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A List of Integral Representations for Diagonals of Power Series of Rational Functions

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Abstract. In this paper we present integral representations for the diagonals of power series. Such representations are obtained by lowering the multiplicity of integration for the previously known integral representation. The procedure for reducing the order of integration is carried out in the framework of the Leray theory of multidimensional residues. The concept of the amoeba of a complex analytic hypersurface plays a special role in the construction of new integral representations.

Keywords: multidimensional power series, complex integral, integral representation, amoeba, Taylor series, diagonal of a power series.

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Introduction

A range of problems associated with branching of parametric integrals is concerned with a study of the diagonals of power series [1, 2] and [3]. It should be noted that much earlier the concept of the diagonal of a power series was used by A. Poincare [4] to study the anomalies of planetary motion.

The diagonal of a Laurent power series

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \tag{1}$$

is defined as the generating function of a subsequence of coefficients $\{c_\alpha\}_{\alpha \in L}$ numbered by elements α of some sublattice $L \subset \mathbb{Z}^n$ (see [1] and [5]). Such diagonals are called complete. Diagonals are graded according to the dimension (rank) of the sublattice.

Following [1], we describe the specifics of the problem on the properties of the diagonals of series for rational functions of n variables

$$F(z) = \frac{P(z)}{Q(z)} = \frac{P(z_1, \ldots, z_n)}{Q(z_1, \ldots, z_n)}, \tag{2}$$

where P and Q are irreducible polynomials. Consider an arbitrary Laurent series for F centered at zero:

$$F(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}.$$

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It is known that such a series converges in domain $\mathrm{Log}^{-1}(E)$, where E is a connected component of the complement $R^n \backslash A_Q$ of amoeba of the denominator Q [6]. Recall that *amoeba* A_Q of the polynomial Q or of the algebraic hypersurface

$$V = \{z \in (\mathbb{C}\backslash 0)^n : Q(z) = 0\}$$

is called the image of V under the mapping $\mathrm{Log} : (\mathbb{C}\backslash 0)^n \to \mathbb{R}^n$, defined by the formula

$$\mathrm{Log} : (z_1, \ldots, z_n) \to (\log|z_1|, \ldots, \log|z_n|).$$

Sometimes instead of the designation A_Q we write A_V. According to the result of the article [7], there is an injective order function

$$\nu : E \to \mathbb{Z}^n \bigcap N_Q,$$

mapping each connected component E of the complement $\mathbb{R}^n \backslash A_Q$ to integer vector $\nu = \nu(E)$, belonging to the Newton polytope N_Q of the polynomial Q. Thus, all connected components can be indexed as $\{E_\nu\}$, where ν runs over a some subset of integer points from N_Q. For example, the Taylor series of a rational function $\dfrac{P}{Q}$, $Q(0) \neq 0$ converges in the component E_0.

Let us consider in more detail the p-dimensional diagonal of the series (1). Consider a p-dimensional sublattice $l \subset L$, with a basis $q^{(1)}, \ldots, q^{(p)}$. We assume that this basis can be extended of L by $n-p$ integer vectors $q^{(p+1)}, \ldots, q^{(n)}$ (this assumption equivalent to say that the totality of all $(p \times p)$-minors of the matrix $\tilde{A} = (q^{(1)}, \ldots, q^{(p)})$ are mutually prime) (see [8] or [9, Proposition 4.2.13]). Obviously the matrix

$$A = (q^{(1)}, \ldots, q^{(n)})$$

is unimodular, and we can assume it's determinant equals 1. Directions $q^{(1)}, \ldots, q^{(p)}$ define a diagonals subsequence $\{c_{lq}\}_{l \in \mathbb{Z}^p_+}$, where $l \cdot q$ means the product of the $(1 \times p)$-matrix l and $(p \times n)$-matrix $\tilde{A}$: $lq = l_1 q^{(1)} + \cdots + l_p q^{(p)}$.

The generating function

$$d_q(t) = \sum_{l \in \mathbb{Z}^p_+} c_{l_q} t_1^{l_1} \cdots t_p^{l_p}$$

of the subsequence $\{c_{lq}\}_{l \in \mathbb{Z}^n_+}$ is called *the one-sided q-diagonal* of the series (1).

We assume that the denominator Q in (2) is not zero at $z = 0$, so the origin $O \in \mathbb{Z}^n$ belongs to the Newton polytope and there is nonempty component E_0 of $\mathbb{R}^n \setminus A_Q$. We start by the Laurent series for the function (1) in $\mathrm{Log}^{-1}(E_0)$, which is in fact the Taylor series of (1) at $z = 0$. It is not hard to prove the following. If $\rho \in \mathrm{Log}^{-1}(E_0)$, then $d_q(t)$ admits the integral representation

$$d_q(t) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\rho} F(z) \frac{z^{q^{(1)}} \cdots z^{q^{(p)}}}{(z^{q^{(1)}} - t_1) \ldots (z^{q^{(p)}} - t_p)} \frac{dz_1}{z_1} \ldots \frac{dz_n}{z_n}, \tag{3}$$

where z^q is a monomial $z_1^{q_1} \ldots z_n^{q_n}$, and cycle

$$\Gamma_\rho = \{z \in \mathbb{C}^n : |z_1| = e^{\rho_1}, \ldots, |z_n| = e^{\rho_n}\}$$

is chosen so that

a) poles of $F(z)$ don't intersect the closed polydisc

$$\overline{U_\rho} = \{z \in \mathbb{C}^n : |z_1| \leqslant e^{\rho_1}, \ldots, |z_n| \leqslant e^{\rho_n}\};$$

b) parameters $t = (t_1, \dots, t_p)$ satisfy the inequalities $|t_i| < e^{\langle q_i, \rho \rangle}$, $i = 1, \dots, p$.

Integration loop Γ_ρ is a preimage $Log^{-1}\rho$ of the point ρ from the connected component E_0 of the amoeba A_Q complements. Here we prove that the integral which represent the diagonal $d_q(t)$ admits a decrease of the order of integration while preserving the rationality of the integrand.

We will assume that $N_Q \subset R^n_u$, and the image $A^{-1}(N_Q) \subset R^n_v$, here R^n_v and R^n_u are the n-dimensional real variable spaces u и v respectively. Let us denote by N' projection of the polyhedron $A^{-1}N_Q$ on the coordinate $(n-p)$-dimensional plane $\{v \in \mathbb{R}^n : v_1 = 0, \dots, v_p = 0\}$, and by $Q'(t, w')$ Laurent polynomial $Q[(t, w')^{A^{-1}}]$ from variables $w' = (w_{p+1}, \dots, w_n)$, wherein $t_1, \dots, t_p$ are parameters.

Theorem 1. *Diagonal $d_q(t)$ in (3) is represented by an integral in the $(n-p)$-dimensional complex algebraic torus $(\mathbb{C}\backslash 0)^{n-p}$ of variables $w_{p+1}, \dots, w_n$ according to the formula*

$$d_q(t) = \frac{1}{(2\pi i)^{n-p}} \int\limits_{\mathrm{Log}^{-1}(\rho')} F[(t_1, \dots, t_p, w_{p+1}, \dots, w_n)^{A^{-1}}] \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}, \tag{4}$$

where

$$\rho' = ((A\rho)_{p+1}, \dots, (A\rho)_n)$$

belongs to the connected component E_0' of the amoeba $A_{Q'}$ supplement of hypersurface $V' = \{w' \in (\mathbb{C}\backslash 0)^{n-p} : Q'(t, w') = 0\}$.

Proof of the theorem

Under the conditions of the theorem it is assumed that the diagonal (3) is considered for the Taylor series of the rational function $F = \dfrac{P}{Q}$. Therefore, it is automatically assumed that means $Q(0) \neq 0$, and that means that the origin 0 is a vertex of the Newton polytope.

Because the determinant of the integer matrix A is equal to one, inverse matrix to it

$$A^{-1} = \begin{pmatrix} b_1^{(1)} & \dots & b_n^{(1)} \\ \vdots & \ddots & \vdots \\ b_1^{(n)} & \dots & b_n^{(n)} \end{pmatrix} = (b_j^{(i)})$$

is an integer and its elements $b_j^{(i)}$ are algebraic complements to the elements $q_j^{(i)}$. The rows and columns of this matrix will be denoted by $b^{(i)}$ and b_j respectively. Let us make in the integral (3) the change of variables

$$z = w^{A^{-1}} = (w^{b_1}, \dots, w^{b_n}),$$

or, in a more detail:

$$(z_1, \dots, z_n) = (w_1^{b_1^{(1)}} \cdots w_n^{b_1^{(n)}}, w_1^{b_2^{(1)}} \cdots w_n^{b_2^{(n)}}, \dots, w_1^{b_n^{(1)}} \cdots w_n^{b_n^{(n)}}).$$

First, note that z^{q^i} will pass to w_i

$$z^{q^{(i)}} = z_1^{q_1^{(i)}} \dots z_n^{q_n^{(i)}} = \left(w_1^{b_1^{(1)}} \dots w_n^{b_1^{(n)}}\right)^{q_1^{(i)}} \dots \left(w_1^{b_n^{(1)}} \dots w_n^{b_n^{(n)}}\right)^{q_n^{(i)}} =$$

$$= w_1^{\langle b^{(1)}, q^{(i)} \rangle} \cdots w_n^{\langle b^{(n)}, q^{(i)} \rangle} = w_i,$$

since $\langle b^{(i)}, q^{(j)}\rangle = \delta_{ij}$ is the Kronecker symbol.

Applying our change of variables to the logarithmic differentials, we obtain

$$\frac{dz_i}{z_i} = \frac{d(w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}})}{w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}}} = \frac{\sum_{k=1}^n b_i^{(k)} w_1^{b_i^{(1)}} \dots w_k^{b_i^{(k)}-1} \dots w_n^{b_i^{(n)}} dw_k}{w_1^{b_i^{(1)}} \dots w_n^{b_i^{(n)}}}.$$

Multiplying the obtained expressions for the logarithmic differentials $\frac{dz_i}{z_i}$ (taking into account the properties of the external product of differentials: $dw_i dw_i = 0$ и $dw_i dw_j = -dw_j dw_i$), we get

$$\frac{|A^{-1}| w_1^{\sum_{i=1}^n b_i^{(1)}-1} \dots w_n^{\sum_{i=1}^n b_i^{(n)}-1} dw_1 \dots\ dw_n}{w_1^{\sum_{i=1}^n b_i^{(1)}} \dots w_n^{\sum_{i=1}^n b_i^{(n)}}} = \frac{dw_1 \wedge \dots \wedge dw_n}{w_1 \dots w_n}.$$

Let us apply the formula for change of variables to the integral (3):

$$d_q(t) = \frac{1}{(2\pi i)^n} \int\limits_{\varphi_\sharp(\Gamma_\rho)} F[(w_1, \dots, w_n)^{A^{-1}}] \frac{w_1 \cdots w_p}{(w_1 - t_1) \dots (w_p - t_p)} \frac{dw_1 \dots dw_n}{w_1 \dots w_n}, \tag{5}$$

where $\varphi_\sharp$ is the homomorphism induced by the mapping $\varphi : z \to w = z^A$.

The cycle Γ_ρ is parameterized in the form

$$\mathrm{Log}^{-1}(\rho) = \{z = e^{\rho + iA^{-1}\theta} : \theta \in A([0, 2\pi)^n)\}.$$

Hence,

$$\varphi_\sharp(\Gamma_\rho) = \{w = z^A : z \in \Gamma_\rho\} = \{w = e^{A\rho + AiA^{-1}\theta}\} = \mathrm{Log}^{-1}(A\rho).$$

In this way,

$$\varphi_\sharp(\Gamma_\rho) = \{w : |w_1| = e^{(A\rho)_1}, \dots, |w_n| = e^{(A\rho)_n}\},$$

where $(A\rho)_i$ is the i-th component of the vector $A\rho$.

By the Cauchy formula

$$d_q(t) = \frac{1}{(2\pi i)^n} \int\limits_{\varphi_\sharp(\Gamma_\rho)} F[(w_1, \dots, w_n)^{A^{-1}}] \frac{dw_1}{w_1 - t_1} \cdots \frac{dw_p}{w_p - t_p} \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}$$

we get

$$d_q(t) = \frac{1}{(2\pi i)^{n-p}} \int\limits_{\mathrm{Log}^{-1}(\rho')} F[(t_1, \dots, t_p, w_{p+1}, \dots, w_n)^{A^{-1}}] \frac{dw_{p+1} \dots dw_n}{w_{p+1} \dots w_n}, \tag{6}$$

where

$$\rho' = ((A\rho)_{p+1}, \dots, (A\rho)_n)$$

belongs to the connected component E_0' of the complement of the amoeba $A_{Q'}$ of the hypersurface $V' = \{w' \in (\mathbb{C}\backslash 0)^{n-p} : Q'(t, w') = 0\}$.

The theorem is proved.

Let me make the following comment on the reduction of the formula (3) to (6). It is not difficult to see that the integrand in (3) admits representation in the form

$$\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_p}{f_p} \wedge \psi,$$

where $\psi = \psi_p$ is a rational differential form of degree $n-p$, and $f_i = z^{q^{(i)}} - t_i$. The system of binomial equations $f_1 = 0, \ldots, f_p = 0$ defines an $(n-p)$-dimensional complex torus $\mathbb{T}^{n-p}$ (embedded in the torus $\mathbb{T}^n = (\mathbb{C} \setminus 0)^n$). In this case, the real torus Γ_ρ is a p-fold tube over a real torus $\gamma \subset \mathbb{T}^{n-p}$(in the coordinates w, it is $Log^{-1}(A\rho'))$. Thus, we are in the conditions of the multiple Leray residue formula (see [10, 11]), according to that the integrals in (3) and (6) coincide.

Example

Consider the example of applying of the theorem to find the integral representation of the diagonal defined by the vectors $q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ of the Taylor series of the function $F(z) = \dfrac{1}{1 + z_1 + z_2 + z_3 + z_2 z_3}$. Newton polytope of the denominator of the function F has the form shown in Fig. 1.

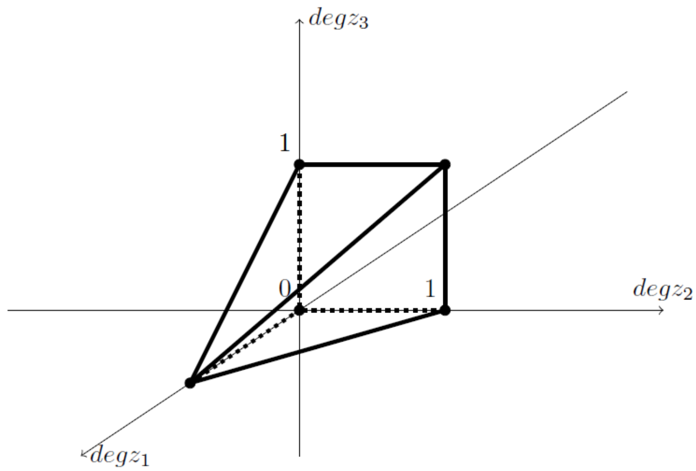

Fig. 1. Newton polytope $1 + z_1 + z_2 + z_3 + z_2 z_3$

For the two-dimensional diagonal $d_{q_1,q_2}(t_1, t_2) = \sum\limits_{l \in \mathbb{Z}^2_+} c_{l_1 q^{(1)} + l_2 q^{(2)}} t_1^{l_1} t_2^{l_2}$ in the set $\mathrm{Log}^{-1}(E_0)$, one has the following integral representation

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^3} \int\limits_{\Gamma_\rho} \frac{1}{1 + z_1 + z_2 + z_3 + z_2 z_3} \cdot \frac{z_1 z_2 z_3 * z_1 z_2^2 z_3^2}{(z_1 z_2 z_3 - t_1)(z_1 z_2^2 z_3^2 - t_2)} \cdot \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}, \quad (7)$$

where the cycle

$$\Gamma_\rho = \{z \in \mathbb{C}^n : |z_1| = e^{\rho_1}, |z_2| = e^{\rho_2}, |z_3| = e^{\rho_3}\}$$

is chosen so that

a) poles of $F(z)$ don't intersect the closed polydisc

$$\overline{U_\rho} = \{z \in \mathbb{C}^n : |z_1| \leqslant e^{\rho_1}, |z_2| \leqslant e^{\rho_2}, |z_3| \leqslant e^{\rho_3}\};$$

b) parameters $t = (t_1, t_2)$ satisfy the inequalities $|t_i| < e^{\langle q_i, \rho\rangle}$, $i = 1, 2$.

Now let's form the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. Using replacement $z^{A^{-1}} = w$, get $z_1 = w_1^2 w_2^{-1}$; $z_2 = w_1^{-1} w_2^1 w_3^{-1}$; $z_3 = w_3$. The denominator of the function F after replacement is converted to $1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1$ and Newtonian polytope is shown in Fig. 2

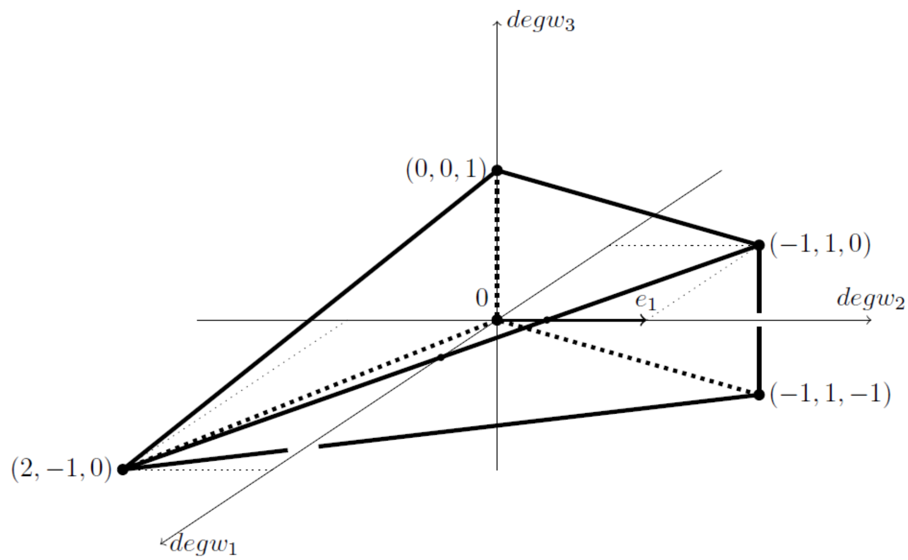

Fig. 2. Newton polytope $1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1$

The integral after replacement $z^{A^{-1}} = w$ looks like this

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^3} \int\limits_{\Gamma_\rho} \frac{1}{1 + w_1^2 w_2^{-1} + w_1^{-1} w_2^1 w_3^{-1} + w_3 + w_1^{-1} w_2^1} \cdot \frac{w_1 * w_2}{(w_1 - t_1)(w_2 - t_2)} \cdot \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3}.$$

After integration by the Cauchy formula with the variable w_1 we obtain the following form of the diagonal

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_{\rho'}} \frac{1}{1 + t_1^2 w_2^{-1} + t_1^{-1} w_2^1 w_3^{-1} + w_3 + t_1^{-1} w_2^1} \cdot \frac{w_2}{w_2 - t_2} \cdot \frac{dw_2}{w_2} \frac{dw_3}{w_3}.$$

We construct the Newton polytope of the denominator of the function $F(t_1, w_2, w_3)$ (Fig. 3). After integrating with the variable w_2 we obtain

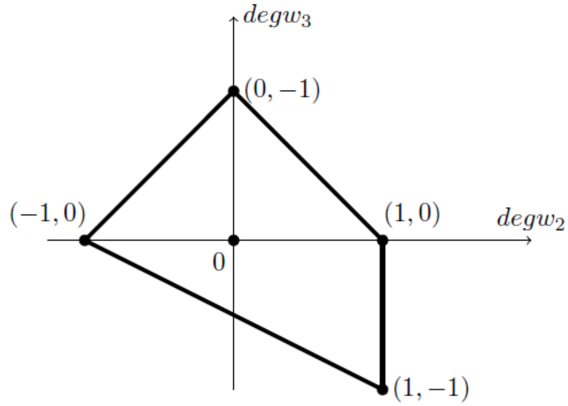

Fig. 3. Newton polytope $1 + t_1^2 w_2^{-1} + t_1^{-1} w_2^1 w_3^{-1} + w_3 + t_1^{-1} w_2^1$

$$d_q(t_1, t_2) = \frac{1}{(2\pi i)} \int\limits_{\Gamma_{\rho''}} \frac{1}{1 + t_1^2 t_2^{-1} + t_1^{-1} t_2^1 w_3^{-1} + w_3 + t_1^{-1} t_2^1} \cdot \frac{dw_3}{w_3}. \tag{8}$$

Therefore the integral (7) admits a reduction to the one-dimensional integral (8) with rational integrand. It is known ([12], Section 10.2) that such integral is an algebraic function in variables t_1, t_2.

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Список интегральных представлений для диагонали степенного ряда рациональной функции

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Аннотация. В работе приводятся интегральные представления для диагоналей степенных рядов. Такие представления получаются понижением кратности интегрирования для известного ранее интегрального представления. Процедура понижения кратности реализуется в рамках многомерной теории вычетов Лере. Особую роль в конструкции новых интегральных представлений играет понятие амебы комплексной аналитической гиперповерхности.

Ключевые слова: многомерные степенные ряды, комлексный интеграл, интегральное представление, амеба, ряд Тейлора, диагональ степенного ряда.