On Estimation of the Convergence Rate to Invariant Measures in Markov Branching Processes with Possibly Infinite Variance and Immigration

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Abstract. The continuous-time Markov Branching Process with Immigration is discussed in the paper. A critical case wherein the second moment of offspring law and the first moment of immigration law are possibly infinite is considered. Assuming that the non-linear parts of the appropriate generating functions are regularly varying in the sense of Karamata, theorems on convergence of transition functions of the process to invariant measures are proved. The rate of convergence is determined provided that slowly varying factors are with remainder.

Keywords: Markov branching process, generating functions, immigration, transition functions, slowly varying function, invariant measures, convergence rate.

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1. Introduction and preliminaries

The discussion of the population growth model called the continuous-time Markov Branching Process with Immigration (MBPI) which was considered in [5] is continued in this paper. Recall that this process has simple physical interpretation: the population size changes not only as a result of reproduction and disappearance of existing individuals but also as a result of the random influx of "extraneous" individuals of the same type from the outside. Namely, the process develops according to the following scheme. Each individual existing at time $t \in \mathcal{T} := [0, +\infty)$ independently of his history and of each other for a small time interval $(t, t + \varepsilon)$ is transformed into $j \in \mathbb{N}_0 \backslash \{1\}$ individuals with probability $a_j \varepsilon + o(\varepsilon)$, and with probability $1 + a_1 \varepsilon + o(\varepsilon)$ stays to live or makes evenly one descendant (as $\varepsilon \downarrow 0$). Here $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N}$ is the set of natural numbers, and $\{a_j\}$ are intensities of individual transformation, $a_j \geqslant 0$ for $j \in \mathbb{N}_0 \backslash \{1\}$ and $0 < a_0 < -a_1 = \sum\limits_{j \in \mathbb{N}_0 \backslash \{1\}} a_j < \infty$. Independently of these for each time interval $j \in \mathbb{N}$ new individuals enter the population with probability $b_j \varepsilon + o(\varepsilon)$, and immigration does not occur with probability $1 + b_0 \varepsilon + o(\varepsilon)$. Immigration intensities $b_j \geqslant 0$ for $j \in \mathbb{N}$ and $0 < -b_0 = \sum\limits_{j \in \mathbb{N}} b_j < \infty$. Newly arrived individuals undergo transformation in accordance with the reproduction law generated by intensities $\{a_j\}$; see [11, p. 217]. Thus, the process under consideration is completely determined by infinitesimal generating functions(GFs)

$$f(s) := \sum_{j \in \mathbb{N}_0} a_j s^j \quad \text{and} \quad g(s) := \sum_{j \in \mathbb{N}_0} b_j s^j \quad \text{for} \ \ s \in [0, 1).$$

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Let us denote the population size at the time $t \in \mathcal{T}$ in MBPI by $X(t)$. This is homogeneous continuous-time Markov chain with state space $\mathcal{S} \subset \mathbb{N}_0$ and transition functions

$$p_{ij}(t) := \mathbb{P}_i\big\{X(t) = j\big\} = \mathbb{P}\big\{X(t + \tau) = j \mid X(\tau) = i\big\}$$

for all $i, j \in \mathcal{S}$ and $\tau, t \in \mathcal{T}$.

Only critical case is considered in the paper, i.e., $f'(1-) = \sum_{j \in \mathbb{N}} j a_j = 0$, and limit behaviours of transition functions $p_{ij}(t)$ as $t \to \infty$ is observed. Pakes [9] was one of the first who studied invariant measures for MBPI with finite variance and found an integral form of GF of invariant measures. He has proved that limits $\pi_j := \lim_{t \to \infty} t^\lambda p_{ij}(t)$ exist independently on j, iff $\sum_{j \in \mathbb{N}} a_j j^2 \ln j < \infty$ and $\sum_{j \in \mathbb{N}} b_j j \ln j < \infty$, where $\lambda = 2g'(1-)/f''(1-)$, besides the set $\{\pi_j, j \in \mathcal{S}\}$ presents an invariant measure for MBPI. The invariant measure of MBPI can also be constructed by the strong ratio limit property of transition functions but slightly different [7]. Namely, the set of positive numbers $\big\{ \upsilon_j := \lim_{t \to \infty} p_{0j}(t)/p_{00}(t) \big\}$ is an invariant measure. Moreover one can see a close relation between the sets $\{\pi_j, j \in \mathcal{S}\}$ and $\{\upsilon_j, j \in \mathcal{S}\}$, and their GFs $\pi(s) = \sum_{j \in \mathcal{S}} \pi_j s^j$ and $\mathcal{U}(s) = \sum_{j \in \mathcal{S}} \upsilon_j s^j$. In fact, they are really only different versions of the same limit law. So, it is easy to see that $\mathcal{U}(s) = \pi(s)/\pi(0)$, and this is consistent with uniqueness, up to a multiplicative constant, of the invariant measure of MBPI.

An estimation of the rate of convergence to invariant measures is of exceptional interest. The rate of convergence of $t^\lambda p_{ij}(t)$ to π_j for all $i, j \in \mathcal{S}$ was studied under the condition $\max\{f'''(1-), g''(1-)\} < \infty$ [5]. It was found that the convergence rate is $\mathcal{O}(\ln t / t)$ as $t \to \infty$.

Throughout the paper, the following *Basic assumptions* for $f(s)$ and $g(s)$ are used

$$f(s) = (1 - s)^{1+\nu} \mathcal{L}\left(\frac{1}{1-s}\right), \qquad\qquad [f_\nu]$$

and

$$g(s) = -(1 - s)^\delta \ell\left(\frac{1}{1-s}\right) \qquad\qquad [g_\delta]$$

for all $s \in [0, 1)$, where $0 < \nu, \delta < 1$ and $\mathcal{L}(\cdot), \ell(\cdot)$ are *slowly varying at infinity* $(\mathbf{SV}_\infty)$ in the sense of Karamata (see, for instance, [2] and [10]). Basic assumptions imply that the offspring distribution belongs to the domain of attraction of the $(1 + \nu)$-stable law, and the immigration distribution belongs to the domain of attraction of the δ-stable law. In the critical case assumption $[f_\nu]$ implies that $2b := f''(1-) = \infty$. If $b < \infty$ then representation $[f_\nu]$ holds with $\nu = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$. Similarly, GF $g(s)$ of the form $[g_\delta]$ generates the immigration law with the δ-order moment. However, if $g'(1-) < \infty$ then assumption $[g_\delta]$ is fulfilled with $\delta = 1$ and $\ell(t) \to g'(1-)$ as $t \to \infty$.

An additional requirement for $\mathcal{L}(x)$ and $\ell(x)$ is introduced:

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = 1 + \mathcal{O}\big(\alpha(x)\big) \quad \text{as} \ \ x \to \infty \qquad\qquad [\mathcal{L}_\nu]$$

for each $\lambda > 0$, where $\alpha(x)$ is known positive decreasing function so that $\alpha(x) \to 0$ as $x \to \infty$. In this case $\mathcal{L}(x)$ is called SV_∞ with remainder $\mathcal{O}\big(\alpha(x)\big)$ (see [2, p. 185, condition SR1]). When employing condition $[\mathcal{L}_\nu]$ it is assumed that

$$\alpha(x) = \mathcal{O}\left(\frac{\mathcal{L}(x)}{x^\nu}\right) \quad \text{as} \ \ x \to \infty.$$

Similarly, the condition

$$\frac{\ell\,(\lambda x)}{\ell(x)} = 1 + \mathcal{O}\big(\beta(x)\big) \quad \text{as} \ \ x \to \infty \qquad\qquad [\ell_\delta]$$

is also allowed for each $\lambda > 0$, where

$$\beta(x) = \mathcal{O}\left(\frac{\ell\,(x)}{x^\delta}\right) \quad \text{as} \ \ x \to \infty.$$

It was shown that the asymptotes of the transition functions depend on the sign of the parameter $\gamma := \delta - \nu$ [3]. In addition, the limit functions $U(s) := \lim\limits_{t\to\infty} \mathcal{P}(t;s)$ for $\gamma > 0$ and $\pi(s) := \lim\limits_{t\to\infty} e^{T(t)}\mathcal{P}(t;s)$ for $\gamma < 0$ and for some $T(t)$ were found.

In this paper the rate of convergence is determined provided that conditions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold.

The rest of this paper is organized as follows. Section 2. contains main results. Auxiliary statements that are used in the proof of theorems are considered in Section 3.. Proof of main results is presented in Section 4..

2.   Main results

Let us consider GF $\mathcal{P}_i(t;s) := \sum\limits_{j\in\mathcal{S}} p_{ij}(t)s^j$. It is not difficult to see that (see [9])

$$\mathcal{P}_i(t;s) = \Big(F(t;s)\Big)^i \exp\left\{\int_0^t g\big(F(u;s)\big)du\right\}, \qquad (1)$$

where $F(t;s)$ is GF of Markov Branching Process initiated by single individual without immigration. Since $F(t;s) \to 1$ as $t \to \infty$ uniformly in $s \in [0,d]$, $d < 1$ (see Lemma 1 below), it is sufficient to consider $\mathcal{P}(t;s) := \mathcal{P}_0(t;s)$. Then taking into account Basic assumptions and the Kolmogorov backward equation $\partial F/\partial t = f\,(F)$, it follows from (1) that

$$\mathcal{P}(t;s) = \exp\left\{\int\limits_s^{F(t;s)} \frac{g(u)}{f(u)}\,du\right\}. \qquad (2)$$

Taking into account Basic assumptions, the integrand is

$$\frac{g(u)}{f(u)} = -(1-u)^{\gamma-1}\,L\left(\frac{1}{1-u}\right), \qquad (3)$$

where $\gamma := \delta - \nu$ and

$$L(t) := \frac{\ell(t)}{\mathcal{L}(t)}\ .$$

State space $\mathcal{S}$ can be classified in accordance with the sign of γ. By virtue of (3), integral $\int\limits_s^1 \left[g(u)/f(u)\right]du$ converges if $\gamma > 0$, and diverges if $\gamma < 0$. It was shown that $\mathcal{S}$ is *positive-recurrent* if $\gamma > 0$, and it is *transient* if $\gamma < 0$. The special case $\gamma = 0$ implies that $g(s) = f'(s)$ and $L(t) \to 1+\nu$ as $t \to \infty$. It is another population process called *Markov Q-process* (see [4], [6], [1, pp. 56–58] and [8] for the discrete-time case).

Main results are formulated only for the case $\gamma \neq 0$ in the following two theorems. Let

$$\tau(t) := \frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)} \qquad \text{and} \qquad T(t) := \big(\tau(t)\big)^{|\gamma|},$$

where $\mathcal{N}(x)$ is SV_∞ defined in Lemma 1 below.

Theorem 2.1. *Let $\gamma > 0$. Then $\mathcal{P}(t; s)$ converges to the function $U(s) = $
$= \exp\left\{\int\limits_{s}^{1}\left[g(u)/f(u)\right] du\right\}$ for $s \in [0, 1)$, and its power series expansion $U(s) = \sum\limits_{j \in \mathcal{S}} u_j s^j$ gener-
ates an invariant distribution $\{u_j, j \in \mathcal{S}\}$ for MBPI. The convergence is uniform over compact
subsets of $[0, 1)$. In addition, if assumptions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold then*

$$\mathcal{P}(t; s) = U(s)\left(1 + \Delta(t; s)\mathcal{K}\left(\tau(t)\right)\right), \tag{4}$$

where $\mathcal{K}(x) = \mathcal{L}^{-\delta/\nu}(x)\ell(x)$, function $\mathcal{N}(x)$ is SV_∞ defined in (13) below and

$$\Delta(t; s) = \frac{1}{\gamma} \frac{1}{\left(\lambda(t; s)\right)^{\gamma/\nu}} + \mathcal{O}\left(\frac{\ln\left[\Lambda(1 - s)\lambda(t; s)\right]}{\left(\lambda(t; s)\right)^{\delta/\nu}}\right) \quad as \ \ t \to \infty,$$

where $\lambda(t; s) = \nu t + \Lambda^{-1}(1 - s)$ and $\Lambda(y) = y^\nu \mathcal{L}(1/y)$. The transition functions are

$$p_{ij}(t) = u_j \left(1 + \mathcal{O}\left(\frac{K(t)}{t^{\gamma/\nu}}\right)\right) \quad as \ \ t \to \infty, \tag{5}$$

where $K(t)$ is SV_∞.

Another asymptotic property comes out for $\mathcal{P}(t; s)$ when $\gamma < 0$. Taking into account Basic assumptions, one can easily verify that

$$-\frac{\ln p_{00}(t)}{T(t)} \sim \frac{1}{|\gamma|} L\left(\tau(t)\right) \quad as \ \ t \to \infty.$$

This asymptotic relation shows that $\left(T(t)\right)^{-1} \ln p_{00}(t)$ is asymptotically SV_∞. Then one should consider the limit of the function $e^{T(t)}\mathcal{P}(t; s)$ as $t \to \infty$. First one needs to consider SV_∞ property of $L(t)$. In accordance with the slowly varying theory, functions $\ell(\cdot)$ and $\mathcal{L}(\cdot)$ are positive. Then by virtue of [2, p. 185, Theorem 3.12.2 (SR1)], one can obtain the following propositions:

▶ $[\mathcal{L}_\nu] \iff \mathcal{L}(x) = C_{\mathcal{L}} + \mathcal{O}\left(\alpha(x)\right) \quad as \ \ t \to \infty,$ \hfill $[C_{\mathcal{L}}]$

▶ $[\ell_\delta] \iff \ell(x) = C_\ell + \mathcal{O}\left(\beta(x)\right) \quad as \ \ t \to \infty,$ \hfill $[C_\ell]$

where $C_{\mathcal{L}}, C_\ell$ are positive constants and functions $\alpha(x), \beta(x)$ are in $[\mathcal{L}_\nu]$ and $[\ell_\delta]$. Then

$$L(t) = \frac{\ell(t)}{\mathcal{L}(t)} = C_L + \mathcal{O}\left(\frac{\ell(t)}{t^\delta}\right) \quad as \ \ t \to \infty, \tag{6}$$

since $\delta < \nu$, where $C_L = C_\ell / C_{\mathcal{L}}$. This requirement for $L(t)$ is quite possible. Especially, one can obtain an "excellent result" if $C_L = |\gamma|$ is chosen. The following explicit form of $\pi(s) = \lim_{t \to \infty} e^{T(t)}\mathcal{P}(t; s)$ was found in [3]:

$$\pi(s) = \exp\left\{\frac{1}{(1 - s)^{|\gamma|}} + \int_s^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1 - u)^{1+|\gamma|}}\right] du\right\}. \tag{7}$$

Now the convergence rate of $e^{T(t)}\mathcal{P}(t; s)$ to $\pi(s)$ is determined in the following theorem.

Theorem 2.2. *Let $\gamma < 0$ and $C_L = |\gamma|$ in (6). If $\mu := 2\delta - \nu > 0$ then*

$$e^{T(t)}\mathcal{P}(t; s) = \pi(s)\left(1 + \rho(t; s)\right), \tag{8}$$

where $\rho(t; s) \to 0$ as $t \to \infty$ uniformly in $s \in [0, r]$, $r < 1$, and the limiting GF $\pi(s)$ can be expressed in the form given in (7). In addition, if assumptions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold then

$$\rho(t; s) = \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^\mu}\right) \quad as \;\; t \to \infty \tag{9}$$

uniformly in $s \in [0, r]$, $r < 1$. Denoting the power series expansion of $\pi(s)$ by $\sum\limits_{j \in \mathcal{S}} \pi_j s^j$, transition functions have the form

$$p_{ij}(t) = \pi_j \left(1 + \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^\mu}\right)\right) \quad as \;\; t \to \infty, \tag{10}$$

and $\{\pi_j, j \in \mathcal{S}\}$ is an invariant measure for MBPI.

Remark 1. *The form of limiting GF $\pi(s)$ given in the first part of Theorem 2.2 is compatible with the results presented in [9] and [5] where the case $\max\{f''(1-), g'(1-)\} < \infty$ was considered. Thus, this theorem essentially strengthens last-mentioned results.*

Remark 2. *The conditions $C_L = |\gamma|$ and $\mu > 0$ in Theorem 2.2 are essential because they ensure the convergence of the integral in (7). In fact, due to Basic assumptions and (6) the majorizing function for the integrand is $(1 - u)^{\mu - 1}$. Then function*

$$\mathcal{B}(s) := \exp\left\{\int_s^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1 - u)^{1 + |\gamma|}}\right] du\right\} \tag{11}$$

is bounded for $s \in [0, 1]$.

The following result is a consequence of Theorem 2.2.

Corollary 1. *Under the conditions of Theorem 2.2*

$$e^{T(t)} p_{00}(t) = \mathcal{B}(0)\left(1 + \mathcal{O}\left(\frac{\ell(\tau(t))}{(\tau(t))^\mu}\right)\right) \quad as \;\; t \to \infty,$$

where function $\mathcal{B}(s)$ is defined in (11).

Remark 3. *Further reasoning imply that functions $\mathcal{L}(x)$ and $\ell(x)$ can be omitted in estimations of error terms of asymptotic relations in given above Theorems. Taking into account assertions $[C_\mathcal{L}]$ and $[C_\ell]$, these functions are asymptotically constant.*

3. Auxiliaries

In this section, some auxiliary assertions are provided. They are essential for the proof of theorems.

First, the asymptotic representation of GF of Markov branching processes $Z(t)$ without immigration is considered. Let $F(t; s) = \mathbb{E}\left[s^{Z(t)} \big| Z(0) = 1\right]$ be GF of the process initiated by single individual. Let $R(t; s) := 1 - F(t; s)$. The following result called the Basic lemma of the theory of critical Markov branching processes [4]. It is presented in slightly different form below.

Lemma 1. *If condition $[f_\nu]$ holds then*

$$\frac{1}{R(t; s)} = \frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)} \cdot \left[1 + \frac{\mathcal{M}(s)}{t}\right]^{1/\nu} \tag{12}$$

for all $s \in [0, 1)$, where $\mathcal{N}(x)$ is SV_∞ such that

$$\mathcal{N}^\nu(t) \cdot \mathcal{L}\left(\frac{(\nu t)^{1/\nu}}{\mathcal{N}(t)}\right) \longrightarrow 1 \quad as \ \ t \to \infty, \tag{13}$$

and $\mathcal{M}(s)$ is GF of invariant measures of MBP that has the form

$$\mathcal{M}(s) = \int\limits_1^{1/(1-s)} \frac{dx}{x^{1-\nu}\mathcal{L}(x)} \ .$$

Let us introduce the following function

$$\Lambda(y) := y^\nu \mathcal{L}\left(\frac{1}{y}\right) = \frac{f(1-y)}{y}$$

for $y \in (0, 1]$. Let us note that function $y\Lambda(y)$ is positive, tends to zero and it has the monotone derivative so that $y\Lambda'(y)/\Lambda(y) \to \nu$ as $y \downarrow 0$ (see [2, p. 401]). Then it is natural to write

$$\frac{y\Lambda'(y)}{\Lambda(y)} = \nu + \delta(y), \tag{14}$$

where $\delta(y)$ is continuous and $\delta(y) \to 0$ as $y \downarrow 0$. Since $\Lambda(1) = \mathcal{L}(1) = a_0$ it follows from (14) that

$$\Lambda(y) = a_0 y^\nu \exp \int_1^y \frac{\delta(u)}{u} du.$$

Therefore

$$\mathcal{L}\left(\frac{1}{y}\right) = a_0 \exp \int_1^y \frac{\delta(u)}{u} du.$$

Substituting $u = 1/t$ in last integrand, one can obtain

$$\mathcal{L}(x) = a_0 \exp \int_1^x \frac{\varepsilon(t)}{t} dt,$$

*** where $\varepsilon(t) = -\delta(1/t)$ and $\varepsilon(t) \to 0$ as $t \to \infty$. Considering the last equation together with $[\mathcal{L}_\nu]$, one can obtain

$$\int_x^{\lambda x} \frac{\varepsilon(t)}{t} dt = \ln\left[1 + \mathcal{O}(\alpha(x))\right] = \mathcal{O}(\alpha(x)) \quad as \ \ x \to \infty$$

for each $\lambda > 0$. Applying the mean value theorem to the left-hand side of the last equality, we have that $\varepsilon(x) = \mathcal{O}(\alpha(x))$. Then condition $[\mathcal{L}_\nu]$ gives

$$\delta(y) = \mathcal{O}\left(\alpha\left(\frac{1}{y}\right)\right) \quad as \ \ y \downarrow 0. \tag{15}$$

The following result is a modification of Lemma 1 and it is required in the subsequent discussions.

Lemma 2. *Let assumptions $[f_\nu]$ and $[\mathcal{L}_\nu]$ hold. Then*

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \mathcal{O}(\ln \nu(t;s)) \quad as \ \ t \to \infty, \tag{16}$$

where $\nu(t; s) = \Lambda(1-s)\nu t + 1$.

Proof. One can write from (14) that

$$\frac{R\Lambda'(R)}{\Lambda(R)} = \nu + \delta(R) \tag{17}$$

since $R := R(t;s) \to 0$ as $t \to \infty$. Using the backward Kolmogorov equation $\partial F/\partial t = f(F)$ and considering representation $[f_\nu]$, relation (17) becomes

$$\frac{d\Lambda(R)}{dt} = -\frac{\Lambda(R)}{R}f(1-R)(\nu + \delta(R)) = -\Lambda^2(R)(\nu + \delta(R)).$$

Therefore

$$d\left[\frac{1}{\Lambda(R)} - \nu t\right] = \delta(R)\,dt. \tag{18}$$

Integrating (18) over $[0, t]$, the following equation is obtained

$$\frac{1}{\Lambda(R(t;s))} - \frac{1}{\Lambda(1-s)} = \nu t + \int_0^t \delta(R(u;s))du, \tag{19}$$

where $\delta(y)$ is in (14). Now one should take integral in (19). Considering (15), one can write

$$\int_0^t \delta(R(u;s))du = \int_0^t \mathcal{O}\big(\Lambda(R(u;s))\big)du. \tag{20}$$

One should mention that due to (12) $R(t;s) \to 0$ as $t \to \infty$ uniformly in $s \in [0,1)$. Therefore, since $\Lambda(y) \to 0$ as $y \downarrow 0$, the integral in the right-hand side of (20) is $o(t)$ as $t \to \infty$. Hence

$$\Lambda(R(t;s)) = \frac{1}{\lambda(t;s)} + o\left(\frac{1}{\lambda(t;s)}\right) \quad as \ \ t \to \infty,$$

where $\lambda(t;s) = \nu t + \Lambda^{-1}(1-s)$. Therefore

$$\int_0^t \mathcal{O}\big(\Lambda(R(u;s))\big)du = \mathcal{O}\left(\int_0^t \Lambda(R(u;s))du\right) = \mathcal{O}\big(\ln\nu(t;s)\big) \quad as \ \ t \to \infty.$$

Together with (19) and (20) this gives relation (16). □

Lemma 3. *Let $L(t)$ be SV_∞ with remainder $\varrho(t)$. Then for $\sigma > 0$*

$$\int_t^\infty y^{-(1+\sigma)}L(y)dy = \frac{1}{\sigma}\frac{1}{t^\sigma}L(t)\big(1 + \mathcal{O}(\varrho(t))\big) \quad as \ \ t \to \infty. \tag{21}$$

Proof. Undoubtedly $\int_1^\infty u^{-(1+\sigma)}du = 1/\sigma$. Considering this fact and making the substitution $y := ut$ in the integrand of (21), one can write

$$\int_t^\infty y^{-(1+\sigma)}L(y)dy = \frac{1}{\sigma}\frac{L(t)}{t^\sigma}\left[1 + \sigma\int_1^\infty \left[\frac{L(ut)}{L(t)} - 1\right]u^{-(1+\sigma)}du\right]. \tag{22}$$

By definition of SV_∞-function with remainder, the expression in brackets of the integrand on the right-hand side of (22) tends to 0 as $t \to \infty$ uniformly in $u > 1$ (by Uniform Convergence Theorem for SV_∞-functions [2, Theorem 1.5.2]) with the rate $\mathcal{O}(\varrho(t))$. Thus relation (21) is obtained.

The Lemma is proved. □

Lemma 4. *Let conditions $[\mathcal{L}_\nu]$ and $[\ell_\delta]$ hold and $\gamma > 0$. Then*

$$\int_x^1 \frac{g(u)}{f(u)}\, du = \frac{1}{\gamma} \frac{g(x)}{\Lambda(1-x)} \left(1 + \mathcal{O}\big(\Lambda(1-x)\big)\right) \quad as \;\; x \uparrow 1. \tag{23}$$

Proof. It follows from Basic assumption that

$$\mathcal{I}(x) := \int_x^1 \frac{g(u)}{f(u)}\, du = - \int_{1/(1-x)}^\infty y^{-(1+\gamma)} L(y)\, dy, \tag{24}$$

where $L(t) = \ell(t)/\mathcal{L}(t)$ as before. One can easily show that

$$\frac{L(ut)}{L(t)} - 1 = \mathcal{O}\left(\frac{\mathcal{L}(t)}{t^\nu}\right) \quad as \;\; t \to \infty$$

uniformly in $u > 0$. Considering the right-hand side of (24), one can directly use (21) with $t = 1/(1-x)$ and $r(t) = \mathcal{O}\left(\mathcal{L}(t)/t^\nu\right)$. Then

$$\mathcal{I}(x) = -\frac{1}{\gamma} \frac{L(t)}{t^\gamma} \left[1 + \mathcal{O}\left(\frac{\mathcal{L}(t)}{t^\nu}\right)\right] \quad as \;\; t \to \infty.$$

Now returning to primary designations, relation (23) is obtained.
The Lemma is proved. □

4.   Proof of Theorems

In this final section the Main results are consistently proved.

Proof of Theorem 2.1. Let us rewrite (2) as follows

$$\mathcal{P}(t; s) = U(s) \exp\left\{ \int_1^{F(t;s)} \frac{g(u)}{f(u)}\, du \right\}, \tag{25}$$

where

$$U(s) = \exp\left\{ \int_s^1 \frac{g(u)}{f(u)}\, du \right\}. \tag{26}$$

Considering (3), the integral in (25) converges for $s \in [0,1)$ and becomes 0 as $t \to \infty$. Therefore $\mathcal{P}(t; s)$ converges to $U(s)$ as $t \to \infty$ uniformly over compact subsets. Now, using the functional equation $F(t + \tau; s) = F\big(t; F(\tau; s)\big)$ (see [9, p. 134]), it follows that

$$\begin{aligned}
\mathcal{P}(t + \tau; s) &= \mathcal{P}(\tau; s) \cdot \exp\left\{ \int_\tau^{t+\tau} g\left(F(u; s)\right) du \right\} \\[2mm]
&= \mathcal{P}(\tau; s) \cdot \exp\left\{ \int_0^t g\left(F\big(u; F(\tau; s)\big)\right) du \right\} = \mathcal{P}(\tau; s) \cdot \mathcal{P}\big(t; F(\tau; s)\big).
\end{aligned}$$

Taking limit as $t \to \infty$, one can obtain the following Schröder type functional equation

$$U\big(F(\tau; s)\big) = \frac{1}{\mathcal{P}(\tau; s)} U(s) \quad \text{for any} \;\; \tau \in \mathcal{T}. \tag{27}$$

Writing the power series expansion $U(s) = \sum\limits_{j \in \mathcal{S}} u_j s^j$, equation (27) has an invariant property $u_j = \sum\limits_{i \in \mathcal{S}} u_i p_{ij}(\tau)$. Obviously $U(1-) = 1$ and hence the function in (26) generates an invariant distribution $\{u_j, j \in \mathcal{S}\}$ for MBPI.

Let us prove now of of (4). Considering (25) and using (23), one can obtain

$$\mathcal{P}(t; s) = U(s) \exp\left\{-I(t; s)\right\}, \tag{28}$$

where

$$I(t; s) = \frac{1}{\gamma} \frac{g\big(F(t; s)\big)}{\Lambda\big(R(t; s)\big)} \left(1 + \mathcal{O}\big(\Lambda(R(t; s))\big)\right) \quad \text{as} \ \ t \to \infty. \tag{29}$$

Next, let us use the asymptotic expansion of $R(t; s)$. Relation (16) implies

$$\frac{1}{\Lambda\left(R(t; s)\right)} = \lambda(t; s) \left(1 + \mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t; s)\right]}{\lambda(t; s)}\right)\right) \quad \text{as} \ \ t \to \infty, \tag{30}$$

and therefore

$$R(t; s) = \frac{\mathcal{N}(t; s)}{(\lambda(t; s))^{1/\nu}} \left(1 + \mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t; s)\right]}{\lambda(t; s)}\right)\right) \quad \text{as} \ \ t \to \infty, \tag{31}$$

where $\lambda(t; s) = \nu t + \Lambda^{-1}(1 - s)$ and $\mathcal{N}(t; s) = \mathcal{L}^{-1/\nu}\big(1/R(t; s)\big)$. Let us note that $g(s)$ has the form of $[g_\delta]$. Using (30) and (31), one can obtain

$$\frac{g\big(F(t; s)\big)}{\Lambda\big(R(t; s)\big)} = -\frac{\mathcal{N}^\delta(t; s)}{(\lambda(t; s))^{\gamma/\nu}} \ell\left(\frac{1}{R(t; s)}\right) \left(1 + \mathcal{O}\left(\frac{\ln\left[\Lambda(1-s)\lambda(t; s)\right]}{\lambda(t; s)}\right)\right) \tag{32}$$

as $t \to \infty$. It is easy to verify that the function $\mathcal{N}(t; s)$ is asymptotically equivalent to the SV_∞-function $\mathcal{N}(t)$ defined in Lemma 1.

Asymptotic formula (4) now follows from a combination of (28), (29) and (32). Equation (5) follows from the continuity theorem for power series.

The Theorem is proved. $\qquad\square$

Proof of Theorem 2.2. Let us write

$$e^{T(t)}\mathcal{P}(t; s) = \exp\left\{(\tau(t))^{|\gamma|} + \int_0^t g\left(F(u; s)\right) du\right\} =$$

$$= \exp\left\{\Delta(t; s) + \big(\tau(t; s)\big)^{|\gamma|} + \int_s^{F(t;s)} \frac{g(x)}{f(x)} \, dx\right\}, \tag{33}$$

where $\Delta(t; s) = \big(\tau(t)\big)^{|\gamma|} - \big(\tau(t; s)\big)^{|\gamma|}$ and $\tau(t; s) = R^{-1}(t; s)$. Standard integration yields

$$\big(\tau(t; s)\big)^{|\gamma|} = \frac{1}{(1 - s)^{|\gamma|}} + \int_s^{F(t;s)} \frac{|\gamma|}{(1 - u)^{1+|\gamma|}} du.$$

Therefore, relation (33) can be written as follows

$$e^{T(t)}\mathcal{P}(t; s) = \pi(s) \cdot \exp\left\{\Delta(t; s) - \int_{F(t;s)}^1 \left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1 - u)^{1+|\gamma|}}\right] du\right\}, \tag{34}$$

where $\pi(s)$ has the form of (7). An exponential factor in (34) defines the convergence rate $\rho(t;s)$ in (8). Let us first evaluate $\Delta(t;s)$ as $t \to \infty$. According to Lemma 1 $\mathcal{M}(0) = 0$ and, therefore, $\tau(t) = \tau(t;0)$. Hence, asymptotic representation (12) gives

$$
\begin{aligned}
\Delta(t;s) &= \left(\tau(t)\right)^{|\gamma|}\left[1 - \left(1 + \frac{\mathcal{M}(s)}{t}\right)^{|\gamma|/\nu}\right] \\
&\sim -|\gamma|\left(\tau(t)\right)^{|\gamma|}\frac{\mathcal{M}(s)}{\nu t} = -|\gamma|\frac{\mathcal{M}(s)}{(\nu t)^{\delta/\nu}\mathcal{N}^{|\gamma|}(t)} \quad \text{as } t \to \infty.
\end{aligned}
$$

On the other hand $\mathcal{M}(s)$ is bounded for $s \in [0,r]$, $r < 1$. Thus

$$
\Delta(t;s) = \mathcal{O}\left(\frac{\mathcal{L}_\gamma(t)}{t^{\delta/\nu}}\right) \quad \text{as } t \to \infty, \tag{35}
$$

uniformly in $s \in [0,r]$, $r < 1$, where $\mathcal{L}_\gamma(t) = \mathcal{N}^{-|\gamma|}(t)$.

Let us observe the integral in (34). Taking into account relations (3) and (6), the integrand in brackets becomes $\mathcal{O}\left((1-u)^{\mu-1}\ell\left(1/(1-u)\right)\right)$ in the neighbourhood of the point $u = 1$. Let us examine the integral $\int_{F(t;s)}^{1}\left[(1-u)^{\mu-1}\ell\left(1/(1-u)\right)\right]du$ as $t \to \infty$. Substitution $y = (1-u)^{-1}$ gives the alternative form

$$
\int_{1/R(t;s)}^{\infty} y^{-(1+\mu)}\ell(y)dy.
$$

The direct application of Lemma 3 transforms the last integral to the form

$$
\int_{1/R(t;s)}^{\infty} y^{-(1+\mu)}\ell(y)dy = \frac{1}{\mu}R^{\mu}(t;s)\ell\left(\frac{1}{R(t;s)}\right)\left(1 + o(1)\right) \quad \text{as } t \to \infty.
$$

But $R(t;s) = \tau^{-1}(t;s)$ and $\tau(t;s)\tau^{-1}(t) \to 1$ as $t \to \infty$ uniformly in $s \in [0,1)$. Thus

$$
\int_{F(t;s)}^{1}\left[\frac{g(u)}{f(u)} + \frac{|\gamma|}{(1-u)^{1+|\gamma|}}\right]du = \mathcal{O}\left(\frac{\ell\left(\tau(t)\right)}{\left(\tau(t)\right)^{\mu}}\right) \quad \text{as } t \to \infty. \tag{36}
$$

Taking into account that $\mu < \delta$ and comparing relations (35) and (36), one can obtain that $\Delta(t;s)$ decreases to zero faster than last integral, i.e., $\Delta(t;s) = o\left(\ell\left(\tau(t)\right)/\left(\tau(t)\right)^{\mu}\right)$ as $t \to \infty$. So, asymptotic relation (8) with the error part $\rho(t;s)$ in form (9) is found from (34)–(36). Equation (10) follows from the continuity theorem for power series.

Finally, one can verify that function $\pi(s)$ satisfies equation (14). Therefore, denoting its power series representation by $\pi(s) = \sum_{j \in \mathcal{S}} \pi_j s^j$, an invariant property $\pi_j = \sum_{i \in \mathcal{S}} \pi_i p_{ij}(\tau)$ is obtained for any $\tau > 0$. Thus $\{\pi_j, j \in \mathcal{S}\}$ is an invariant measure for MBPI $X(t)$.

The Theorem is proved. $\qquad\square$

Proof of Corollary 1. The statement follows immediately from (8) by setting $x = 0$. $\qquad\square$

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Об оценке скорости сходимости к инвариантным мерам в марковских ветвящихся процессах с возможной бесконечной дисперсией и иммиграцией

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Аннотация. В работе исследуется марковский ветвящийся случайный процесс с непрерывным временем и с иммиграцией. Мы рассматриваем критический случай, в котором второй момент закона размножения частиц и первый момент закона иммиграции бесконечны. Предполагая, что нелинейные части соответствующих производящих функций правильно меняются в смысле Карамата, мы доказываем теоремы о сходимости переходных вероятностей процесса к инвариантным мерам. Мы определим скорости этой сходимости при условии, что медленно меняющиеся части являются функциями с остатком.

Ключевые слова: марковский ветвящийся процесс, производящие функции, иммиграция, переходные вероятности, медленно меняющаяся функция, инвариантные меры, скорость сходимости.