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Iterations and Groups of Formal Transformations

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Abstract. In this paper, we consider the problem of formal iteration. We construct an area preserving mapping which does not have any square root. This leads to a counterexample to Moser's existence theorem for an interpolation problem. We give examples of formal transformation groups such that the iteration problem has a solution for every element of the groups.

Keywords: iteration, formal transformations, functional equations.

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Introduction

Iterated functions are objects of study in computer science, fractals, dynamical systems and renormalization group physics [1–4]. Here we will consider continuous iterations of mappings. Let $\mathbb{K}$ denote either the set of real numbers or the set of complex numbers. Suppose we are given a local diffeomorphism u of a neighborhood of the origin $0 \in \mathbb{K}^n$ onto another and leaves 0 fixed. The problem of continuous iteration consists in finding a one-parameter family of mappings (a flow) $f(t,x) = f^t(x)$ such that

$$f^t \circ f^s = f^{t+s}, \qquad f^1 = u, \qquad f^0(x) = x \qquad \forall\, t, s \in \mathbb{R}. \tag{1}$$

The iteration problem was investigated by Koenigs, Lewis, Baker, Chen, Sternberg and others. Bibliographical references can be found in [4–6].

Every smooth flow f^t is defined by a system of ordinary differential equations

$$y' = X(y)$$

with initial condition $y(0) = x$. Thus the iteration problem is equivalent the following question. Given a a local diffeomorphism u, does there exist a system of ordinary differential equations such that $y(1) = u$? If the answer to this question is affirmative then we say that the map u is embedded in the flow f^t.

The problem is of great interest in the study of the exponential mapping of infinite-dimensional Lie algebras of vector fields [7–9]. Let $\exp(tX)$ denote an one-parameter group generated by a vector field X, then the map $\exp : X \mapsto \exp(X)$ is called the exponential map or time-one map. Let G be a group of smooth (or formal) maps, and we are given the mapping $u \in G$. The question which arises is this: under what conditions is there a vector field X such that $u = \exp(X)$? If such a vector field X exists, then it is called the logarithm of u. We will also say that the formal transformation u possesses a logarithm.

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Let us denote by $GS_n(\mathbb{K})$ the group of formal power series transformations [9]. Lewis [10] proved that if a transformation $u \in GS_n(\mathbb{K})$ satisfies so-called pseudo-incommensurable condition, then the iteration problem has a formal power series solution. This Lewis result has been repeatedly proved by different authors [5, 6, 9].

In this paper we discuss the iteration problem for some subgroups of the group $GS_n(\mathbb{K})$. It turns out that there are mappings u to which the problem does not even have formal solution, namely, we give an example of a polynomial mapping $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ preserving the area such that there does not exist a 2-tuple $g = (g_1, g_2)$ of formal power series $g_1, g_2 \in \mathbb{R}[[x, y]]$ with $g \circ g = u$. This is a counterexample to Moser's statement [3] about the existence of a solution to the iteration problem for area-saving mappings. We present sufficient conditions for the existence of a solution of the iteration problem. These conditions allow to indicate some groups of formal transformations such that any element of a group possesses a logarithm and the corresponding iteration problem has a formal solution.

1. Examples and condition for the existence of solutions

We begin with the case of a linear mapping

$$u(x) = Ux, \quad x \in \mathbb{K}^n,$$

where U is an invertible matrix. In this case, a solution of the iteration problem has the form

$$u^t(x) = U^t x = e^{t \ln(U)} x$$

whenever the matrix $\ln(U)$ is correctly defined. When $\mathbb{K} = \mathbb{C}$ the matrix $\ln(U)$ exists but in general it is not unique. If $\mathbb{K} = \mathbb{R}$ and U is positive definite then $\ln(U)$ is a real matrix. Some details of the linear case can be found in [10]. Sometimes a nonlinear problem (1) can be reduced to a linear one. This is true if an analytical map u is conjugate to a linear map. Some of the most known results in this direction are Poincaré and Siegel-Sternberg theorems [11–13].

We now consider the groups of formal transformations. Let $\mathbb{K}[[x]]$ denote the ring of formal power series in indeterminate $x_1, \dots, x_n$ with coefficients in $\mathbb{K}$. The ring has a maximal ideal $\mathfrak{M}_1$ and a ideal $\mathfrak{M}_2$ consisting of series without constant and linear terms. Denote by $\mathfrak{M}_i^n$ $(i = 1, 2)$ the n-ary Cartesian product of $\mathfrak{M}_i$. Obviously $\mathfrak{M}_1^n$ is a monoid under substitution of series. We denote by $GS_n(\mathbb{K})$ the set of all invertible elements of $\mathfrak{M}_1^n$. We shall call elements of $GS_n(\mathbb{K})$ formal transformations. It is clear that $GS_n(\mathbb{K})$ is a group. As usual, the general linear group of degree n over $\mathbb{K}$ is denoted by $GL_n(\mathbb{K})$.

Example 1. Let us consider the group $GS_1(\mathbb{C})$ and a polynomial map

$$u = e^{i\pi/3} z + z^7.$$

It is easy to see that there is no a formal power series

$$g = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots$$

such that

$$g \circ g = u. \tag{2}$$

Actually, comparing coefficients of z in (2), we have

$$c_1^2 = e^{i\pi/3}.$$

Then comparing coefficients of $z^2, \dots, z^6$ yields $c_2 = \dots = c_6 = 0$. Finally, comparing coefficients of z^7, we obtain

$$c_1 c_7 (c_1^6 + 1) = 1.$$

This is a contradiction, because $c_1^6 + 1 = 0$. This example shows that there is no one-parameter group passing through the polynomial $e^{i\pi/3}z + z^7$. If such a group f^t exists, then $f^{1/2} \circ f^{1/2} = u$. But it is not possible as we just proved. This example shows that polynomial map $u = e^{i\pi/3}z + z^7 \in GS_1(\mathbb{C})$ does not possess a logarithm.

We remark that such examples have been known for a long time (see, for example [4,9]).

Example 2. Let $SS_n(\mathbb{K})$ denote the set $\{f \in GS_n(\mathbb{K}) : det(Df) = 1\}$, where Df is the Jacobian matrix of f, i.e. $SS_n(\mathbb{K})$ is a group of volume preserving formal transformations. Consider an area preserving polynomial mapping $v \in SS_2(\mathbb{R})$ given by

$$\tilde{x}_1 = x_1 + x_2^{m+1}, \quad \tilde{x}_2 = x_2$$

and the rotation matrix

$$M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\alpha = 2\pi/m$ and $m \geqslant 2$ is an even number. Thus $u = Mv$ is an area preserving mapping.

It is convenient to use the complex variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. Then the mapping u has the form

$$u(z, \bar{z}) = e^{\frac{2i\pi}{m}} \left(z + \left(\frac{z - \bar{z}}{2i} \right)^{m+1} \right). \tag{3}$$

Let us show that there does not exist a formal series

$$g(z, \bar{z}) = c_{10}z + c_{01}\bar{z} + c_{20}z^2 + c_{11}z\bar{z} + c_{02}\bar{z}^2 + \dots$$

satisfying the condition (2). We assume that such series exists and try to find his coefficients.

Collect all terms belonging to $z, \bar{z}$ in (2). Then we have two equations

$$e^{i\frac{2\pi}{m}} = c_{10}^2 + |c_{01}|^2, \tag{4}$$

$$c_{01}(c_{10} + \bar{c}_{10}) = 0.$$

It follows that

$$c_{01} = 0, \quad c_{10} = \pm\exp(i\pi/m).$$

Then comparing coefficients of $z^k\bar{z}^l$ ($1 < k + l < m + 1$) yields equation

$$c_{kl}(c_{10} + c_{10}^k \bar{c}_{10}^l) = 0.$$

Obviously, the following inequality holds

$$c_{10} + c_{10}^k \bar{c}_{10}^l \neq 0$$

whenever $1 < k + l < m + 1$. Thus we have $c_{kl} = 0$.

Finally, we collect all terms belonging to z^{m+1} and obtain equalities

$$\frac{\exp(2i\pi/m)}{(2i)^{m+1}} = c_{(m+1)0}c_{10}(1 + c_{10}^m) = 0,$$

since $c_{10} = \pm\exp(i\pi/m)$ and m is an even number. This contradiction proves our assertion.

This example implies that Moser's theorem [3] on the solvability of the iteration problem in the class of formal series is not true even for polynomial mappings. Moreover, it is impossible to find the square root of a area preserving mapping in the general case. This example shows that the polynomial map (3) does not possess a logarithm. We shall see that the above examples are related to resonances.

Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ be characteristic values of a matrix $U \in GL_n(\mathbb{K})$. We recall that an identity of the form

$$\lambda_s = \lambda_1^{m_1} \cdots \lambda_n^{m_n}, \quad m_i \in \mathbb{N}, \quad \sum_{i=1}^{n} m_i > 1 \tag{5}$$

is called the resonance (induced by U). We say that the resonance (5) is not obstructive if

$$\lambda_s^t = \lambda_1^{tm_1} \cdots \lambda_n^{tm_n} \qquad \forall\, t \in \mathbb{R}. \tag{6}$$

It is easy to see that we have resonances of the form

$$\lambda = \lambda^{m+1},$$

in Examples 1 and 2 above. These resonances are obstructive since

$$\lambda^{\frac{1}{2}} \neq \lambda^{\frac{m+1}{2}}.$$

Using the theory of normal forms we proved the following statement in [14].

Lemma. *Let $u = Ux + g \in GS_n(\mathbb{C})$ be a formal transformation with $U \in GL_n(\mathbb{C})$ and $g \in \mathfrak{M}_2^n$. If any resonance induced by the matrix U is not obstructive then u possesses a logarithm.*

Now we show that the conditions (5), (6) are equivalent to Lewis's ones. Indeed, it follows from (5) that

$$\exp(\log \lambda_s) = \exp(m_1 \log \lambda_1 + \cdots + m_n \log \lambda_n).$$

The last equality is equivalent to

$$\log(\lambda_s) - \sum_{j=1}^{n} m_j \log(\lambda_j) \in 2\pi i \mathbb{Z}. \tag{7}$$

Similarly, the condition (6) yields

$$t(\log(\lambda_s) - \sum_{j=1}^{n} m_j \log(\lambda_j)) \in 2\pi i \mathbb{Z} \quad \forall\, t \in \mathbb{R}.$$

It follows that

$$\log(\lambda_s) = \sum_{j=1}^{n} m_j \log(\lambda_j). \tag{8}$$

Conversely, it is easy to see that the equality (8) gives (6) and (7) implies (5).

We recall that Lewis's condition means that any relation (7) implies the equality (8) (see [9, 10]).

One can apply Lemma to obtain subgroups G of $GS_n(\mathbb{K})$ such that any $u \in G$ possesses a logarithm. For example, consider subgroup B_l which consists of formal transformations

$$u = Ux + g, \quad g \in \mathfrak{M}_2^n,$$

where U is a lower triangular matrix with real positive eigenvalues.

Corollary. *Any formal transformation $u \in B_l$ possesses a logarithm.*

The analogous result holds for subgroup of formal transformations B^u with upper triangular matrices.

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Итерации и группы формальных преобразований

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Аннотация. В работе рассматривается задача формальной итерации. Строится сохраняющее площадь отображение, которое не допускает извлечения квадратного корня, что, в свою очередь, приводит к контрпримеру — к теореме Мозера для задачи интерполяции. Даны примеры групп формальных преобразований, для которых задача итерации имеет решение для произвольного элемента группы.

Ключевые слова: итерация, формальное преобразование, функциональное уравнение.