DOI: 10.17516/1997-1397-2021-14-5-589-598
УДК 517.55

Laurent-Hua Loo-Keng Series with Respect to the Matrix Ball from Space $\mathbb{C}^n[m \times m]$

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Received 10.08.2020, received in revised form 10.09.2020, accepted 20.10.2020

Abstract. The aim of this work is to obtain multidimensional analogs of the Laurent series with respect to the matrix ball from space $\mathbb{C}^n[m \times m]$. To do this, we first introduce the concept of a "layer of the matrix ball" from $\mathbb{C}^n[m \times m]$, then in this layer of the matrix ball we use the properties of integrals of the Bochner-Hua Loo-Keng type to obtain analogs of the Laurent series.

Keywords: matrix ball, Laurent series, holomorphic function, Shilov's boundary, Bochner-Hua Loo Keng integral, orthonormal system.

Citation: G.Kh. Khudayberganov, J.Sh. Abdullayev, Laurent-Hua Loo Keng Series with Respect to the Matrix Ball from Space $\mathbb{C}^n[m \times m]$, J. Sib. Fed. Univ. Math. Phys., 2021, 14(5), 589–598.
DOI: 10.17516/1997-1397-2021-14-5-589-598.

1. Introduction and preliminaries

In classical complex analysis the Laurent expansions play an important role in studies (in the study) of holomorphic functions in a neighborhood of isolated singular points (in a ring). Analogs of Laurent series have already been constructed in multivariable complex analysis, for example, in the product of circular rings

$$\{z \in \mathbb{C}^n : r_\nu < |z_\nu - a_\nu| < R_\nu,\ \nu = 1, 2, \ldots, n\} \tag{1}$$

or in the domains of Hartogs

$$\{z = ('z, z_n) \in \mathbb{C}^n : 'z \in {'D},\, r('z) < |z_n - a_n| < R('z)\}, \tag{2}$$

where $'D$ is a domain of $\mathbb{C}^{n-1}$ (Hartogs-Laurent series). Namely, any function f holomorphic in (1) can be represented as a multiple Laurent series

$$f(z) = \sum_{|k|=-\infty}^{\infty} c_k (z-a)^k, \tag{3}$$

where $k = (k_1, k_2, \ldots, k_n)$ integer vectors, and

$$c_k = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{k+1}},$$

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$$\Gamma = \{\zeta \in \mathbb{C}^n : |\zeta_\nu - a_\nu| = \rho_\nu,\, r_\nu < \rho_\nu < R_\nu,\, \nu = 1, 2, \ldots, n\}\,.$$

In this case, the domains of convergence of series (3) are relatively complete Reinhart domains.

In the works of É. Cartan [1], C. L. Siegel [2], Hua Loo-Keng [3], I. I. Pjateckiĭ-Šapiro [4], as well as in [5] the matrix approach of presenting the theory of multivariable complex analysis is widely used. It mainly deals with the classical domains and related questions of function theory and geometry. The importance of studying classical domains is that they are not reducible, i.e. these domains are, in a sense, model domains of multidimensional space.

Recently, scientists have obtained many significant results in the classical fields, and at the same time, a number of open problems have been formulated. For example, in [6] the regularity and algebraicity of mappings in classical domains are studied, and in [7] harmonic Bergman functions in classical domains are studied from a new point of view. In the paper [8], holomorphic and pluriharmonic functions are defined for classical domains of the first type, the Laplace and Hua Loo-Keng operators are studied also. A connection was found between these operators.

In addition, scientific works in matrix balls associated with classical domains from the space $\mathbb{C}^n\,[m \times m]$ are developing.

Consider the space of complex m^2 variables denoted by $\mathbb{C}^{m^2}$. In some questions, it is convenient to represent the point Z of this space in the form of a square $[m \times m]$ matrix, that is, in the form $Z = (z_{ij})_{i,j=1}^m$. With this representation of points, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m \times m]$. The direct product $\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$ that have n copies of $[m \times m]$ matrices we denote by $\mathbb{C}^n[m \times m]$.

Let $Z = (Z_1, \ldots, Z_n)$ be a vector composed of square matrices Z_j of order m, considered over the field of complex numbers $\mathbb{C}$. Let us write the elements of the vector $Z = (Z_1, \ldots, Z_n)$ as points z of the space $\mathbb{C}^{nm^2}$:

$$z = (z_{11}^{(1)}, \ldots, z_{1m}^{(1)}, \ldots, z_{m1}^{(1)}, \ldots, z_{mm}^{(1)}, \ldots, z_{11}^{(n)}, \ldots, z_{1m}^{(n)}, \ldots, z_{m1}^{(n)}, \ldots, z_{mm}^{(n)}) \in \mathbb{C}^{nm^2}. \qquad (4)$$

Hence, we can assume that Z is an element of the space $\mathbb{C}^n\,[m \times m]$, that is, we arrive at the isomorphism $\mathbb{C}^n\,[m \times m] \cong \mathbb{C}^{nm^2}$.

Let us define the matrix "scalar" product:

$$\langle Z, W\rangle = Z_1 W_1^* + \cdots + Z_n W_n^*,$$

where W_j^* is the conjugate and transposed matrix for the matrix W_j.

It is known (see [9, 10]) that the matrix balls $\mathbb{B}_{m,n}^{(1)}$, $\mathbb{B}_{m,n}^{(2)}$ and $\mathbb{B}_{m,n}^{(3)}$ of the first, second and third types, respectively, have the form:

$$\mathbb{B}_{m,n}^{(1)} = \{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z\rangle > 0\}\,,$$

$$\mathbb{B}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\,[m \times m] : I - \langle Z, Z\rangle > 0, \quad \forall Z'_\nu = Z_\nu,\ \nu = 1, \ldots, n\}\,,$$

and

$$\mathbb{B}_{m,n}^{(3)} = \{(Z \in \mathbb{C}^n\,[m \times m] : I + \langle Z, Z\rangle > 0,\ \forall\, Z_\nu^{'} = -Z_\nu,\ \ \nu = 1, \ldots, n\Big\}\,.$$

The skeletons (the Shilov boundaries) of matrix balls $\mathbb{B}_{m,n}^{(k)}$, denoted by $\mathbb{X}_{m,n}^{(k)}$, $\ k = 1, 2, 3$, i.e.,

$$\mathbb{X}_{m,n}^{(1)} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z\rangle = I\}\,,$$

$$\mathbb{X}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z\rangle = I, \quad Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\}\,,$$

$$\mathbb{X}^{(3)}_{m,n} = \left\{Z \in \mathbb{C}^n\left[m \times m\right] : I + \langle Z, Z\rangle = 0, \;\; Z^{'}_{\nu} = -Z_{\nu}, \;\; \nu = 1, 2, \ldots, n\right\}.$$

Note that, $\mathbb{B}^{(1)}_{1,1}$, $\mathbb{B}^{(2)}_{1,1}$ and $\mathbb{B}^{(3)}_{2,1}$ are unit disks, and $\mathbb{X}^{(1)}_{1,1}$, $\mathbb{X}^{(2)}_{1,1}$, and $\mathbb{X}^{(3)}_{2,1}$ are unit circles in the complex plane $\mathbb{C}$.

If $n = 1$, $m > 1$, then $\mathbb{B}^{(k)}_{m,1}$, $k = 1, 2, 3$ are the classical domains of the first, second and third type (according to the classification of E. Cartan (see [1])), and the skeletons $\mathbb{X}^{(1)}_{m,1}$, $\mathbb{X}^{(2)}_{m,1}$, and $\mathbb{X}^{(3)}_{m,1}$ are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

Note that the matrix balls $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ are complete circular convex bounded domains. In addition, the domains $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ and their skeletons $\mathbb{X}^{(1)}_{m,n}$, $\mathbb{X}^{(2)}_{m,n}$, $\mathbb{X}^{(3)}_{m,n}$ are invariant under unitary transformations (see [10, 12]).

The first type of matrix ball was considered by A. G. Sergeev in [11], and by G. Khudayberganov in [9]. In [10], formulas for the volume of a matrix ball of the first type and its skeleton are obtained, the holomorphic automorphisms for a matrix ball of the first type are described, and integral formulas for matrix balls of the second and third types are obtained. In [13] the volumes of the third type matrix ball and the generalized Lie ball are calculated. The total volumes of these domains are necessary to find the kernels of the integral formulas for these domains (Bergman, Cauchy–Szegő, Poisson kernels, etc. (see, for example, [14–17])). In addition, they are useful for the integral representation of functions holomorphic in these domains in the mean value theorem and other important concepts. In the papers [18, 19] analogs of Laurent series with respect to the classical Cartan domains of the first, second, and third types are obtained.

The aim of this work is to obtain analogs of the Laurent series[‡] with respect to the matrix ball from space $\mathbb{C}^n\left[m \times m\right]$. To do this, we first introduced the concept of a "layer of the matrix ball" from $\mathbb{C}^n\left[m \times m\right]$, then in this layer of the matrix ball, we used the properties of integrals of the Bochner–Hua Loo-Keng type to obtain analogs of the Laurent series.

1. Laurent-Hua Loo-Keng series with respect to the matrix ball $\mathbb{B}_{m.n}$

Let $\mathbb{B}_{m.n}$[§] be a matrix ball. For functions $f(Z) = f(z^{(1)}_{11}, \ldots, z^{(1)}_{1m}, \ldots, z^{(n)}_{m1}, \ldots, z^{(n)}_{mm})$ holomorphic in $\mathbb{B}_{m.n}$ and continuous on $\overline{\mathbb{B}}_{m,n}$ ($\overline{\mathbb{B}}_{m,n} = \mathbb{B}_{m,n} \cup \partial\mathbb{B}_{m,n}$) the Bochner–Hua Loo-Keng integral formula is valid [10, 20]:

$$f(Z) = \int_{\mathbb{X}_{m,n}} \det{}^{-mn}\left(I^{(m)} - \langle Z, U\rangle\right) f\left(U\right) d\mu, \tag{5}$$

where $f\left(U\right)$ is an integrable function, $d\mu$ is the Haar measure on $\mathbb{X}_{m,n}$.

Let

$$\mathbb{B}^{-}_{m,n} = \left\{Z \in \mathbb{C}^n\left[m \times m\right] : \;\; I - \langle Z, Z\rangle < 0\right\}.$$

The integral (5) of Bochner–Hua Loo-Keng type makes sense in each of the domains $\mathbb{B}_{m,n}$ and $\mathbb{B}^{-}_{m,n}$ ([21]).

Let us write the elements of vector $Z = (Z_1, \ldots, Z_n) \in \mathbb{B}_{m,n}$ in the form (4) and by $z^{[\alpha]}$ we will denote a vector with components

$$\sqrt{\frac{|\alpha|!}{\alpha_1!\alpha_2!\ldots\alpha_{nm^2}!}}\left(z^{(1)}_{11}\right)^{\alpha_1}\cdots\left(z^{(1)}_{1m}\right)^{\alpha_m}\cdots\left(z^{(n)}_{mm}\right)^{\alpha_{nm^2}}, \;\; |\alpha| = \sum_{i=1}^{nm^2}\alpha_i, \;\; \alpha_i \geqslant 0. \tag{6}$$

[‡]In what follows, we will call these series Laurent-Hua Loo-Keng series.

[§]For convenience, we denote $\mathbb{B}^{(1)}_{m,n}$ by $\mathbb{B}_{m,n}$, and $\mathbb{X}^{(1)}_{m,n}$ by $\mathbb{X}_{m,n}$.

The dimension of the subspace generated by the vector $z^{[\alpha]}$ is equal to the dimension of the direct sum of subspaces with dimensions (see [3, 22, 23])

$$q(\alpha_1, \alpha_2, \ldots, \alpha_m) = N(\alpha_1, \alpha_2, \ldots, \alpha_m) \cdot N(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0)$$

and it is equal to

$$\sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_m=|\alpha| \\ \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m \geqslant 0}} N(\alpha_1, \alpha_2, \ldots, \alpha_m) \cdot N(\alpha_1, \alpha_2, \ldots, \alpha_m, 0, \ldots, 0) = \frac{(nm^2 + |\alpha| - 1)!}{\alpha!(nm^2 - 1)!},$$

where

$$N(\alpha_1, \alpha_2, \ldots, \alpha_m) = \frac{D(\alpha_1 + m - 1, \alpha_2 + m - 2, \ldots, \alpha_{m-1} + 1, \alpha_m)}{D(m-1, m-2, \ldots, 1, 0)},$$

$$D(\alpha_1, \alpha_2, \ldots, \alpha_m) = \prod_{1 \leqslant i < j \leqslant m} (\alpha_i - \alpha_j), \quad m \geqslant 2.$$

Obviously, (6) contains all monomials of degree α, that is, any polynomial in

$$z_{11}^{(1)}, \ldots, z_{1m}^{(1)}; \ldots; z_{m1}^{(1)}, \ldots, z_{mm}^{(1)}; \ldots; z_{11}^{(n)}, \ldots, z_{1m}^{(n)}; \ldots; z_{m1}^{(n)}, \ldots, z_{mm}^{(n)}$$

is a linear combination of expressions like (6), if α takes values $0, 1, 2, \ldots$.

Let us denote by

$$\varphi^{(p)}_{\alpha_1, \alpha_2, \ldots, \alpha_m}(Z), \;\; p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m)$$

the components of the vector $z^{[\alpha]}$.

In [24] it was proved that the system of functions

$$(\rho_\alpha)^{-\frac{1}{2}} \varphi^{(p)}_\alpha(Z), \quad p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m), \;\; \alpha = 0, 1, 2, \ldots$$

is an orthonormal system in the domain $\mathbb{B}_{m,n}$, where

$$\rho_\alpha = \int_{\mathbb{B}_{m,n}} \left| \varphi^{(p)}_\alpha(Z) \right|^2 d\nu, \quad d\nu = \prod_{k=1}^{n} \prod_{1 \leqslant i \leqslant j \leqslant m} dx^{(k)}_{i,j} dy^{(k)}_{i,j}$$

and the system of functions

$$(\delta_\alpha)^{-\frac{1}{2}} \varphi^{(p)}_\alpha(U), \quad p = 1, 2, \ldots, q\ (\alpha_1, \alpha_2, \ldots, \alpha_m), \;\; \alpha = 0, 1, 2, \ldots. \tag{7}$$

forms a complete orthonormal system on $\mathbb{X}_{m,n}$, where $\varphi^{(p)}_\alpha(U)$, $p = 1, 2, \ldots, q(\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\alpha = 0, 1, 2, \ldots$ are components of the vector $u^{[\alpha]}$ $(u = (u^{(1)}_{11}, \ldots, u^{(1)}_{1m}, \ldots, \ldots, \ldots, u^{(n)}_{m1}, \ldots, u^{(n)}_{mm}))$ and

$$\delta_\alpha = \int_{\mathbb{X}_{m,n}} \left| \varphi^{(p)}_\alpha(U) \right|^2 d\mu.$$

Theorem 1.1 (see [24]). *Let $f(U)$ be an integrable function in $\mathbb{X}_{m,n}$ and let*

$$a^p_\alpha = \frac{V(\mathbb{X}_{m,n})}{\sqrt{\delta_\alpha}} \cdot \int_{\mathbb{X}_{m,n}} f(U) \overline{\varphi^{(p)}_\alpha(U)} d\mu \tag{8}$$

be the Fourier coefficients of this function with respect to the orthonormal system (7)*. Then, in* $\mathbb{B}_{m,n}$ *the integral* (5) *represents a holomorphic function that expands in this domain in a series*

$$\sum_{\alpha\geqslant 0}\sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}}. \tag{9}$$

If we denote

$$F^{\pm}(Z) = \begin{cases} F(Z), Z \in \mathbb{B}_{m,n}, \\ F(Z), Z \in \mathbb{B}^-_{m,n}, \end{cases}$$

then, by Theorem 1.1, for all $Z \in \mathbb{B}_{m,n}$ we have

$$F^+(Z) = \sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}}, \tag{10}$$

with coefficients (8). Therefore, $F^+(Z) \in \mathcal{O}(\mathbb{B}_{m,n})$, i.e. $F^+(Z)$ is holomorphic in $\mathbb{B}_{m,n}$. Now let $Z \in \mathbb{B}^-_{m,n}$. Then we have

$$F^-(Z) = \int\limits_{X_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \left\langle (\langle Z, Z\rangle)^{-1} Z, U\right\rangle\right)} d\mu. \tag{11}$$

The Cauchy-Szegő kernel has the following form (see [24]):

$$C(\widetilde{Z}, U) = \frac{1}{V(\mathbb{X}_{m,n})} \det^{-mn}(I^{(m)} - \left\langle \widetilde{Z}, U\right\rangle = \sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)} \varphi_{i,j}^{(\alpha)}(\widetilde{Z})\overline{\varphi_{i,j}^{(\alpha)}(U)}.$$

Using the equalities from [3, p.114] we obtain

$$\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(U) = \varphi^{(p)}_{\alpha_1-\alpha_m,\alpha_2-\alpha_m,\ldots,\alpha_{m-1}-\alpha_m,0}(U)(\det U)^{\alpha_m},$$

and noticing that $\widetilde{Z} = (\langle Z, Z\rangle)^{-1} Z$, we have

$$\frac{1}{\det^{mn}\left(I^{(m)} - \left\langle (\langle Z, Z\rangle)^{-1} Z, U\right\rangle\right)} = V(\mathbb{X}_{m,n}) \sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)} \varphi_{i,j}^{(\alpha)}\left((\langle Z, Z\rangle)^{-1} Z\right) \overline{\varphi_{i,j}^{(\alpha)}(U)}.$$

Multiplying the last expression by $f(U)$ and integrating term by term against the measure $d\mu$, we obtain

$$\begin{aligned} F^-(Z) &= \int_{\mathbb{X}_{m,n}} f(U)\, V(\mathbb{X}_{m,n}) \sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)} \varphi_{i,j}^{(\alpha)}\left((\langle Z, Z\rangle)^{-1} Z\right) \overline{\varphi_{i,j}^{(\alpha)}(U)} d\mu = \\ &= \sum_{\alpha\geqslant 0}\sum_{i,j=1}^{q(\alpha)} \varphi_{i,j}^{(\alpha)}\left((\langle Z, Z\rangle)^{-1} Z\right) \left[V(\mathbb{X}_{m,n}) \int_{\mathbb{X}_{m,n}} f(U)\overline{\varphi_{i,j}^{(\alpha)}(U)} d\mu\right] = \\ &= (-1)^{mn} \sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m} \frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left((\langle Z, Z\rangle)^{-1} Z\right)}{\sqrt{\delta_\alpha}}. \end{aligned} \tag{12}$$

Therefore, in the domain $\mathbb{B}^{-}_{m,n}$ the integral (11) represents a holomorphic function of $(\langle Z,Z\rangle)^{-1}Z$, which has an expansion

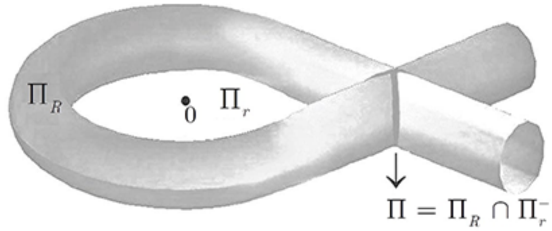
$$F^{-}(Z)=(-1)^{mn}\sum_{\alpha_1\geqslant\ldots\geqslant\alpha_m\geqslant 0}\sum_{p=1}^{q(\alpha)}a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left((\langle Z,Z\rangle)^{-1}Z\right)}{\sqrt{\delta_\alpha}}. \tag{13}$$

Consider the matrix domains of the form:

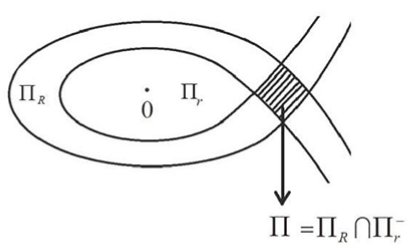
$$\Pi_R=\left\{Z\in\mathbb{C}^n\left[m\times m\right]: R^2I^{(m)}-\langle Z,Z\rangle>0\right\},$$
$$\Pi_r=\left\{Z\in\mathbb{C}^n\left[m\times k\right]: r^2I^{(m)}-\langle Z,Z\rangle<0\right\},$$

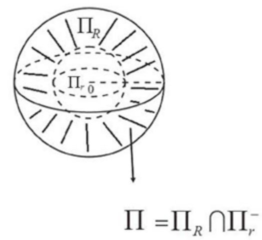
where R,r are real numbers such that $0<r<R<\infty$. We denote $\Pi=\Pi_R\cap\Pi_r^{-}$ and the sets Π will call the "*layer of the matrix ball*".

The following diagram shows a layer of a matrix ball from the space $\mathbb{C}^n\left[m\times m\right]$



and, in particular, for $n=1$ and $m=1$ we have:





The following theorem holds

Theorem 1.2. *If $F(Z)\in\mathcal{O}\left(\Pi\right)\cap C\left(\bar{\Pi}\right)$, then for $Z\in\Pi$ the Laurent-Hua Loo-Keng expansion*

$$F(Z)=F^{+}(Z)+F^{-}(Z).$$

where the coefficients in (13) are calculated by the formula

$$a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}=V\left(\mathbb{X}_\rho\right)\int_{\mathbb{X}_\rho}f\left(U\right)\overline{\varphi^{(\alpha)}_{i,j}}(U)d\mu, \tag{14}$$

and

$$\mathbb{X}_\rho=\left\{U\in\mathbb{C}^n\left[m\times m\right]:\ \langle U,U\rangle=\rho^2I^{(m)},r<\rho<R.\right\}.$$

Proof. Fix an arbitrary point $Z \in \Pi$ and construct a layer Π' such that $Z \in \Pi' \subseteq \Pi$ ($\Pi' = {\Pi'}_R \cap {\Pi'}_r, r < r' < R' < R$). Then, by virtue of the Bochner-Hua Loo-Keng integral formula (5) the following expansion takes place:

$$F(Z) = \int_{\Gamma'} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U \rangle\right)} d\mu + \int_{-\gamma'} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U \rangle\right)} d\mu, \tag{15}$$

where

$$\Gamma' = \left\{ U \in \mathbb{C}^n [m \times m] : \langle U, U \rangle = (R')^2 I^{(m)} \right\},$$

and

$$\gamma' = \left\{ U \in \mathbb{C}^n [m \times m] : \langle U, U \rangle = (r')^2 I^{(m)} \right\}.$$

Therefore, by virtue of (10)

$$\int_{\Gamma'} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U \rangle\right)} d\mu = \sum_{\alpha_1 \geqslant \ldots \geqslant \alpha_m \geqslant 0} \sum_{p=1}^{q(\alpha)} a_\alpha^p \frac{\varphi_\alpha^{(p)}(Z)}{\sqrt{\delta_\alpha}} = F^+(Z), \tag{16}$$

where

$$a_\alpha^p = \frac{V(\Gamma')}{\sqrt{\delta_\alpha}} \cdot \int_{\Gamma'} f(U) \overline{\varphi_\alpha^{(p)}(U)} d\mu.$$

Assuming

$$\varphi_{\alpha_1, \ldots, \alpha_m}^{(p)}(U) = \varphi_{\alpha_1 - \alpha_m, \alpha_2 - \alpha_m, \ldots, \alpha_{m-1} - \alpha_m, 0}^{(p)}(U) (\det U)^{\alpha_m},$$

for any $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_m$ from (13) we get

$$\int_{-\gamma'} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U \rangle\right)} =$$

$$= (-1)^{mn} \sum_{\alpha_1 \geqslant \ldots \geqslant \alpha_m \geqslant 0} \sum_{p=1}^{q(\alpha)} a_{-\alpha_m - m, \ldots, -\alpha_1 - m}^{(p)} \frac{\varphi_{-\alpha_m - m, \ldots, -\alpha_1 - m}^{(p)}\left(\left(\langle Z, Z \rangle\right)^{-1} Z\right)}{\sqrt{\delta_\alpha}} = F^-(Z), \tag{17}$$

where

$$a_{-\alpha_m - m, \ldots, -\alpha_1 - m}^{(p)} = V(\gamma') \int_{-\gamma'} f(U) \overline{\varphi_{i,j}^{(\alpha)}(U)} d\mu.$$

Now, substituting (16) and (17) into (15), we obtain the required expansion

$$F(Z) = F^+(Z) + F^-(Z).$$

It remains to note that by Cauchy's homotopy theorem in formulas for calculating the coefficients a_α^p and $a_{-\alpha_m - m, \ldots, -\alpha_1 - m}^{(p)}$ can be replaced with any

$$\mathbb{X}_\rho = \left\{ U \in \mathbb{C}^n [m \times m] : \langle U, U \rangle = \rho^2 I^{(m)}, r < \rho < R \right\},$$

and then these formulas will take the form (14). The theorem is proved. $\square$

Corollary 1. *In Theorem 1.2, when $n = 1$ the expansion (14) coincides with the Laurent expansion in the "matrix ring" defined in the Cartan classical domains (in the space $\mathbb{C}[m \times m]$):*

$$\Pi = \Pi_R \cap \Pi_r^-,$$

where matrix domains

$$\Pi_R = \left\{ Z \in \mathbb{C}\,[m \times k] : R^2 I^{(m)} - ZZ^* > 0 \right\},$$

$$\Pi_r = \left\{ Z \in \mathbb{C}\,[m \times k] : r^2 I^{(m)} - ZZ^* < 0 \right\},$$

and R, r are real numbers such that $0 < r < R$ (see [18]).

Corollary 2. *When $m = 1$ then the expansion (14) coincides with Laurent expansion of holomorphical function in a ball layer (in the space $\mathbb{C}^n$):*

$$\Pi = \{z \in \mathbb{C}^n : r < |z| < R\}$$

(by Severi's theorem, this expansion coincides with the Taylor expansion in the ball $\mathbb{B}_{1,n} = \{z \in \mathbb{C}^n : |z| < 1\}$).

Corollary 3. *When $m = n = 1$ we obtain the Laurent expansion on the complex plane.*

2. Open problems

We present some unsolved problems related to the matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$, associated with the classical domains of the second and third types:

1. Obtain analogs of the expansion of the Laurent–Hua Loo-Keng series for the matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$.

$2_{1,2,3}$. Describe domains of convergence of Laurent series with respect to matrix balls $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$.

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Ряды Лорана-Хуа Ло-кена относительно матричного шара из пространства $\mathbb{C}^n\,[m \times m]$

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Аннотация. Целью данной работы является получение аналогов ряда Лорана относительно матричного шара из пространства $\mathbb{C}^n\,[m \times m]$. Для этого сначала введены понятие "слоя матричного шара" из $\mathbb{C}^n\,[m \times m]$, затем в этом слое матричного шара использовались свойства интегралов типа Бохнера-Хуа Ло-кена для получения аналогов ряда Лорана.

Ключевые слова: матричной шар, ряд Лорана, голоморфная функция, граница Шилова, интеграл Бохнера-Хуа Ло-кена, ортонормальная система.