ of a cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$, $r \geqslant m$, we have $\xi_k = 0$ for all $k > m-1$, at that $\partial\xi_{m-1} = 0$. We assume that the ∂-cycle ξ_{m-1} is the end term of this resolution, ignoring the following zero terms. We will proceed similarly with the resolution of cycle $\xi \in Z_p(C_{*,q}^{\partial})$.

The following statement is the last step to the construction of the desired generalization of a connecting homomorphism.

Lemma 2.3. *Let $\xi \in Z_{m-3}(C_{*,q}^{\partial})$, $q \geqslant 1$, and let $\{\xi_{m-2}, \xi_{m-1}\}$ be the resolution of ξ. The correspondence of classes $[\xi] \in H_{m-3}(C_{*,q}^{\partial})$ and $[\xi_{m-1}] \in H_{q-1}(C_{m-1,*})$ defines correctly a homomorphism of homology groups*

$$\psi_{m-1} = (\delta^{-1}\partial\delta^{-1})_*\colon H_{m-3}(C_{*,q}^{\partial}) \to H_{q-1}(C_{m-1,*}).$$

Proof. The action of the homomorphism ψ_{m-1} is illustrated by the following diagram:

$$\begin{tikzcd}
0 & \xi \arrow[l, "\delta"'] \arrow[d, "\partial"'] \arrow[dr, dashed, "\psi_{m-1}"] & \xi_{m-2} \arrow[l, "\delta"'] \arrow[d, "\partial"] & \\
 & 0 & \partial\xi_{m-2} \arrow[l, "\delta"] & \xi_{m-1} \arrow[l, dashed, "\delta"]
\end{tikzcd}$$

First, we show that the image $[\xi_{m-1}]$ does not depend on the choice of the resolution. Let $\{\xi_{m-2}, \xi_{m-1}\}$ and $\{\xi'_{m-2}, \xi'_{m-1}\}$ are resolutions of $\xi \in Z_{m-3}(C_{*,q}^{\partial})$. We have $\delta(\xi_{m-2} - \xi'_{m-2}) = \delta\xi_{m-2} - \delta\xi'_{m-2} = \xi - \xi = 0$, so there is (the only one) $\mathfrak{U}$-chain $\tau \in C_{m-1,q}$ for which $\delta\tau = \xi_{m-2} - \xi'_{m-2}$. Hence $\xi'_{m-2} = \xi_{m-2} - \delta\tau$, and

$$\delta\xi'_{m-1} = \partial\xi'_{m-2} = \partial(\xi_{m-2} - \delta\tau) = \partial\xi_{m-2} - \partial(\delta\tau) = \delta\xi_{m-1} - \delta(\partial\tau) = \delta(\xi_{m-1} - \partial\tau).$$

Since $\delta\colon C_{m-1,*} \to C_{m-2,*}$ is a monomorphism, we get $\xi'_{m-1} = \xi_{m-1} - \partial\tau$. Therefore $[\xi'_{m-1}] = [\xi_{m-1}]$ in $H_{q-1}(C_{m-1,*})$.

Next, we will show that the image $[\xi_{m-1}]$ also does not depend on the choice of the cycle representing the class in $H_{m-3}(C_{*,q}^{\partial})$. If $[\zeta] = [\xi]$ then $\zeta = \xi + \delta\sigma$, where $\partial\sigma = 0$. We put $\zeta_{m-2} = \xi_{m-2} + \sigma$, $\zeta_{m-1} = \xi_{m-1}$. We have $\delta\zeta_{m-2} = \delta(\xi_{m-2} + \sigma) = \delta\xi + \delta\sigma = \xi + \delta\sigma = \zeta$,

$$\partial\zeta_{m-2} = \partial(\xi_{m-2} + \sigma) = \partial\xi_{m-2} + \partial\sigma = \partial\xi_{m-2} = \delta\xi_{m-1} = \delta\zeta_{m-1}.$$

Hence, $\{\zeta_{m-2}, \zeta_{m-1}\}$ is the resolutions for ζ, at that $\zeta_{m-1} = \xi_{m-1}$.

In order to prove that this correspondence of homology classes is a homomorphism, it suffices to note that a resolution for the sum of $\mathfrak{U}$-chains is the sum (term-by-term) of resolutions. $\square$

Remark 2.6. Let $m = 2$. For the cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$ and its resolution $\{\xi_0, \xi_1\}$ similarly the correspondence $[\xi] \mapsto [\xi_1]$ gives the homomorphism

$$\psi_1 = (\delta^{-1}\partial\varepsilon^{-1})_*\colon H_r(S_*^{\mathfrak{U}}) \to H_{r-1}(C_{1,*}),$$

which is the connecting homomorphism φ for the usual long exact Mayer–Vietoris sequence.

Consider the homomorphisms $\varphi_0, \varphi_1, \dots, \varphi_{m-3}$ from the sequence (6) and complement them with the homomorphism ψ_{m-1} from Lemma 2.3. We obtain a sequence of homomorphisms whose action, according to Proposition 2.1 and Lemma 2.3, is described in terms of a resolution of the cycle as follows.

Theorem 2.1. *For any $r \geqslant m$ there exists a sequence of homomorphisms*

$$H_r(S_*^{\mathfrak{U}}) \xrightarrow{\varphi_0} H_0(C^{\partial}_{*,r-1}) \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{m-3}} H_{m-3}(C^{\partial}_{*,r-m+2}) \xrightarrow{\psi_{m-1}} H_{r-m+1}(C_{m-1,*}), \tag{7}$$

given by the following sequence of images:

$$[\xi] \longmapsto [\partial\xi_0] \longmapsto \dots \longmapsto [\partial\xi_{m-3}] \longmapsto [\xi_{m-1}],$$

where $\{\xi_p\}$ is the resolution for the cycle $\xi \in Z_r(S_*^{\mathfrak{U}})$.

Finally, taking the composition of all homomorphisms from (7), we obtain the desired connecting homomorphism $\varphi = \psi_{m-1}\varphi_{m-3}\dots\varphi_1\varphi_0 = (\delta^{-1}\partial\delta^{-1}\dots\partial\delta^{-1}\partial\varepsilon^{-1})_*$.

Theorem 2.2. *Let $\mathfrak{U} = \{U_i\}$ be a finite open cover of a topological space X, consisting of $m \geqslant 2$ elements. Then the correspondence of homology classes $[\xi] \to [\xi_{m-1}]$, where $\xi \in Z_r(S_*^{\mathfrak{U}})$ and $\{\xi_p\}$ is arbitrary $\mathfrak{U}$-resolution of cycle ξ, defines a connecting homomorphism*

$$\varphi\colon H_r(S_*^{\mathfrak{U}}) \to H_{r-m+1}(C_{m-1,*}).$$

For $m > 2$ this homomorphism generates a semi-exact long sequence of homology groups

$$\begin{aligned} \dots \longleftarrow H_{q-m+1}(C_{m-2,*}) \xleftarrow{\delta_*} H_{q-m+1}(C_{m-1,*}) \xleftarrow{\varphi} H_q(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_q(C_{0,*}) \xleftarrow{\delta_*} \dots \\ \dots \xleftarrow{\delta_*} H_q(C_{m-1,*}) \xleftarrow{\varphi} H_{q+m-1}(S_*^{\mathfrak{U}}) \xleftarrow{\varepsilon_*} H_{q+m-1}(C_{0,*}) \longleftarrow \dots \end{aligned} \tag{8}$$

Proof. It remains to show that $\operatorname{im}\varphi \subset \ker\delta_*$ and $\operatorname{im}\varepsilon_* \subset \ker\varphi$. The first inclusion follows from the equality $\delta_*[\xi_{m-1}] = [\delta\xi_{m-1}] = [\partial\xi_{m-2}] = 0$. Let us prove the second inclusion. Let $[\xi] \in \operatorname{im}\varepsilon_*$. Then $\xi = \varepsilon\xi_0$ for some $\mathfrak{U}$-chain $\xi_0 \in C_{0,q}$, and $\partial\xi_0 = 0$. Therefore $\varphi_0[\xi] = [\partial\xi_0] = 0$, and hence $\varphi[\xi] = 0$. □

Remark 2.7. In what follows, we will assume that the codomain of the connecting homomorphism is the subgroup $H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*}) = \ker\delta_* \subset H_{q-m+1}(C_{m-1,*})$. The notation $H^{\mathrm{sep}}_q(C_{m-1,*})$ will be discussed later. In this way,

$$\varphi\colon H_q(S_*^{\mathfrak{U}}) \to H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*}).$$

Similarly, from the previous proof, we can conclude that $\operatorname{im}\psi_{m-1} \subset \ker\delta_*$. Therefore, we further assume that $\psi_{m-1}\colon H_{m-3}(C^{\partial}_{*,q}) \to H^{\mathrm{sep}}_{q-1}(C_{m-1,*})$.

The proofs of the following properties of the homomorphisms φ_0, φ_{p+1} and ψ_{m-1}, from which the connecting homomorphism is "glued", are completely standard. Moreover, conditions on the homology appear as sufficient conditions for inverting the required vertical arrows of the complex (4) at diagrammatic search.

Lemma 2.4. *If $H_{q-1}(C_{0,*}) \cong 0$, then the homomorphism $\varphi_0\colon H_q(S_*^{\mathfrak{U}}) \to H_0(C^{\partial}_{*,q-1})$ is an epimorphism. If $H_q(C_{0,*}) \cong H_{q-1}(C_{1,*}) \cong 0$, then it is a monomorphism.*

Lemma 2.5. *If $H_{q-1}(C_{p+1,*}) \cong 0$, then the homomorphism $\varphi_{p+1}\colon H_p(C^{\partial}_{*,q}) \to H_{p+1}(C^{\partial}_{*,q-1})$ is an epimorphism. If $H_{q-1}(C_{p+2,*}) \cong 0$, then it is a monomorphism.*

Lemma 2.6. *The homomorphism* $\psi_{m-1}\colon H_{m-3}(C^{\partial}_{*,q}) \to H^{sep}_{q-1}(C_{m-1,*})$ *is an isomorphism.*

Remark 2.8. For $m = 2$ the homomorphism $\psi_1\colon H_q(S^{\mathfrak{U}}_*) \to H^{\mathrm{sep}}_{q-1}(C_{1,*})$ is only an epimorphism in general case.

Considering that the composition of epimorphisms is an epimorphism, and the composition of monomorphisms is a monomorphism, we obtain the following property of the connecting homomorphism.

Theorem 2.3. *For the connecting homomorphism*

$$\varphi\colon H_q(S^{\mathfrak{U}}_*) \to H^{sep}_{q-m+1}(C_{m-1,*})$$

to be an epimorphism, it suffices to satisfy the condition

$$H_{q-1}(C_{0,*}) \cong H_{q-2}(C_{1,*}) \cong \ldots \cong H_{q-m+2}(C_{m-3,*}) \cong 0, \tag{9}$$

and for φ *to be a monomorphism, it suffices to satisfy the condition*

$$H_q(C_{0,*}) \cong H_{q-1}(C_{1,*}) \cong \ldots \cong H_{q-m+2}(C_{m-2,*}) \cong 0. \tag{10}$$

Remark 2.9. The homomorphism $\varphi\colon H_q(S^{\mathfrak{U}}_*) \to H^{\mathrm{sep}}_{q-m+1}(C_{m-1,*})$ is surjective if and only if the sequence (8) is exact in the term $H_{q-m+1}(C_{m-1,*})$.

Remark 2.10. The condition (9) can be replaced by the following weaker condition: if $\xi \in C_{p,q-p-1}$ such that $\partial\xi = 0$ and $\delta\xi = 0$ ($\varepsilon\xi = 0$), then $[\xi] = 0$ in $H_{q-p-1}(C_{p,*})$, $p = 0, \ldots, m-3$.

3. Separating cycles

The notion of the *separating cycle* appeared in complex analysis in connection with a property of the Grothendieck residue. Let ω be a meromorphic n-form on an n-dimensional complex-analytic manifold M, and $F_1, \ldots, F_n$ are polar hypersurfaces of ω, $F = F_1 \cup \ldots \cup F_n$. In a sufficiently small neighborhood U_a of an isolated point a of the intersection $Z = F_1 \cap \ldots \cap F_n$ the form ω is given by

$$\omega = \frac{h(z)\, dz_1 \wedge \ldots \wedge dz_n}{f_1(z) \ldots f_n(z)}, \tag{11}$$

where $h, f_1, \ldots f_n$ are holomorphic germs at a, $F_k|_{U_a} = \{f_k = 0\}$. The *grothendieck residue* of the form ω at the point a is represented by the integral

$$\operatorname{res}_a \omega = \frac{1}{(2\pi i)^n} \int_{\gamma^{(a)}} \omega, \tag{12}$$

where $\gamma^{(a)}$ is a *local cycle* at a having the form

$$\gamma^{(a)} = \{z \in U_a\colon |f_1(z)| = \varepsilon_1, \ldots, |f_n(z)| = \varepsilon_n\}. \tag{13}$$

The orientation of $\gamma^{(a)}$ is determined by the condition $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) \geqslant 0$. It is not hard to see that $\gamma^{(a)} \in Z_n(M \setminus F)$.

The mentioned property of the residue (12) is as follows: it is zero if h belongs to the ideal generated by $f_1, \ldots, f_n$ in the ring $\mathcal{O}_a$ of germs of holomorphic functions. This property is of a topological nature. Indeed, if $h = h_j f_j$, then ω has poles only on the $n-1$ hypersurfaces $F_k = \{f_k = 0\}$, $k = 1, \ldots [j] \ldots, n$, in the complement of which the n-cycle $\gamma^{(a)}$ becomes homologically trivial. Indeed, $\gamma^{(a)}$ is a boundary of the $(n+1)$-chain $\sigma_j = \{|f_1| = \varepsilon_1, \ldots, |f_j| \leqslant \varepsilon_j, \ldots, |f_n| = \varepsilon_n\}$ taken with a suitable orientation, so $\gamma^{(a)} \sim 0$. Therefore, according to Stokes' formula, the integral (12) for the $h = h_j f_j$ is zero.

Definition 3.1. *A n-dimensional cycle $\Gamma \in Z_n(M \setminus F)$ separates hypersurfaces $F_1, \dots, F_n$, if it satisfies the conditions*

$$\Gamma \sim 0 \text{ in } M \setminus (F_1 \cup \dots [j] \dots \cup F_n) \quad \text{for all} \quad j = 1, \dots, n.$$

By the above, the local cycle $\gamma^{(a)}$ from the definition of the Grothendieck residue separates the set of polar hypersurfaces of ω.

An important argument for the use of Grothendieck residues of meromorphic forms is their rational computability in terms of a finite number of Taylor coefficients of functions $h, f_1, \dots, f_n$ at the point a. In this connection there is a problem of the representation of the integral of a meromorphic form ω by residues. The topological formulation of this problem is as follows. Let $\mathcal{F} = \{F_1, \dots, F_n\}$ be a set of hypersurfaces in an n-dimensional complex-analytic manifold M. Let us denote by F the union of these hypersurfaces, and by Z_0 the discrete part of their intersection $Z = F_1 \cap \dots \cap F_n$. It is required to find out in which case the given n-cycle Γ in $M \setminus F$ is homologically expressed in terms of local cycles $\gamma^{(a)}$, $a \in Z_0$. In view of the above, for this it is necessary the cycle Γ separates the given set of hypersurfaces $\mathcal{F}$.

We denote by $H_n^{\mathrm{loc}}(M \setminus F)$ the subgroup in $H_n(M \setminus F)$ generated by the classes of all local cycles $\gamma^{(a)}$, $a \in Z_0$. We also denote by $H_n^{\mathrm{sep}}(M \setminus F)$ the subgroup of classes of all cycles separating the set of hypersurfaces $\mathcal{F}$. We have

$$H_n^{\mathrm{loc}}(M \setminus F) \subset H_n^{\mathrm{sep}}(M \setminus F).$$

We are interested in sufficient conditions on the manifold M and the collection of hypersurfaces $\mathcal{F}$ under which $H_n^{\mathrm{sep}}(M \setminus F) = H_n^{\mathrm{loc}}(M \setminus F)$, that is, in which any separating cycle is homologically represented in terms of local cycles.

Consider the space $X = M \setminus Z$ and its cover $\mathfrak{U}$ formed by open sets $U_j = M \setminus F_j$, $j = 1, \dots, n$. We get the corresponding extended double complex (4) and the semi-exact sequence (8) for $q = m = n$. Given the isomorphism of Theorem 1.1, this sequence can be written as

$$\dots \longleftarrow H_n(C_{m-2,*}) \xleftarrow{\delta_*} H_n(M \setminus F) \xleftarrow{\varphi} H_{2n-1}(M \setminus Z) \xleftarrow{\varepsilon_*} H_{2n-1}(C_{0,*}) \longleftarrow \dots. \tag{14}$$

The condition for separating the set $\mathcal{F}$ by a cycle Γ means that $\delta\Gamma$ is an ∂-boundary in the group $C_{n-2,n}$. So $H_n^{\mathrm{sep}}(M \setminus F) = \ker \delta_*$, which explains the previously used (see Remark 2.7) notation for the subgroup $H_{q-m+1}^{\mathrm{sep}}(C_{m-1,*}) = \ker \delta_* \subset H_{q-m+1}(C_{m-1,*})$. Let us show that $H_n^{\mathrm{loc}}(M \setminus F) \subset \operatorname{im} \varphi$.

It suffices to show that each generator $[\gamma^{(a)}]$, $a \in Z_0$, of the group $H_n^{\mathrm{loc}}(M \setminus F)$ have preimage in $H_{2n-1}(M \setminus Z)$. For a fixed point $a \in Z_0$, consider the $(2n-1)$-dimensional sphere S_a centred at the point a of a small radius. The class $[S_a]$ can be represented as a cycle $\partial \Pi_a$, where

$$\Pi_a = \{z \in U_a \colon |f_i(z)| < \varepsilon_i,\ i = 1, \dots, n\}, \tag{15}$$

where the orientation of the *special analytical polyhedron* Π_a is induced by the orientation of the manifold M. Moreover, the boundary $\partial \Pi_a$ of the polyhedron Π_a is the sum of its $(n-1)$-dimensional faces $\tau_j = \{|f_1| \leqslant \varepsilon_1, \dots, |f_j| = \varepsilon_j, \dots, |f_n| \leqslant \varepsilon_n\}$, $j = 1, \dots, n$, taken with suitable orientation, at that $\operatorname{supp} \tau_j \subset U_j$. Therefore, $\partial \Pi_a \in Z_{2n-1}(S_*^{\mathfrak{U}})$, and for the cycle $\partial \Pi_a$ can be built the $\mathfrak{U}$-resolution $\{\xi_p\}$. It is directly verified that terms of the resolution can be taken as follows:

$$\xi_p(i_0, i_1, \dots, i_p) = \pm \tau_{i_0} \cap \tau_{i_1} \cap \dots \cap \tau_{i_p}.$$

Moreover, the final term $\xi_{n-1} = \xi_{n-1}(1, \ldots, n)$ of the resolution is the local cycle $\gamma^{(a)}$. So $\varphi[S_a] = \varphi[\Pi_a] = [\gamma^{(a)}]$, as required to prove. It also follows from the last reasoning that if the group $H_{2n-1}(M \setminus Z)$ is generated by the classes of cycles S_a, $a \in Z_0$, in particular if $H_{2n-1}(M) \cong 0$ and $Z = Z_0$, then $H_n^{\mathrm{loc}}(M \setminus F) = \operatorname{im} \varphi$. This fact, with considering Remark 2.10, proves the following theorem.

Theorem 3.1. *Let $H_{2n-1}(M) \cong 0$ and let the intersection $Z = F_1 \cap \ldots \cap F_n$ be discrete. Then the groups $H_n^{sep}(M \setminus F)$ and $H_n^{loc}(M \setminus F)$ are coincide if and only if the semi-exact sequence* (14) *is exact in the term $H_n(M \setminus F)$.*

Remark 3.1. For $n = 2$ the sequence (14) turns into a long exact Mayer–Vietoris sequence. Therefore, under the assumptions made on the manifold and the set of hypersurfaces, the equality $H_2^{\mathrm{loc}}(M \setminus F) = H_2^{\mathrm{sep}}(M \setminus F)$ is always the case.

Consider (see Remark 2.7) the connecting homomorphism φ from the sequence (14) as the homomorphism

$$\varphi \colon H_{2n-1}(M \setminus Z) \to H_n^{\mathrm{sep}}(M \setminus F).$$

Sequence (14) is exact in the term $H_n(M \setminus F)$ if and only if φ is an epimorphism (see Remark 2.9). Considering Theorem 2.3, we obtain the following sufficient condition.

Theorem 3.2. *Let $H_{2n-1}(M) \cong 0$ and let the intersection $Z = F_1 \cap \ldots \cap F_n$ be discrete. Then for the equality of groups $H_n^{sep}(M \setminus F) = H_n^{loc}(M \setminus F)$ it suffices to satisfy the condition*

$$H_{2n-2}(C_{0,*}) \cong H_{2n-3}(C_{1,*}) \cong \ldots \cong H_{n+1}(C_{n-3,*}) \cong 0. \tag{16}$$

Remark 3.2. In last two theorems, instead the triviality of the group $H_{2n-1}(M)$ and the discreteness of the intersection $Z = F_1 \cap \ldots \cap F_n$, we can assume the following weaker condition: the group $H_{2n-1}(M \setminus Z)$ is generated by classes of cycles S_a, $a \in Z_0$. The condition (16) can also be replaced (see Remark 2.10) by the following weaker condition: if $\xi \in C_{p,2n-p-2}$ such that $\partial\xi = 0$ and $\delta\xi = 0$ ($\varepsilon\xi = 0$), then $[\xi] = 0$ in group $H_{2n-p-2}(C_{p,*})$, $p = 0, \ldots, n-3$.

As a consequence of Theorem 3.2, it is easy to obtain the following theorem on separating cycles in Stein manifolds which was proved by Tsikh.

Theorem 3.3 (see [5])**.** *Let M be a Stein manifold of dimension n. Then the equality of groups $H_n^{sep}(M \setminus F) = H_n^{loc}(M \setminus F)$ holds for any set $\mathcal{F} = \{F_1, \ldots, F_n\}$ of hypersurfaces in M.*

Proof. As it was noted in [5], it suffices to prove the statement of the theorem under the following assumptions: 1) $H_{2n-1}(M) \cong 0$; 2) $M \setminus F_j$, $j = 1, \ldots, n$, are the Stein manifolds; 3) the intersection $Z = F_1 \cap \ldots \cap F_n$ is discrete. It remains to note that all possible intersections of the sets $U_j = M \setminus F_j$ are also Stein manifolds. Condition (16) follows from the fact that for an arbitrary Stein manifold X the homology groups (with coefficients in the field) $H_q(X)$ are trivial for $q > \dim X$. □

Remark 3.3. In the case of the Stein manifold M and an arbitrary set of hypersurfaces $\mathcal{F}$ in M the connecting homomorphism $\varphi \colon H_{2n-1}(M \setminus Z) \to H_n^{\mathrm{sep}}(M \setminus F)$ is an isomorphism. The injectivity follows from the fulfillment of conditions of the form (10).

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Связывающий гомоморфизм и разделяющие циклы

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Аннотация. Обсуждается построение длинной полуточной последовательности Майера–Виеториса для гомологий объединения конечного числа открытых подпространств. Эта последовательность применяется для получения топологических условий, при которых интеграл от мероморфной дифференциальной формы в многомерном комплексном многообразии представляется в виде суммы вычетов Гротендика. Для существования такого представления интеграла необходимо, чтобы цикл интегрирования разделял семейство полярных гиперповерхностей формы. Условие разделения в ряде случаев оказывается достаточным условием для представления интеграла в виде суммы вычетов. Ранее при описании таких случаев (в работах А. К. Циха, А. П. Южакова, Р. В. Ульверта и др.) ключевым оказывалось условие штейновости многообразия. Основным результатом данной статьи является ослабление этого условия.

Ключевые слова: последовательность Майера–Виеториса, вычет Гротендика, разделяющий цикл.