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On the Zeta-Function of Zeros of an Entire Function

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Abstract. This article is devoted to the study of the properties of the zeta-function of zeros of an entire function. We obtain an explicit expression for the kernel of the integral representation of the zeta-function in one case.

Keywords: zeta-function of zeros, integral representation, canonical product.

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Introduction

The purpose of this article is to correct a mistake in the work [1]. Namely, in the article [1] an incorrect statement was given that for an entire function f, satisfying some additional conditions, the following equality holds on the positive part of the real axis

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2x}. \tag{1}$$

It is easy to see that for any entire function f this equality cannot be true on the whole positive semiaxis. Indeed, the function $\dfrac{f'(z)}{f(z)}$ is meromorphic in the whole complex plane. By virtue of the uniqueness theorem, the equality (1) holds not only on $\mathbb{R}^+$, but also in $\mathbb{C} \setminus \{0\}$. However, the function

$$\frac{\sqrt{\pi}}{2\sqrt{z}} - \frac{1}{2z}$$

is not meromorphic in a neighborhood of the origin.

Our article is devoted to correcting the relation (1) and some of its consequences. Note that this result is related to the study of a generalized zeta-function constructed by zeros of some entire function.

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1. Auxiliary results

Let $f(z)$ be an entire function of order ρ in $\mathbb{C}$. Consider the equation

$$f(z) = 0. \tag{2}$$

Denote by $N_f = f^{-1}(0)$ the set of all solutions to (2) (we take every zero as many times as its multiplicity). The numbers of roots is at most countable.

The zeta-function $\zeta_f(s)$ of Eq. (2) is defined in the following way:

$$\zeta_f(s) = \sum_{z_n \in N_f} (-z_n)^{-s},$$

where $s \in \mathbb{C}$.

In [2], using the residue theory, V. I. Kuzovatov and A. A. Kytmanov obtained two integral representation for the zeta-function constructed by zeros of an entire function of finite order on the complex plane. With the help of these representations, they described a domain which the zeta-function can be extended to.

Theorem 1.1 ([2]). *Let $f(z)$ be an entire function of the zero order in $\mathbb{C}$ and satisfy the condition*

$$\frac{f'(z)}{f(z)} - \omega_0 = O\left(\frac{1}{|z|}\right), \quad |z| \to \infty.$$

Suppose that $0 < \operatorname{Re} s < 1$*. Then*

$$\zeta_f(s) = \frac{\sin \pi s}{\pi} \int_0^\infty \left(\frac{f'(x)}{f(x)} - \omega_0\right) x^{-s}\, dx, \tag{3}$$

where ω_0 is the limit value of $\dfrac{f'(x)}{f(x)}$ at infinity.

The method of proof of Theorem 1.1 shows that the statement remains valid in the case when $f(z)$ is an entire function of order less than 1.

Now we will give an integral representation for the zeta-function $\zeta_f(s)$ of zeros z_n of f which are $z_n = -q_n + is_n$, $q_n > 0$. Let us denote

$$F(f, x) = \sum_{n=1}^{\infty} e^{z_n x}. \tag{4}$$

We will assume that $\operatorname{Re} s = \sigma > 1$ and the following conditions hold:

$$\varliminf_{n \to \infty} \frac{q_n}{n} > 0, \tag{5}$$

$$\text{the series} \quad \sum_{n=1}^{\infty} \left(\frac{1}{q_n}\right)^{\sigma - 1} \quad \text{converges}. \tag{6}$$

For the convergence of the series (4), using condition (5), it is necessary and sufficient (for real x) that $x > 0$ [2].

Theorem 1.2 ([2]). *Suppose that the conditions* (5) *and* (6) *are satisfied and* $\operatorname{Re} s > 1$*. Then*

$$\zeta_f(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F(f, x)\, dx,$$

where $F(f, x)$ is defined by formula (4)*, and $\Gamma(s)$ is the Euler gamma-function.*

Our goal is to obtain an explicit expression for the kernel of the integral representation (3) in case $z_n = -\pi n^2$. This choice of zeros z_n is due to the fact that for series

$$F(f,x) = \sum_{n=1}^{\infty} e^{z_n x} = \sum_{n=1}^{\infty} e^{-\pi n^2 x} := \psi(x)$$

for $x > 0$ it is known (see, for example, [3, Chapter II, S. 6]) that

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left\{ 2\psi\left(\frac{1}{x}\right) + 1 \right\}.$$

2. The main result

Theorem 2.1. *Let $f(z)$ be an entire function of order $\rho < 1$ with zeros $z_n = -\pi n^2$. Then for real $x \in (0; +\infty)$ the following holds*

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}} \operatorname{cth} \sqrt{\pi x} - \frac{1}{2x}.$$

Proof. Since the order of f is less than 1, it has the form

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \tag{7}$$

The representation (7) is true, for example, for entire functions of order less than 1 or for entire functions of the first order with the additional condition, i.e. the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|}$ is convergent. In particular, the representation (7) is true for functions of the zero genus.

It is easy to show that in this case we obtain

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \tag{8}$$

if $z \neq z_n$.

Since the order of the canonical product (7) is equal to the index of convergence ρ_1 of its zeros and for given values of z_n

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln |z_n|} = \frac{1}{2},$$

representations (7) and (8) are true for considered function $f(z)$.

To further prove the assertion of the theorem, we use the standard decomposition (see, for example, [4, formula 5.1.25.4])

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + a^2} = \frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cth} \pi a.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = -\frac{1}{2a^2} + \frac{\pi}{2a} \operatorname{cth} \pi a.$$

Thus

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{\infty} \frac{1}{x+\pi n^2} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2+x/\pi} = \frac{1}{\pi}\left(-\frac{1}{2x/\pi} + \frac{\pi}{2\sqrt{x/\pi}}\operatorname{cth}\pi\sqrt{x/\pi}\right) =$$

$$= -\frac{1}{2x} + \frac{\sqrt{\pi}}{2\sqrt{x}}\operatorname{cth}\sqrt{\pi x}. \qquad \square$$

Corollary 1. *Suppose that the conditions of Theorem 2.1 are satisfied. If* ω_0 *is the limit value of* $\dfrac{f'(x)}{f(x)}$ *at infinity, i.e.*

$$\omega_0 = \lim_{x\to+\infty} \frac{f'(x)}{f(x)},$$

then $\omega_0 = 0$.

Proof. To prove the statement, we note that

$$\lim_{x\to+\infty} \operatorname{cth} x = \lim_{x\to+\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to+\infty} \frac{e^x\left(1+e^{-2x}\right)}{e^x\left(1-e^{-2x}\right)} = 1. \qquad \square$$

Remark 1. If f is an arbitrary entire function of order $1 \leqslant \rho < \infty$, with zeros $z_n = -\pi n^2$, then the ratio can be represented as

$$\frac{f(z)}{\prod\limits_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)} = e^{g(z)},$$

where $g(z)$ is an entire function. Since $1 \leqslant \rho < \infty$, $g(z)$ is a polynomial, $\deg g = \rho$, and $\rho \in \mathbb{N}$ [5]. Therefore,

$$f(z) = \Pi(z)e^{g(z)}, \quad \Pi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right),$$

and

$$\frac{f'(z)}{f(z)} = \frac{\Pi'(z)e^{g(z)} + \Pi(z)e^{g(z)}g'(z)}{\Pi(z)e^{g(z)}} = \frac{\Pi'(z)}{\Pi(z)} + g'(z).$$

Consequently in this case we take

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{\pi}}{2\sqrt{x}}\operatorname{cth}\sqrt{\pi x} - \frac{1}{2x} + g'(x), \quad 1 \leqslant \rho < \infty.$$

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О дзета-функции нулей целой функции

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Аннотация. Данная статья посвящена исследованию свойств дзета-функции нулей целой функции. Получено явное выражение для ядра интегрального представления дзета-функции в одном случае.

Ключевые слова: дзета-функция нулей, интегральное представление, каноническое произведение.