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Inverse Problems of Finding the Lowest Coefficient in the Elliptic Equation

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Abstract. The article is devoted to the study of problems of finding the non-negative coefficient $q(t)$ in the elliptic equation

$$u_{tt} + a^2\Delta u - q(t)u = f(x,t)$$

($x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $t \in (0,T)$, $0 < T < +\infty$, Δ — operator Laplace on $x_1, \dots, x_n$). These problems contain the usual boundary conditions and additional condition (spatial integral overdetermination condition or boundary integral overdetermination condition). The theorems of existence and uniqueness are proved.

Keywords: elliptic equation, unknown coefficient, spatial integral condition, boundary integral condition, existence, uniqueness.

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The problems studied in this work belong to the class of nonlinear inverse coefficient problems for elliptic differential equations.

Various aspects of the theory of linear and nonlinear inverse coefficient problems for differential equations are well covered in the world literature — see, for example, monographs [1–8], articles [9–19]. Directly for elliptic equations inverse coefficient problems were studied in [15–19] (a more detailed bibliography can be found in [17]).

The nonlinear inverse coefficient problems for elliptic equations studied in this work, the results obtained in it will be essentially differ either in the formulations (in particular, in the given redefinition conditions), or in the results from the statements and results from the works of predecessors.

The problems studied in this work have a model form. More general cases and also possible generalization of the obtained results will be discussed at the end of the article.

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1. Statement of the problems

Let Ω be a bounded domain of variables $(x_1, \dots, x_n)$ of space $\mathbb{R}^n$, Γ is the boundary of Ω. We assume that Γ is a compact infinitely differentiable manifold. Next, Q is a cylinder $\Omega \times (0, T)$ of finite height T, $S = \Gamma \times (0, T)$ is the lateral surface of Q. Let $f(x,t)$, $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ be given functions defined for $x \in \overline{\Omega}$, $t \in [0, T]$; let a be given positive number.

Inverse Problem I: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation*

$$u_{tt} + a^2 \Delta u - q(t)u = f(x,t) \tag{1}$$

provided that $u(x,t)$ satisfies the conditions

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x), \quad x \in \Omega; \tag{2}$$

$$u(x,t)|_S = 0, \tag{3}$$

$$\int_\Omega N(x)u(x,t)\,dx = \mu(t), \quad t \in (0,T). \tag{4}$$

Inverse Problem II: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1) provided that $u(x,t)$ satisfies (2), (4) and also the condition*

$$\left.\frac{\partial u(x,t)}{\partial \nu}\right|_S = 0. \tag{5}$$

Inverse Problem III: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1) provided that $u(x,t)$ satisfies (2), (5), and also the condition*

$$\int_\Gamma N(x)u(x,t)\,ds_x = \mu(t). \tag{6}$$

In Inverse Problems I and II conditions (2) and (3), (2) and (5) are the conditions of a correct boundary value problem for second-order differential elliptic equation in a cylinder Q, whereas condition (4) is space-integral overdetermination condition. In Inverse Problem III conditions (2) and (5) are also the conditions of a correct boundary value problem for second-order differential elliptic equations, whereas condition (6) is an boundary-integral overdetermination condition.

All constructions and arguments in this paper will be carried out using the Lebesgue spaces L_p and Sobolev spaces W_p^l. The necessary information about the functions from these spaces can be found in the books [20–22].

The goal of this article is to prove the existence and uniqueness of regular solutions to the problems under study, that is, of solutions having all the weak derivatives in the sense of Sobolev involved in the equation.

2. Solvability of the inverse Problems I и II

Perform some auxiliary constructions for Inverse Problem I. Given the function $w(x,t)$, we define the function $\Phi(t;w)$: $\Phi(t;w) = a^2 \int\limits_\Omega N(x)\Delta w(x,t)\,dx$.

Put $\quad v_0(x,t) = \frac{t}{T}u_1(x) + \frac{T-t}{T}u_0(x), \qquad f_1(x,t) = f(x,t) - a^2\Delta v_0(x,t),$

$$f_0(t) = \int_\Omega N(x)f_1(x,t)\,dx, \quad \varphi(t) = \frac{1}{\mu(t)}, \quad \psi(t) = \varphi(t)[\mu''(t) - f_0(t)],$$

$$f_2(x,t) = f_1(x,t) + [\psi(t) + \varphi(t)\Phi(t;v_0)]v_0(x,t).$$

Consider the boundary value problem: *Find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\varphi(t)\Phi(t;v+v_0) + \psi(t)]v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{1'}$$

and satisfies condition

$$v(x,0) = v(x,T) = 0, \quad x \in \Omega, \tag{2'}$$

and also the condition (3). Using a solution $v(x,t)$ of this boundary value problem we can establish the solvability of the inverse problem I.

Integro-differential equation (1′) is called loaded equation [23, 24].

Put $\varphi_0 = \max\limits_{[0,T]} |\varphi(t)|, \psi_0 = \min\limits_{[0,T]} \psi(t)$, $N_1 = 2\sum\limits_{i=1}^{n}\int\limits_{\Omega}\left[u_{0x_i}^2 + u_{1x_i}^2\right]dx$, $N_2 = \dfrac{1}{2}\varphi_0^2 N_1 T\|N\|^2_{L_2(\Omega)}$, $N_3 = \sum\limits_{i=1}^{n}\int\limits_{Q} f_{2x_i}^2\,dx\,dt$, $N_4 = \dfrac{2N_3}{a^2(1-N_2)}$, $N_5 = a^2\|N\|_{L_2(\Omega)}\,(TN_4)^{1/2} + |\Phi(0,u_0)| + |\Phi(0,u_1)|$.

Theorem 2.1. *Suppose the fulfillment of conditions*

$$N(x) \in L_2(\Omega), \quad \mu(t) \in C^2([0,T]); \quad f(x,t) \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)) \cap L_\infty(0,T;L_2(\Omega)),$$

$$u_0(x) \in W_2^3(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad u_1(x) \in W_2^3(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad \Delta u_0(x) = \Delta u_1(x) = 0 \quad \text{for} \quad x \in \Gamma;$$

$$\varphi_0 > 0, \quad \psi_0 > 0, \quad N_2 < 1, \quad N_5 \leqslant \frac{\psi_0}{\varphi_0};$$

$$\mu(0) = \int_\Omega N(x)u_0(x)\,dx, \quad \mu(T) = \int_\Omega N(x)u_1(x)\,dx.$$

Then the inverse problem I has a solution $\{u(x,t), q(t)\}$ such that $u(x,t) \in W_2^2(Q)$, $\Delta u(x,t) \in W_2^1(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geqslant 0$ for $t \in [0,T]$.

Proof. We establish the solvability of the boundary value problem (1′), (2′), (3) in the space $W_2^2(Q)$. We use the regularization method and method of cut-off functions.

Let γ be a number from the interval $\left(0, \frac{\varphi_0}{\psi_0}\right]$. Define the cut-off function $G_\gamma(\xi)$:

$$G_\gamma(\xi) = \begin{cases} \xi, & \text{if} \quad |\xi| \leqslant \gamma, \\ \gamma, & \text{if} \quad \xi \geqslant \gamma, \\ -\gamma, & \text{if} \quad \xi \leqslant -\gamma. \end{cases}$$

Next, let ε be a positive number. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t;v+v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{$1'_\varepsilon$}$$

in the cylinder Q and satisfies conditions (2′) *and* (3′) *and also the condition*

$$\Delta v(x,t)|_S = 0. \tag{7}$$

Show that for a fixed number ε this problem has a solution belonging to $W_2^{4,2}(Q)$. Let's use the fixed point method.

Let $w(x,t)$ be a function from the space $W_2^{4,2}(Q)$. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t;w+v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{$1'_{\varepsilon,w}$}$$

in the cylinder Q and satisfies conditions $(2')$*, (3), (7).*

This problem is the first boundary value problem for linear loaded quasi-elliptic equation. Using method of continuation in the parameter (see [25]), it is not difficult to establish its solvability in the space $W_2^{4,2}(Q)$.

Let λ be a number from the segment $[0,1]$. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; w + v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \lambda\varphi(t)v_0(x,t)\Phi(t; v) \quad (1'_{\varepsilon,w,\lambda})$$

in the cylinder Q and satisfies conditions $(2')$*, (3) and (7).* For $\lambda = 0$, this problem is solvability in the space $W_2^{4,2}(Q)$ (this is not difficult to prove using the classical Galerkin method with the choice of a special basis [21]). Next, all possible solutions $v(x,t)$ of the boundary value problem $(1'_{\varepsilon,w,\lambda})$, $(2')$, (3), (7) at a fixed ε satisfy estimate

$$\left(1 - \frac{\varphi_0^2 N_1 T}{2}\|N\|^2_{L_2(\Omega)}\right)\int_Q (\Delta v_t)^2\,dx\,dt + \frac{a^2}{2}\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2\,dx\,dt +$$

$$+\varepsilon\int_Q \left(\Delta^2 v\right)^2\,dx\,dt \leqslant C_1\int_Q f_2^2\,dx\,dt, \tag{8}$$

with the constant C_1 defined only by ε. In order to prove this estimate we multiply the equation $(1'_{\varepsilon,w,\lambda})$ by the function $-\Delta^2 v$ and integrate on the cylinder Q. Using $\psi(t) + \varphi(t)G_\gamma(\Phi(t; w + v_0)) \geqslant 0$ and applying Hölder's and Young's inequality, and also inequality

$$\int_\Omega [\Delta v(x,t)]^2\,dx \leqslant T\int_Q [\Delta v_t(x,t)]^2\,dx\,dt$$

we obtain the estimate (8). From estimate (8) and the second main inequality for elliptic operator (see [21]) it follows that all possible solutions $v(x,t)$ to boundary value problem $(1'_{\varepsilon,w,\lambda})$, $(2')$, (3), (7) for a fixed ε satisfy the a priori estimate

$$\|v\|_{W_2^{4,2}(Q)} \leqslant C_2\|f_2\|_{L_2(Q)} \tag{9}$$

with the constant C_2 defined only by the domain Ω, the functions $\mu(t)$, $N(x)$, $u_0(x)$ and $u_1(x)$ and numbers a, T,ε. According to the theorem on the method of continuation in a parameter [25, ch. III, Sec. 14], solvability of the boundary value problem $(1'_{\varepsilon,w,0})$, $(2')$, (3), (7) in $W_2^{4,2}(Q)$ and estimate (9) imply that the problem $(1'_{\varepsilon,w})$, $(2')$, (3), (7) has a solution $v(x,t)$ lying in the space $W_2^{4,2}(Q)$.

Held arguments signify that the boundary value problem $(1'_{\varepsilon,w})$, $(2')$, (3), (7) generates the operator A, taking the space $W_2^{4,2}(Q)$ to itself: $A(w) = v$. We show that for the operator A, all the conditions of Schauder's fixed point theorem are satisfied.

Observe first of all that from the estimate (9) it follows that the operator A takes a closed ball of radius $R_0 = C_2\|f_2\|_{L_2(Q)}$ of space $W_2^{4,2}(Q)$ to itself.

We now show that the operator A will be continuous on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

Let $\{w_m(x,t)\}_{m=1}^\infty$ be a sequence of functions from this ball converging in the space $W_2^{4,2}(Q)$ to the function $\overline{w}(x,t)$. Let $v_m(x,t)$, $\overline{v}(x,t)$ be images of functions $w_m(x,t)$ and $\overline{w}(x,t)$ under action of the operator A. There are equalities

$$v_{mtt} - \overline{v}_{tt} + a^2\Delta(v_m - \overline{v}) - \varepsilon\Delta^2(v_m - \overline{v}) - [\varphi(t)G_\gamma(\Phi(t; w_m + v_0)) + \psi(t)](v_m - \overline{v}) =$$

$$= \varphi(t)[G_\gamma(\Phi(t; w_m + v_0)) - G_\gamma(\Phi(t; \overline{w} + v_0))]\overline{v} + \varphi(t)v_0(x,t)\Phi(t; v_m - \overline{v}), \quad (x,t) \in Q,$$

$$v_m(x,0) - \overline{v}(x,0) = v_m(x,T) - \overline{v}(x,T) = 0, \quad x \in \Omega,$$

$$v_m(x,t) - \overline{v}(x,t)|_S = \Delta(v_m(x,t) - \overline{v}(x,t))|_S = 0.$$

These equalities mean that the functions $v_m(x,t) - \overline{v}(x,t)$ are solutions to the first boundary value problem for the linear quasi-elliptic "loaded" equation $(1'_{\varepsilon,w})$. Note that the function $G_\gamma(\xi)$ satisfies the Lipschitz condition and $\overline{v}(x,t) \in W_2^{4,2}(Q)$. Repeating the proof of the estimate (9) and applying the Holder's inequality, we get inequality

$$\|v_m - \overline{v}\|_{W_2^{4,2}(Q)} \leqslant C_3 \|w_m - \overline{w}\|_{L_2(Q)} \tag{10}$$

with constant C_3, defined by the functions $\mu(t)$, $N(x)$, $u_0(x)$ and $u_1(x)$, as well as the numbers a, T, and ε. From this inequality and from the convergence of the sequence $\{w_m(x,t)\}_{m=1}^\infty$ in space $W_2^{4,2}(Q)$ to the function $\overline{w}(x,t)$ it follows that the sequence $\{v_m(x,t)\}_{m=1}^\infty$ converges in the same space to the function $\overline{v}(x,t)$. This means that the operator A is continuous on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

We show that the operator A is compact on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

Let $\{w_m(x,t)\}_{m=1}^\infty$ be a family of functions from this ball. Let $\{v_m(x,t)\}_{m=1}^\infty$ be a family of images of functions $w_m(x,t)$ under the action of the operator A. Boundedness of families $\{w_m(x,t)\}_{m=1}^\infty$ in the space of $W_2^{4,2}(Q)$ and the classical embedding theorems [20–22] imply that there is a subsequence $\{w_{m_k}(x,t)\}_{k=1}^\infty$, strongly convergent in the space $L_2(Q)$. Repeating for the difference $v_{m_k}(x,t) - v_{m_{k+l}}(x,t)$ proof of the estimate (10), it is easy to obtain, that the sequence $\{v_{m_k}(x,t)\}_{k=1}^\infty$ is the fundamental in the space $W_2^{4,2}(Q)$. And this means that the operator A is compact on the ball of radius R_0 of the space $W_2^{4,2}(Q)$.

So, the operator A takes a ball of radius R_0 of the space $W_2^{4,2}(Q)$ to itself. The operator A is continuous and compact on this ball. According to Schauder's theorem, in the indicated ball there is at least one function $v(x,t)$, for which holds $A(v) = v$. This function $v(x,t) \in W_2^{4,2}(Q)$ is solution of the boundary value problem $(1'_\varepsilon)$, $(2')$, (3), (7). Show that the solutions $v(x,t)$ satisfy a priori estimates uniform in ε.

Consider the equality

$$-\int_Q \{v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; v + v_0))]v - \varepsilon\Delta^2 v\}\Delta^2 v \, dx\, dt =$$

$$= -\int_Q f_2 \Delta^2 v \, dx\, dt - \int_Q \varphi(t)v_0(x,t)\Phi(t; v)\Delta^2 v \, dx\, dt.$$

Integrating by parts and applying the Cauchy–Bunyakovsky and Young inequalities, we conclude that this equality implies the estimate

$$(1 - N_2)\int_Q (\Delta v_t)^2 \, dx\, dt + \frac{a^2}{2}\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx\, dt \leqslant N_3^{1/2}\left(\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx\, dt\right)^{1/2}.$$

It is easy to show that there are estimates

$$\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx\, dt \leqslant \frac{4N_3}{a^4}, \quad \int_Q (\Delta v_t)^2 \, dx\, dt \leqslant \frac{2N_3}{a^2(1 - N_2)} = N_4.$$

Summing up the inequalities and also using the second main inequality for elliptic operators, we obtain that solutions solutions$v(x,t)$ to the boundary value problem $(1'_\varepsilon)$, $(2')$, (3), (7) satisfy the estimates

$$|\Phi(t; v+v_0)| \leqslant N_5, \tag{11}$$

$$\|v\|_{W_2^2(Q)} + \|\Delta v\|_{W_2^1(Q)} + \sqrt{\varepsilon}\|\Delta^2 v\|_{L_2(Q)} \leqslant C_4, \tag{12}$$

with the constant C_4 в (12) defined by the functions $\mu(t)$, $N(x)$, $u_0(x)$ и $u_1(x)$, and the numbers a и T.

The estimate (12) and the reflexivity of a Hilbert space imply that there exist sequences $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers and $\{v_m(x,t)\}_{m=1}^\infty$ of solutions to the boundary value problem $(1'_{\varepsilon_m})$, $(2')$, (3), (7) and also a function $v(x,t)$ such that, as $m \to \infty$, the convergences

$$\varepsilon_m \to 0, \qquad v_m(x,t) \to v(x,t) \quad \text{strongly in} \quad W_2^2(Q),$$

$$\varepsilon_m \Delta^2 v_m(x,t) \to 0 \quad \text{weakly in} \quad L_2(Q)$$

hold.

Obviously, the limit function $v(x,t)$ will be a solution to the boundary value problem $(1'_0)$, $(2')$, (3), and due to estimate (12) for this solution will be the inclusions $v(x,t) \in W_2^2(Q)$, $\Delta v(x,t) \in W_2^1(Q)$.

Let us fix the number γ: $\gamma = \dfrac{\psi_0}{\varphi_0}$. Let us define the functions $u(x,t)$ and $q(t)$:

$$u(x,t) = v(x,t) + v_0(x,t), \quad q(t) = \psi(t) + \varphi(t)\Phi(t;u).$$

Estimate (11) and the inequality from the condition of the theorem for the number N_5 mean that the equality $G_\gamma(\Phi(t;u)) = \Phi(t;u)$ holds, and that $q(t) \geqslant 0$ $\forall t \in [0,T]$. Obviously, the functions $u(x,t)$ and $q(t)$ will be related in the cylinder Q by equation (1). Let's show that for the function $u(x,t)$ the overdetermination condition (4) will be satisfied.

We multiply equation (1) by the function $N(x)$ and integrate over the domain Ω. Taking into account the form of the functions $\varphi(t)$, $\psi(t)$, $\Phi(t;u)$ and consistency conditions for of the functions $u_0(x)$, we obtain that the function $\alpha(t)$ satisfies the problem

$$\alpha''(t) - q(t)\alpha(t) = 0, \qquad \alpha(0) = \alpha(T) = 0. \tag{13}$$

Since $q(t) \geqslant 0$, then $\alpha(t) \equiv 0$. This means that the function $u(x,t)$ satisfies the overdetermination condition (4). The theorem is proved. □

The study of the solvability of the inverse problem II differs only in insignificant details from the above study of the solvability of the inverse problem I.

Let

$$N_6 = \sqrt{2}\varphi_0 \left(\int_\Omega \left[u_0^2(x) + u_1^2(x)\right] dx \right)^{1/2} \|N\|_{L_2(\Omega)},$$

$$N_7 = \frac{aT^{1/2}}{(1-N_6)^{1/2}} \left(\sum_{i=1}^n \int_\Omega N_{x_i}^2\, dx \right)^{1/2} \|f_2\|_{L_2(Q)} + |\Phi(0,u_0)| + |\Phi(0,u_1)|.$$

Theorem 2.2. *Suppose the fulfillment of conditions*

$$N(x) \in W_2^1(\Omega), \quad \mu(t) \in C^2([0,T]); \quad f(x,t) \in L_2(0,T; W_2^1(\Omega)),$$

$$u_0(x) \in W_2^3(\Omega), \quad u_1(x) \in W_2^3(\Omega); \quad \varphi_0 > 0, \quad \psi_0 > 0, \quad N_6 < 1, \quad N_7 \leqslant \frac{\psi_0}{\varphi_0},$$

$$\mu(0) = \int_\Omega N(x) u_0(x)\, dx, \quad \mu(T) = \int_\Omega N(x) u_1(x)\, dx.$$

Then inverse problem II has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^2(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geqslant 0$ *for* $t \in [0,T]$.

3. Solvability of the inverse Problems III

We introduce the function $\Phi_1(t;w)$: $\Phi_1(t;w) = a^2 \int\limits_\Gamma N(x)\Delta w(x,t)\, ds_x$, where $w(x,t)$ is some given function.

Introduce the notations $F_0(t) = \int\limits_\Gamma N(x) f(x,t)\, ds_x$, $\psi_1(t) = \varphi(t)[\mu''(t) - F_0(t)]$,

$$\widetilde{f}_2(x,t) = f_1(x,t) + [\psi_1(t) + \varphi(t)\Phi_1(t;v_0)]v_0(x,t).$$

Consider the boundary value problem: *Find a function* $v(x,t)$ *that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\varphi(t)\Phi_1(t; v + v_0) + \psi(t)]v = \widetilde{f}_2(x,t) + \varphi(t) v_0(x,t)\Phi_1(t;v) \tag{14}$$

in the cylinder Q *and satisfies conditions* $(2')$ *and* (5). A solution $v(x,t)$ to this problem will provide an opportunity construct the required solution to the inverse problem III.

The function $w(x) \in W_2^1(\Omega)$ satisfies the inequality

$$\int_\Gamma w^2(x)\, ds_x \leqslant c_0 \int_\Omega \left[w^2(x) + \sum_{i=1}^n w_{x_i}^2(x) \right] dx \tag{15}$$

with a constant c_0 determined only by the domain Ω (see [20, 21]).

Let us specify again that the function $v_0(x,t)$ satisfies the inequality

$$\sum_{i=1}^n \int_\Omega v_{0x_i}^2(x,t)\, dx \leqslant 2 \sum_{i=1}^n \int_\Omega \left[u_{0x_i}^2(x) + u_{1x_i}^2(x) \right] dx. \tag{16}$$

As before, we define the required constants:

$$\psi_1 = \min_{[0,T]} \psi_1(t), \qquad N_8 = \left(\sum_{i=1}^n \int_Q \widetilde{f}_{2x_i}^2\, dx\, dt \right)^{1/2}, \quad N_9 = \frac{c_0 \varphi_0^2 N_1 T^2}{4a^2} \|N\|^2_{L_2(\Gamma)},$$

$$N_{10} = \frac{c_0 \varphi_0^2 N_1}{4a^2}, \quad N_{11} = \frac{2N_8}{a^2 - 2N_9}, \quad N_{12} = \frac{N_8 N_{11}}{1 - N_{10}},$$

$$N_{13} = \sqrt{2}\varphi_0 \int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2 \right] dx \left(N_9 N_{11}^2 + N_{10} N_{12} \right)^{1/2} + \left(\int_Q \left(\Delta \widetilde{f}_2 \right)^2 dx\, dt \right)^{1/2},$$

$$N_{14} = a^2 \|N\|^2_{L_2(\Gamma)} (c_0 T)^{1/2} \left[N_{12} + \frac{N_{13}^2}{a^2} \right]^{1/2} + |\Phi_1(0;u_0)| + |\Phi_2(0;u_1)|.$$

Theorem 3.1. *Suppose the fulfillment of conditions* $N(x) \in L_2(\Gamma)$, $\mu(t) \in C^2([0,T])$;

$$f(x,t) \in L_2(0,T;W_2^2(\Omega)), \quad u_0(x) \in W_2^3(\Omega), \quad u_1(x) \in W_2^3(\Omega);$$

$$\frac{\partial f(x,t)}{\partial \nu} = \frac{\partial \Delta u_0(x)}{\partial \nu} = \frac{\partial \Delta u_1(x)}{\partial \nu} = 0 \quad \textit{for} \quad x \in \Gamma;$$

$$\varphi_0 > 0, \quad \psi_1 > 0, \quad N_{10} < 1, \quad a^2 - 2N_9 > 0, \quad N_{14} \leqslant \frac{\psi_1}{\varphi_0}.$$

$$\mu(0) = \int_\Gamma N(x) u_0(x)\, dx, \quad \mu(T) = \int_\Gamma N(x) u_1(x)\, dx.$$

Then inverse problem III has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^2(Q)$, $\Delta u(x,t) \in W_2^1(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geqslant 0$ *for* $t \in [0,T]$.

Proof. Let γ be a number from the interval $\left(0, \frac{\varphi_0}{\psi_1}\right]$, $\varepsilon > 0$.

Consider the boundary value problem: *Find a function* $v(x,t)$ *that is a solution to equation*

$$v_{tt} + a^2 \Delta v - [\psi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v + v_0))] v - \varepsilon \Delta^2 v = \widetilde{f}_2(x,t) + \varphi(t) v_0(x,t) \Phi_1(t;v) \qquad (14'_\varepsilon)$$

in the cylinder Q *and satisfies conditions* $(2')$, (5), *and*

$$\left.\frac{\partial(\Delta v)}{\partial \nu}\right|_S = \left.\frac{\partial(\Delta^2 v)}{\partial \nu}\right|_S = 0. \qquad (17)$$

Using the fixed point method and the method of continuation in a parameter, it is easy to show that for a fixed ε and for satisfying the conditions of the theorem, this problem has a solution $v(x,t)$ such that $v(x,t) \in W_2^2(Q)$, $\Delta^2 v(x,t) \in L_2(Q)$. Let us show that the function $v(x,t)$ satisfies a priori estimates uniform in ε.

Multiply equation $(14'_\varepsilon)$ by the function $\Delta^2 v(x,t)$ and integrate over the cylinder Q. We obtain the equality

$$\int_Q (\Delta v_t)^2\, dx\, dt + a^2 \sum_{i=1}^n \int_Q (\Delta v_{x_i})^2\, dx\, dt + \varepsilon \int_Q \left(\Delta^2 v\right)^2\, dx\, dt +$$

$$+ \int_Q [\psi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v + v_0))] (\Delta v)^2\, dx\, dt = \sum_{i=1}^n \int_Q \widetilde{f}_{2x_i} \Delta v_{x_i}\, dx\, dt -$$

$$- \sum_{i=1}^n \int_Q \varphi(t) \Phi_1(t;v) v_{0x_i} \Delta v_{x_i}\, dx\, dt. \qquad (18)$$

Let us introduce the notation: $I_1 = \int\limits_Q (\Delta v_t)^2\, dx\, dt$, $\quad I_2 = \sum\limits_{i=1}^n \int\limits_Q (\Delta v_{x_i})^2\, dx\, dt$.

Taking into account the notation introduced above and using the Young's and Cauchy - Bunyakovsky's inequalities it is easy from the equality (18) go to inequality

$$I_1 + \frac{a^2}{2} I_2 \leqslant N_8 I_2^{1/2} + \frac{\varphi_0^2 N_1}{2a^2} \int_0^T \Phi_1^2(t;v)\, dt. \qquad (19)$$

There is an estimate

$$\int_0^T \Phi_1^2(t;v)\, dt \leqslant N_9 I_2 + N_{10} I_1. \qquad (20)$$

Summing up, we obtain a consequence of inequalities (19) и (20):

$$I_1 + \frac{a^2}{2} I_2 \leqslant N_8 I_2^{1/2} + N_9 I_2 + N_{10} I_1. \tag{21}$$

Elementary calculations allow us to derive from (21) the estimate

$$I_2 \leqslant N_{11}^2, \tag{22}$$

and further, the estimate

$$I_1 \leqslant N_{12}. \tag{23}$$

Equality (18) and estimates (22), (23) imply the boundedness of the first term on the left side of$(14'_\varepsilon)$:

$$\varepsilon \int_Q v_{tt}^2 \, dx \, dt \leqslant C_5. \tag{24}$$

Here the constant C_5 is determined by the functions $f(x,t)$, $u_0(x)$, $u_1(x)$, $N(x)$, $\mu(t)$, numbers a and T (the exact value of the number C_5 is not important).

Multiply equation $(14'_\varepsilon)$ by the function $\Delta^2 v(x,t)$ and integrate over the cylinder Q. We obtain the equality

$$\sum_{i=1}^n \int_Q (\Delta v_{x_i t})^2 \, dx \, dt + a^2 \int_Q \left(\Delta^2 v\right)^2 \, dx \, dt +$$

$$+ \sum_{i=1}^n \int_Q [\phi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v + v_0))] (\Delta v_{x_i})^2 \, dx \, dt + \varepsilon \int_Q \left(\Delta^3 v\right)^2 \, dx \, dt =$$

$$= \int_Q \Delta \widetilde{f}_2 \Delta^2 v \, dx \, dt + \int_Q \varphi(t) \Phi_1(t; v) \Delta v_0 \Delta^2 v \, dx \, dt. \tag{25}$$

An inequality similar to the inequality (18) holds:

$$\int_\Omega [\Delta v_0(x,t)]^2 \, dx \leqslant 2 \int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2\right] \, dx.$$

Using this inequality, Hölder's inequality and estimates (20), (22), (23), we obtain from (25) the inequality:

$$a^2 \int_Q \left(\Delta^2 v\right)^2 \, dx \, dt \leqslant \left(\int_Q (\Delta^2 v)^2 \, dx \, dt \right)^{1/2} \left[\left(\int_Q \left(\Delta \widetilde{f}_2 \right)^2 \, dx \, dt \right)^{1/2} + \right.$$

$$\left. + \sqrt{2} \varphi_0 \left(\int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2\right] \, dx \right)^{1/2} \left(N_9 N_{11}^2 + N_{10} N_{12}\right)^{1/2} \right].$$

This inequality and again from equality (25) imply the estimates

$$\int_Q \left(\Delta^2 v\right)^2 \, dx \, dt \leqslant \frac{N_{13}^2}{a^4}, \tag{26}$$

$$\varepsilon \int_Q \left(\Delta^3 v\right)^2 \, dx \, dt + \sum_{i=1}^n \int_Q (\Delta v_{x_i t})^2 \, dx \, dt \leqslant \frac{N_{13}^2}{a^2}. \tag{27}$$

Estimates (22)–(24), (27), estimates for solutions of elliptic equations (see [21]) and also the reflexivity of a Hilbert space imply that there exist sequences $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers and

$\{v_m(x,t)\}_{m=1}^{\infty}$ to the boundary value problems $(14'_{\varepsilon_m})$, $(2')$, (5), (17) and also a function $v(x,t)$ such that, as $m\to\infty$, the convergences

$$\varepsilon_m \to 0, \qquad v_m(x,t)\to v(x,t) \quad \text{weakly in} \quad W_2^2(Q),$$
$$\Delta v_m(x,t)\to \Delta v(x,t) \quad \text{weakly in} \quad W_2^1(Q),$$
$$\Delta v_m(x,t)\to \Delta v(x,t) \quad \text{strongly in} \quad L_2(\Gamma),$$
$$\varepsilon_m \Delta^3 v_m(x,t)\to 0 \quad \text{weakly in} \quad L_2(Q)$$

hold. The limit function $v(x,t)$ satisfies the equation

$$v_{tt}+a^2\Delta v-[\psi_1(t)+\varphi(t)G_\gamma(\Phi_1(t;v+v_0))]v=\widetilde{f}_2(x,t)+\varphi(t)v_0(x,t)\Phi_1(t;v),$$

and the conditions $(2')$, (5). The function $v(x,t)$ belongs to $W_2^2(Q)$ and $\Delta v(x,t)\in W_2^1(Q)$, $\Delta^2 v(x,t)\in L_2(Q)$, $\Delta v_{x_it}(x,t)\in L_2(Q)$, $i=1,\dots,n$. The following inequalities

$$|\Phi_1(t;v+v_0)|\leqslant|\Phi_1(t;v)|+|\Phi_1(t;v_0)|\leqslant a^2\|N\|_{L_2(\Gamma)}\left(\int_\Gamma(\Delta v)^2\,ds\right)^{1/2}+|\Phi(0;u_0)|+|\Phi(0;u_1)|\leqslant$$
$$\leqslant a^2c_0^{1/2}\|N\|_{L_2(\Gamma)}\left[\int_\Omega(\Delta v)^2\,dx+\sum_{i=1}^n\int_\Omega(\Delta v_{x_i})^2\,dx\right]^{1/2}+|\Phi(0;u_0)|+|\Phi(0;u_1)|\leqslant$$
$$\leqslant a^2(c_0T)^{1/2}\|N\|_{L_2(\Gamma)}\left[\int_Q(\Delta v_t)^2\,dx\,dt+\sum_{i=1}^n\int_Q(\Delta v_{x_it})^2\,dx\right]^{1/2}+|\Phi(0;u_0)|+$$
$$+|\Phi(0;u_1)|\leqslant a^2(c_0T)^{1/2}\|N\|_{L_2(\Gamma)}\left[\frac{N_{13}^2}{a^2}+N_{12}\right]^{1/2}+|\Phi(0;u_0)|+|\Phi(0;u_1)|=N_{14} \tag{28}$$

hold.

Let $\gamma=\dfrac{\psi_1}{\varphi_0}$. Due to the condition $N_{14}\leqslant\dfrac{\psi_1}{\varphi_0}$ it follows from (28) that $G_\gamma(\Phi_1(t;v+v_0))=$ $=\Phi_1(t;v+v_0)$. Let us define the functions $u(x,t)$ и $q(t)$:

$$u(x,t)=v(x,t)+v_0(x,t),\quad q(t)=\psi_1(t)+\varphi(t)\Phi_1(t;u).$$

It is these functions that give the required solution to the inverse problem III (which is shown as in the proof of Theorem 2.1). The theorem is proved. □

4. Uniqueness of solutions

The following theorems give conditions under which the inverse problems I – III can only have one solution.

Let $W_{R_0}=\left\{v(x,t):\,v(x,t)\in W_2^2(Q),\quad \operatorname*{vrai\,max}_{[0,T]}\left(\int\limits_\Omega v^2(x,t)\,dx\right)\leqslant R_0\right\}$.

Theorem 4.1. *Let $\{u_1(x,t),q_1(t)\}$, $\{u_2(x,t),q_2(t)\}$ be two solutions of the inverse problem I such that $u_i(x,t)\in W_{R_0}$, $q_i(t)\in L_\infty([0,T])$, $q_i(t)\geqslant 0$ for $t\in[0,T]$, $i=1,2$. Suppose the fulfillment of the conditions*

$$N(x)\in L_2(\Omega),\quad \mu(t)\in C^2([0,T]),\quad f(x,t)\in L_\infty(0,T;L_2(\Omega));\quad \varphi_0>0,\quad \varphi_0R_0^{1/2}\|N\|_{L_2(\Omega)}<1.$$

Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.

Proof. The function $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the following problem

$$w_{tt} + a^2\Delta w - q_1(t)w = \varphi(t)\Phi(t;w)u_2, \quad (x,t) \in Q;$$

$$w(x,0) = w(x,T) = 0, \quad x \in \Omega;$$

$$w(x,t)|_S = 0.$$

We multiply the equation by the function $\Delta w(x,t)$ and integrate over the cylinder Q. Taking into account the nonnegativity of the function $q_1(t)$ and the boundary conditions, applying Hölder's inequality, we obtain the inequality $\int\limits_Q (\Delta w)^2\,dx\,dt \leqslant 0$. This inequality implies that the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q. But then the functions $q_1(t)$ и $q_2(t)$ coincide for almost of all t from the segment $[0,T]$. The theorem is proved. □

Theorem 4.2. *Let $\{u_1(x,t), q_1(t)\}$, $\{u_2(x,t), q_2(t)\}$ be two solutions of the inverse problem II such that $u_i(x,t) \in W_{R_0}$, $q_i(t) \in L_\infty([0,T])$, $q_i(t) \geqslant 0$ for $t \in [0,T]$, $i = 1,2$. Suppose the assumptions of Theorem 2.2 are fulfilled. Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.*

The proof of this theorem is quite similar to the proof of Theorem 4.1.

Let

$$\widetilde{W}_{R_0} = \left\{ v(x,t) :\, v(x,t) \in W_2^2(Q), \Delta v(x,t) \in W_2^1(Q), \operatorname*{vrai\,max}_{[0,T]} \left(\sum_{i=1}^n \int_\Omega v_{x_i}^2(x,t)\,dx \right) \leqslant R_0 \right\}.$$

Theorem 4.3. *Let $\{u_1(x,t), q_1(t)\}$, $\{u_2(x,t), q_2(t)\}$ be two solutions of the inverse problem III such that $u_i(x,t) \in \widetilde{W}_{R_0}$, $q_i(t) \in L_\infty([0,T])$, $q_i(t) \geqslant 0$ for $t \in [0,T]$, $i = 1,2$. Suppose the fulfillment of the conditions*

$$N(x) \in L_2(\Gamma), \quad \mu(t) \in C^2([0,T]), \quad f(x,t) \in L_2(Q) \cap L_\infty(0,T; L_2(\Gamma));$$

$$\varphi_0 > 0, \quad \varphi_0 (c_0 R_0)^{1/2} \|N\|_{L_2(\Gamma)} < \min\left(\frac{2}{3}, \frac{4}{a^2T^2}\right).$$

Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.

Proof. The function $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the following problem

$$w_{tt} + a^2\Delta w - q_1(t)w = \varphi(t)\Phi_1(t;w)u_2, \quad (x,t) \in Q; \tag{29}$$

$$w(x,0) = w(x,T) = 0, \quad x \in \Omega; \tag{30}$$

$$\left.\frac{\partial w(x,t)}{\partial \nu}\right|_S = 0. \tag{31}$$

Equalities (29) and (31) imply, in particular, the property

$$\left.\frac{\partial \Delta w(x,t)}{\partial \nu}\right|_S = 0. \tag{32}$$

Further, using the procedure for approximating the function $w(x,t)$ by smooth functions while maintaining the property (32), it is easy to show that the equality holds (formally obtained by multiplying equation (29) by the function $\Delta^2 w$ and integrating over Q)

$$\int_Q (\Delta w_t)^2\,dx\,dt + a^2\sum_{i=1}^n \int_Q (\Delta w_{x_i})^2\,dx\,dt + \int_Q q(t)(\Delta w)^2\,dx\,dt =$$

$$= \int_Q \varphi(t)\Phi_1(t;w)\left(\sum_{i=1}^n u_{2x_i}\Delta w_{x_i}\right)dx\,dt. \qquad (33)$$

We obtain an estimate for the right-hand side of the inequality (33). Using the Cauchy-Bunyakovsky's and Hölder's inequalities, the condition (32) and estimate

$$\int_Q (\Delta w)^2\,dx\,dt \leqslant \frac{T^2}{2}\int_Q (\Delta w_t)^2\,dx\,dt,$$

we obtain the inequality

$$\int_Q (\Delta w_t)^2\,dx\,dt + a^2\sum_{i=1}^n \int_Q (\Delta w_{x_i})^2\,dx\,dt \leqslant$$

$$\leqslant \frac{1}{4}a^2T^2\varphi_0(c_0R_0)^{1/2}\|N\|_{L_2(\Gamma)}\int_Q (\Delta w_t)^2\,dx\,dt + \frac{3}{2}a^2\varphi_0(c_0R_0)^{1/2}\|N\|_{L_2(\Gamma)}\sum_{i=1}^n\int_Q (\Delta w_{x_i})^2\,dx\,dt.$$

This inequality and the conditions of the theorem imply the identities $\Delta w_t(x,t) \equiv 0$, $\Delta w_{x_i} \equiv 0$ for $(x,t) \in Q$, $i = 1,\dots,n$, and further follows the identity $w(x,t) \equiv 0$ for $(x,t) \in Q$. The last identity means that he functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q. But then the functions $q_1(t)$ и $q_2(t)$ coincide for almost of all t from the segment $[0,T]$. The theorem is proved. □

5. Comments and appendices

1. Let us show that the set of input data of inverse problems I–III, for which all conditions of the existence and uniqueness theorems are satisfied, is not empty.

Let $u_0(x)$ and $u_1(x)$ be given nonnegative functions in Ω such that, in addition to the conditions of Theorem 2.1, they satisfy the conditions

$$\frac{\partial u_0(x)}{\partial\nu} = \frac{\partial u_1(x)}{\partial\nu} = 0 \quad \textit{for} \quad x \in \Gamma, \qquad \int_\Omega u_0(x)\,dx = \int_\Omega u_1(x)\,dx = 1.$$

Similar functions exist. For example, $u_0(x) = \alpha_0[\rho(x)]^{m_0}$, $u_1(x) = \alpha_1[\rho(x)]^{m_1}$, where $\rho(x)$ is the distance from the point $x \in \Omega$ to the boundary Γ, $m_0 \geqslant 3$, $m_1 \geqslant 3$. The multipliers α_0 and α_1 are selected so that the required integral equalities hold. Or $u_0(x)$ and $u_1(x)$ can be finite in $\overline{\Omega}$.

Let $N(x) \equiv 1$, $\mu(t) \equiv 1$, $f(x,t) = \widetilde{f}_0(x)$, $\widetilde{f}_0(x) < 0$, $x \in \overline{\Omega}$. Then

$$\psi(t) = -\int_\Omega \widetilde{f}_0(x)\,dx = \psi_0 > 0, \quad \Phi(0,u_0) = \Phi(0,u_1) = 0.$$

Obviously, the condition $N_2 < 1$ of Theorem 1 will hold for small numbers T, the number N_5 can also be made arbitrarily small by decreasing the number T. Hence, for the given functions $f(x,t)$, $u_0(x)$, $u_1(x)$, $\mu(t)$ and $N(x)$ for small T all conditions of the Theorem 1 will be satisfied.

Condition $N_6 < 1$ of Theorem 2.2 will be be executed if the functions $u_0(x)$ and $u_1(x)$ or the measure of the region Ω are small, the condition for the number N_7 will be run automatically.

The non-emptiness of the set of input data for which all conditions of Theorem 2.2 are satisfied is also easy to show. Take as $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ are identically constant functions, $f(x,t)$ is negative function in $\overline{Q}$. Condition $N_6 < 1$ of Theorem 2.2 will be be executed if the functions $u_0(x)$ and $u_1(x)$ or the measure of the domain Ω are small, the condition for the number N_7 будет will be run automatically.

Conditions of Theorem 3 are satisfied for small numbers T, if the functions $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ are identically constant functions, $f(x,t) > 0, (t,x) \in \overline{Q}$ and overdetermination conditions hold.

Obviously, the conditions of the uniqueness theorems (Theorems 4.1–4.3) will obviously be satisfied for small numbers R_0.

2. Inverse problems I − III can also be studied for equations that are more general than (1). Thus, the Laplace operator can be replaced an arbitrary second-order elliptic operator with variable coefficients, into the equation (1) low-order terms with first-order derivatives can be added. The essence of the results obtained is a more general form of the equation (1) will not change, but the number of calculations will increase.

3. If the conditions of existence theorems are satisfied, then for solutions $u(x,t)$ of inverse problems I, II, or III it is easy to establish estimates for quantities defining the sets W_{R_0} or $\widetilde{W}_{R_0}$. The constants in these estimates will be determined by the input data. Using further conditions of the respective theorems of uniqueness, it will be easy to obtain theorems that give both the existence and the uniqueness of solutions to inverse problems I, II, or III.

4. Conditions (3) or (5) in inverse problems I, II, or III can be inhomogeneous. Assuming that there are continuations of the given boundary data into the cylinder Q and using the technique of proving Theorems 2.1–2.2, Theorem 3.1, it will be possible to obtain the solvability of the inverse problems with nonzero boundary data.

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Обратные задачи восстановления младшего коэффициента в эллиптическом уравнении

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Abstract. Изучается разрешимость обратных задач восстановления неотрицательного коэффициента $q(t)$ в эллиптическом уравнении

$$u_{tt} + a^2\Delta u - q(t)u = f(x,t)$$

($x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $t \in (0,T)$, $0 < T < +\infty$, Δ — оператор Лапласа, действующий по переменным x_1, ..., x_n). Вместе с естественными для эллиптических уравнений граничными условиями в изучаемых задачах задают также одно из дополнительных условий — либо условие пространственного интегрального переопределения, либо же условие граничного интегрального переопределения. Доказываются теоремы существования и единственности решений.

Ключевые слова: эллиптические уравнения, неизвестный коэффициент, пространтсвенное интегральное переопределение, граничное интегральное переопределение, существование, единственность.