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Boundary Value Problems for Fourth-Order Sobolev Type Equations

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Abstract. The goal of the article is the study of solvability in the Sobolev spaces of boundary value problems for some classes of Sobolev-type fourth-order linear equations. We will prove that an initial boundary value problems well problems with data both at the initial time moment and the final time moments can be well-posed for the equations under study.

Keywords: Sobolev-type fourth-order differential equation, boundary value problem, existence, uniqueness.

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Introduction

The article is devoted to the study of the solvability of boundary value problems for differential equations

$$D_t^4 u + \sum_{k=0}^{3} A_k D_t^k u = f(x,t) \quad \left(D_t^k = \frac{\partial^k}{\partial t^k}, \quad k = \overline{0,4} \right) \tag{$*$}$$

with operators A_k of the form

$$A_k = \frac{\partial}{\partial x_i}\left(a^{ij,k}(x) \frac{\partial}{\partial x_j} \right) + a_{0,k}(x)$$

(here and below, summation over repeated indices from 1 to n is carried out).

The differential equations $(*)$ are recently attributed to the class of Sobolev-type equations. Various aspects of the theory of Sobolev-type equations are reflected in monographs [1–7] and also in numerous journal articles (it is impossible to mention even a small part of such articles just because they are numerous).

For Sobolev-type differential equations, best studied is the solvability of the Cauchy problem and initial boundary value problems. At the same time, as is shown in [3, 8], in some case, for Sobolev-type equations, simultaneously with initial boundary value problems, other problems can also be well-posed; these include problems with data both at the initial and final time moments. In the present article, for equations $(*)$, we study the solvability both of initial boundary value problems and problems with data at different time moments.

Clarify that the goal of the present article is to prove the solvability of some problem for equations $(*)$ in the classes of regular solutions, i.e., solutions having all weak derivatives in the sense of Sobolev [9–11] occurring in the equation.

Formally, equation $(*)$ with the above operators is a fifth-order equation. The use of the term "fourth-order Sobolev equation" in the title and the article means that the equations under study

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are fourth-order equations with respect to the time (distinguished) variable, which is the leading variable and defines the statements of the problems.

One more remark: Equations ($*$) have model and the simplest form. We will speak of some more general equations and of generalizations of the results at the end of the article.

1. Statements of the Problems

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with smooth (for simplicity, infinitely differentiable) boundary Γ, Q is the cylinder $\Omega\times(0,T)$ of finite height T, and $S=\Gamma\times(0,T)$ is the lateral boundary of Q. Furthermore, let $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j=1,\dots,n$, $k=0,\dots,3$, $f(x,t)$ be given functions defined for $x\in\overline{\Omega}$ and $t\in[0,T]$ and let A_k and L be the differential operators whose action at a given function $v(x,t)$ is defined by the equalities

$$A_k v=\frac{\partial}{\partial x_i}\left(a^{ij,k}(x)v_{x_j}\right)+a_{0,k}(x)v,$$

$$Lv=D_t^k v+\sum_{k=0}^{3}A_k D_t^k v.$$

Boundary Value Problem I: *Find a function $u(x,t)$ that is a solution to the equation*

$$Lu=f(x,t) \tag{1}$$

in the cylinder Q such that

$$u(x,t)|_S=0, \tag{2}$$

$$D_t^k u(x,t)\big|_{t=0,\,x\in\Omega}=0,\quad k=0,1,2,3. \tag{3}$$

Boundary Value Problem II: *Find a function $u(x,t)$ that is a solution to equation (1) in Q and satisfies conditions (2) and also the condition*

$$D_t^k u(x,t)\big|_{t=0,\,x\in\Omega}=0,\quad k=0,1,2,\quad D_t^3 u(x,t)\big|_{t=T,\,x\in\Omega}=0. \tag{4}$$

Boundary Value Problem III: *Find a function $u(x,t)$ that is a solution to equation (1) in Q that satisfies conditions (2) and also the condition*

$$u(x,t)|_{t=0,\,x\in\Omega}=D_t^2 u(x,t)\big|_{t=0,\,x\in\Omega}=D_t u(x,t)|_{t=0,\,x\in\Omega}=D_t^3 u(x,t)\big|_{t=0,\,x\in\Omega}=0. \tag{5}$$

Boundary Value Problem I is a usual initial boundary value problem for nonstationary differential equations of the fourth order (with respect to time). Boundary Value Problem II is a modified V. N. Vragov's problem (see [12–14]) for fourth-order quasihyperbolic equations. Finally, Boundary Value Problem III is in fact an elliptic boundary value problem.

In the present article, we propose sufficient conditions on the coefficients of (1) new compared to the previous works that guarantee the existence and uniqueness of regular solutions to boundary value problems I, II, or III.

2. Solvability of boundary value Problems I-III

Theorem 1. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x)\in C^1(\overline{\Omega}),\quad i,j=1,\dots,n,\quad a_{0,k}(x)\in C(\overline{\Omega}),\quad k=0,1,2; \tag{6}$$

$$a^{ij,3}(x)\in C^2(\overline{\Omega}),\quad a^{ij,3}(x)=a^{ji,3}(x),\quad i,j=1,\dots,n,\quad a_{0,3}(x)\in C(\overline{\Omega}), \tag{7}$$

$$-a^{ij,3}(x)\xi_i\xi_j\geqslant m_0|\xi|^2,\quad m_0>0,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n. \tag{8}$$

Then, for every function $f(x,t)$ in $L_2(Q)$, Boundary Value Problem I has a solution $u(x,t)$ such that $D_t^k u(x,t)\in L_2(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $k=0,1,2,3$, $D_t^4 u(x,t)\in L_2(Q)$.

Proof. Make use of the method of continuation in a parameter. Let $\lambda \in [0,1]$. Consider the following problem: *Find a function $u(x,t)$ that is a solution to the equation*

$$D_t^4 u + A_3 D_t^3 u + \lambda \sum_{k=0}^{2} A_k D_t^k u = f(x,t) \tag{9}$$

and Q that satisfies conditions (2) and (3). Note that, for $\lambda = 0$, this problem has a solution $u(x,t)$ belonging to the desired class; this follows from the fact that, for $\lambda = 0$, equation (9) is a usual parabolic equation with respect to $u_{ttt}(x,t)$. Furthermore, by the theorem on the method of extension in a parameter (see [15, Chapter III, Sec. 14], the boundary value problem (9), (2), (3) has a regular solution $u(x,t)$ if $f(x,t) \in L_2(Q)$ and problem (9), (2), (3) is solvable in the class of regular solutions for $\lambda = 0$ if all derivatives occurring in (9) are uniformly bounded in $L_2(Q)$.

For proving the desired boundedness, let us first consider the equality

$$\int_0^t \int_\Omega \left[D_\tau^4 u + A_3 D_\tau^3 u + \lambda \sum_{k=0}^{2} A_k D_\tau^k u \right] D_\tau^3 u \, dx \, d\tau = \int_0^t \int_\Omega f D_\tau^3 u \, dx \, d\tau. \tag{10}$$

Integrating by parts, applying Young's inequality and the inequality

$$\int_\Omega w^2(x,t) \, dx \leqslant T \int_0^t \int_\Omega w_\tau^2(x,\tau) \, dx \, d\tau, \tag{11}$$

which is valid for functions $w(x,t)$ vanishing for $t = 0$, and using conditions (6)–(8) and Gronwall's lemma, it is not hard to obtain from (10) the estimate

$$\int_\Omega \left[D_t^3 u(x,t) \right]^2 \, dx + \sum_{i=1}^{n} \int_0^t \int_\Omega \left(D_\tau^3 u_{x_i} \right)^2 \, dx \, d\tau \leqslant C_1 \int_Q f^2 \, dx \, dt, \tag{12}$$

where the constant C_1 is defined only by the functions $a^{ij,k}(x)$, $i,j = 1,\dots,n$, $a_{0,k}(x)$, $k = 0,1,2,3$, and the number T.

Now, consider the equality

$$-\int_0^t \int_\Omega \left(D_\tau^4 u + A_3 D_\tau^3 u + \lambda \sum_{k=0}^{2} A_k D_\tau^k u \right) A_3 D_\tau^3 u \, dx \, d\tau = -\int_0^t \int_\Omega f A_3 D_\tau^3 u \, dx \, d\tau.$$

Integrating by parts once again, applying Young's inequality, inequality (11), estimate (12), conditions (6)–(8), and also the second main inequality for elliptic operators (see [10, Chapter III, Stc. 8], and Gronwall's lemma, we conclude that solutions $u(x,t)$ to the boundary value problem (9), (2), (3) satisfy the second a priori estimate

$$\sum_{i=1}^{n} \int_\Omega \left[D_t^3 u_{x_i}(x,t) \right]^2 \, dx + \sum_{i,j=1}^{n} \int_0^t \int_\Omega \left(D_\tau^3 u_{x_i x_j} \right)^2 \, dx \, d\tau \leqslant C_2 \int_Q f^2 \, dx \, dt, \tag{13}$$

where the constant C_2 is defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j = 1,\dots,n$, $k = 0,1,2,3$, the domain Ω, and the number T.

Estimates (12) and (13) imply the obvious third estimate

$$\int_0^t \int_\Omega \left(D_\tau^4 u \right)^2 \, dx \, d\tau \leqslant C_3 \int_Q f^2 \, dx \, dt, \tag{14}$$

of solutions $u(x,t)$ to the boundary value problem (9), (2), (3); the constant C_3 in this estimate is again defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j=1,\dots,n$, $k=0,1,2,3$, the domain Ω, and the number T.

Estimates (12)–(14) give the desired uniform boundedness over λ in $L_2(Q)$ of all derivatives occurring in (9). As we already said above, this boundedness and the solvability of the boundary value problem (9), (2), (3) for $\lambda=0$ give the solvability of this problem in the desired class also for $\lambda=1$. This exactly means the validity of the theorem.

The theorem is proved. □

Before proving the following theorem on the solvability of Problem I in the class of regular solutions, we formulate an auxiliary assertion on the nonnegativity of the scalar product of a pair of second-order differential operators.

Let A and B be differential operators whose action is defined by the equality

$$Av = \frac{\partial}{\partial x_i}\left(a^{ij}(x)v_{x_j}\right) + a_0(x)v,$$

$$Bv = \frac{\partial}{\partial x_i}\left(b^{ij}(x)v_{x_j}\right) + b_0(x)v.$$

Proposition 1. *Suppose the fulfillment of the conditions*

$$a^{ij}(x)\in C^2(\overline{\Omega}),\quad b^{ij}(x)\in C^2(\overline{\Omega}),\quad a^{ij}(x)=a^{ji}(x),\quad b^{ij}(x)=b^{ji}(x),\quad x\in\overline{\Omega},\quad i,j=1,\dots,n;$$

$$a_0(x)\in C^1(\overline{\Omega}),\quad b_0(x)\in C^1(\overline{\Omega}),\quad a_0(x)\leqslant -a_0<0,\quad b_0(x)\leqslant -b_0<0,\quad x\in\overline{\Omega};$$

$$\exists\,\alpha^i(x):\ \alpha^i(x)\in C(\overline{\Omega}),\quad \alpha^i(x)\geqslant 0,\quad x\in\overline{\Omega},\quad i=1,\dots,n,$$

$$\alpha^i(x)\xi_i^2\leqslant a^{ij}(x)\xi_i\xi_j\leqslant M_0\alpha^i(x)\xi_i^2,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$|a^{ij}_{x_k}(x)|\leqslant M_1\sqrt{\alpha^i(x)},\quad x\in\overline{\Omega},\quad i,j,k=1,\dots,n;$$

$$a^{ij}(x)\nu_i\nu_j=0\quad \textit{for}\quad x\in\Gamma;$$

$$b^{ij}(x)\xi_i\xi_j\geqslant m_0|\xi|^2,\quad m_0>0,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$\Big[a_0(x)b^{ij}(x)+b_0(x)a^{ij}(x)+\frac{1}{2}\left(a^{ij}_{x_k}(x)b^{kl}(x)\right)_{x_l}+\frac{1}{2}\left(b^{ij}_{x_k}(x)a^{kl}(x)\right)_{x_l}-$$
$$-\left(a^{il}_{x_k}(x)b^{jk}_{x_l}(x)\right)\Big]\xi_i\xi_j\leqslant 0,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$a_0(x)b_0(x)+\frac{1}{2}\left(a_{0x_i}(x)b^{ij}(x)\right)_{x_j}+\frac{1}{2}\left(b_{0x_i}(x)a^{ij}(x)\right)_{x_j}\geqslant 0,\quad x\in\overline{\Omega}.$$

Then every function $v(x)\in W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega)$ satisfies the inequality

$$\int_\Omega AvBv\,dx\geqslant 0$$

This assertion is proved in [16].

We say that operators A and B of the above form satisfy the (A,B)–condition if the coefficients of these operators satisfy all conditions of Proposition 1.

Theorem 2. *Suppose the fulfillment of the $(-A_3,-A_2)$–condition and also of the condition*

$$a^{ij,k}(x)\in C^1(\overline{\Omega}),\quad i,j=1,\dots,n,\quad a_{0,k}(x)\in C(\overline{\Omega}),\quad k=0,1. \tag{15}$$

Then, for every function $f(x,t)$ such that $f(x,t)\in L_2(Q)$, $f_t(x,t)\in L_2(Q)$, $f(x,0)=0$ for $x\in\overline{\Omega}$, Boundary Value Problem I has a solution $u(x,t)$ such that $D_t^k u(x,t)\in L_\infty(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $k=0,1,2,3$, $D_t^4u(x,t)\in L_\infty(0,T;L_2(\Omega))$.

Proof. Observe first of all that the $(-A_3, -A_2)$–condition in particular means that $-A_3$ is an elliptic-parabolic operator in $\overline{\Omega}$ and $-A_2$ is an elliptic operator.

Let ε be a positive number. Define operators $A_{3,\varepsilon}$ and L_ε:

$$A_{3,\varepsilon} = A_3 + \varepsilon A_2, \quad L_\varepsilon = L + \varepsilon A_2 D_t^3.$$

Consider the following boundary value problem: *Find a function $u(x,t)$ that is a solution to the equation $L_\varepsilon u = f$ in Q that satisfies conditions (2) and (3).* Obviously, this boundary value problem is Boundary Value Problem I and that it satisfies all conditions of Theorem 1. Moreover, due to the condition $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, a solution $u(x,t)$ to this problem satisfies the memberships

$$D_t^k u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)), \quad k = 0,1,2,3,4, \quad D_t^5 u(x,t) \in L_2(Q) \tag{16}$$

(this fact stems from its validity for the "shortened" equation $D_t^4 u + A_{3,\varepsilon} D_t^3 u = f(x,t)$ and the corresponding a priori estimates).

Differentiate the equation $L_\varepsilon u = f(x,t)$ with respect to t (this is possible due to memberships (16)), multiply it by $D_t^4 u(x,t)$, and integrate it over the cylinder $\{x \in \Omega, \quad 0 < \tau < t\}$. Involving the ellipticity of the operators $-A_{3,\varepsilon}$ and $-A_2$, applying Young's inequality, inequality (11), and Gronwall's lemma, we obtain the estimate

$$\varepsilon \int_0^t \int_\Omega \left(A_2 D_\tau^4 u\right)^2 dx\, d\tau + \sum_{i=1}^n \int_\Omega \left[D_t^4 u_{x_i}(x,t)\right]^2 dx + \int_\Omega \left[A_2 D_t^3 u(x,t)\right]^2 dx \leqslant C_4 \int_Q f_t^2 dx\, dt, \tag{17}$$

where the constant C_4 is defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j = 1,\dots,n$, $k = 0,1$, and also the number T.

Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of positive numbers converging to zero and let $\{u_m(x,t)\}_{m=1}^\infty$ be a sequence of solutions to the equation $L_{\varepsilon_m} u = f$ satisfying (2) and (3). Estimate (17), the second main inequality for elliptic operators, and the reflexivity of a Hilbert space mean that there exists a sequence $\{u_{m_l}(x,t)\}_{l=1}^\infty$ and a function $u(x,t)$ that satisfy the following weak convergences as $l \to \infty$ in $L_2(Q)$:

$$\varepsilon_{m_l} A_2 D_t^3 u(x,t) \to 0,$$

$$D_t^4 u_{m_l}(x,t) \to D_t^4 u(x,t),$$

$$A_k D_t^k u_{m_l}(x,t) \to A_k D_t^k u(x,t), \quad k = 0,1,2,3.$$

Obviously, the limit function $u(x,t)$ is a solution to Boundary Value Problem I and this solution still satisfies (17). Therefore, the function $u(x,t)$ is the desired solution to the problem under study.

The theorem is proved. □

Turn to investigating the solvability of Boundary Value Problem II.

The main difference of Boundary Value Problem II from Boundary Value Problem I is that, in its study, it is impossible to use Gronwall's lemma. Gronwall's lemma can be replaced by smallness conditions.

We will give the simplest version of the theorem in the solvability of a Boundary Value Problem II, whose prove involves smallness conditions.

Let operators A_0 and A_1 be defined with the use of the parameter β and the operators $\widetilde{A}_0$ and $\widetilde{A}_1$:

$$A_0 = \beta \widetilde{A}_0, \quad A_1 = \beta \widetilde{A}_1, \quad \widetilde{A}_k = \frac{\partial}{\partial x_i}\left(\widetilde{a}^{ij,k}(x) \frac{\partial}{\partial x_j}\right) + \widetilde{a}_{0i}(x), \quad k = 0,1. \tag{18},$$

Theorem 3. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x) \in C^2(\overline{\Omega}), \quad a^{ij,k}(x) = a^{ji,k}(x), \quad i,j = 1,\dots,n, \quad k = 2,3;$$

$$\widetilde{a}^{ij,k}(x) \in C^1(\overline{\Omega}), \quad i,j = 1,\dots,n, \quad \widetilde{a}_{0,k}(x) \in C(\overline{\Omega}), \quad k = 0,1;$$

$$a^{ij,k}(x)\xi_i\xi_j \geqslant m_0|\xi|^2, \quad m_0 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad k = 2,3;$$

$$a_{0,k}(x) \in C(\overline{\Omega}), \quad k = 0,1,2,3, \quad a_{0,k}(x) \leqslant 0, \quad k = 2,3.$$

Then there exists a positive number β_0 such that for $|\beta| < \beta_0$ and $f(x,t) \in L_2(Q)$, Boundary Value Problem II has a solution $u(x,t)$ such that $D_t^k u(x,t) \in L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, $k = 0,1,2,3$, $D_t^4 u(x,t) \in L_2(Q)$.

Proof. For $\lambda = 0$, Boundary Value Problem II for equation (9) has a solution $u(x,t)$ in the desired class; this follows from the fact that for $\lambda = 0$ equation (9) is an inverse parabolic equation with respect to $D_t^3 u(x,t)$. Further, consider (10). Integrating by parts and estimating the last two summands on the left-hand side (10) from above with the use of (11), we infer that there exists a positive number β_1 such that for $|\beta| < \beta_0$ we have the a priori estimate

$$\sum_{i=1}^{n} \int_Q \left(D_t^3 u_{x_i}\right)^2 dx\,dt \leqslant C_5 \int_Q f^2\,dx\,dt \tag{19}$$

with the constant C_5 defined only by the coefficients of the operators A_k, $k = 0,1,2,3$.

At the next step, consider the equality

$$\int_Q \left[D_t^4 u + A_3 D_t^3 u + \lambda \sum_{k=0}^{2} A_k D_t^k u\right] A_2 D_t^3 u\,dx\,dt = \int_Q f A_2 D_t^3 u\,dx\,dt.$$

Reckoning with the ellipticity of A_2 and A_3 and using the second main inequality for a pair of elliptic operators [10, Chapter III, Sec. 8], it is not hard to show that there exists a number β_0 such that $0 < \beta_0 \leqslant \beta_1$, and for $|\beta| < \beta_0$, for solutions $u(x,t)$ to Boundary Value Problem II for equation (9), estimate (13) holds with some constant C_6 on the right-hand side that is defined only by the coefficients of the operators A_k, $k = 0,1,2,3$, and the domain Ω.

Estimate (14) with the corresponding constant C_7 on the right-hand side obviously follows from the previous estimates.

The obtained estimates of solutions to Boundary Value Problem II for equation (9) and the theorem on the method of continuation in a parameter and give the solvability of Boundary Value Problem II for equation (1) in the desired class.

The theorem is proved. □

Theorem 4. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x) \in C^2(\overline{\Omega}), \quad a^{ij,k}(x) = a^{ji,k}(x), \quad a_{0,k}(x) \in C(\overline{\Omega}), \quad i,j = 1,\dots,n, \quad k = 0,1,2,3; \tag{20}$$

$$a^{ij,k}(x)\xi_i\xi_j \geqslant m_0|\xi|^2, \quad m_0 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_{0,k}(x) \leqslant 0, \quad k = 2,3; \tag{21}$$

$$-a^{ij,k}(x)\xi_i\xi_j \geqslant m_1|\xi|^2, \quad m_1 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_{0,k}(x) \geqslant 0, \quad k = 0,1; \tag{22}$$

$$A_0 = \beta \widetilde{A}_0. \tag{23}$$

Then there is a positive number β_0 such that, for $|\beta| < \beta_0$ and $f(x,t) \in L_2(Q)$, Boundary Value Problem III has a solution $u(x,t)$ such that $D_t^k u(x,t) \in L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, $k = 0,1,2,3$, $D_t^4 u(x,t) \in L_2(Q)$.

Proof. Show that solutions $u(x,t)$ to Boundary Value Problem III of the class mentioned in the statement of the theorem satisfy the desired a priori estimates.

Multiply equation (1) by $D_t^2 u(x,t)$. Integrating over Q, applying integration by parts, and using (20)–(22), it is not hard to obtain the first a priori estimate for solutions $u(x,t)$ to Boundary Value Problem III:

$$\int_Q \left[\left(D_t^3 u\right)^2 + \sum_{i=1}^n \left(D_t^2 u_{x_i}\right)^2 \right] dx\, dt \leqslant C_8 \int_Q f^2\, dx\, dt; \tag{24}$$

here the constant C_8 is defined only by the coefficients of the operators A_k, $k = 0, 1, 2, 3$.

At the next step, multiply equation (1) by $A_2 D_t^3 u(x,t)$ and integrate it over Q. Using conditions (20)–(23), inequality (11), and also the second main inequality for a pair of elliptic operators, we conclude that there exists a number β_0 such that for $|\beta| < \beta_0$ we have a second estimate

$$\sum_{i,j=1}^n \int_Q \left(D_t^3 u_{x_i x_j}\right)^2\, dx\, dt \leqslant C_9 \int_Q f^2\, dx\, dt; \tag{25}$$

with the constant C_9 defined only by the coefficients of the operators A_k, $k = 0, 1, 2, 3$, and the domain Ω.

The last a priori estimate

$$\int_Q \left(D_t^4 u\right)^2\, dx\, dt \leqslant C_{10} \int_Q f^2\, dx\, dt \tag{26}$$

obviously stems of the previous two estimates.

Using estimates (24)–(26) and the method of continuation in a parameter (for example, with the use of the equation

$$D_t^4 u + A_2 D_t^2 u + \lambda(A_3 D_t^3 u + A_1 D_t u + A_0 u) = f(x,t)\big),$$

it is not hard to obtain the desired solvability of Boundary Value Problem III.

The theorem is proved. $\square$

3. Conclusion.

Observe first of all that the conditions of Proposition 1 are fulfilled, for instance, if the numbers a_0 and b_0 are large.

Furthermore, it is not hard to generalize the obtained results to equations more general than (1); for example, to equations with general second-order elliptic operators A_k.

Some of the conditions of the proven theorems can be changed: for example, we can discard the sign-definiteness of the operator A_0 from Theorem 4.

Observe finally that conditions (18) and (23) mean that $\widetilde{A}_1$ and $\widetilde{A}_0$ are fixed operators, whereas the number β is a parameter (namely, a smallness parameter).

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Краевые задачи для уравнений соболевского типа четвертого порядка

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Аннотация. Целью статьи является исследование разрешимости в пространствах Соболева краевых задач для некоторых классов линейных уравнений четвертого порядка соболевского типа. Докажем, что начально-краевые задачи с данными как в начальный момент времени, так и в конечные моменты времени могут быть корректными для исследуемых уравнений.

Ключевые слова: дифференциальное уравнение четвертого порядка соболевского типа, краевая задача, существование, единственность.