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A Note on the Conjugacy Between Two Critical Circle Maps

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Abstract. We study a conjugacy between two critical circle homeomorphisms with irrational rotation number. Let f_i, $i = 1, 2$ be a C^3 circle homeomorphisms with critical point $x_{cr}^{(i)}$ of the order $2m_i + 1$. We prove that if $2m_1 + 1 \neq 2m_2 + 1$, then conjugating between f_1 and f_2 is a singular function.

Keywords: circle homeomorphism, critical point, conjugating map, rotation number, singular function.

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1. Introduction and preliminaries

Denjoy's classical theorem [4] states, that if the C^2 circle diffeomorphism f and irrational rotation number $\rho = \rho_f$ then f is topologically conjugate to the linear rotation f_ρ, that is, there exists a circle homeomorphism φ with $f = \varphi^{-1} \circ f_\rho \circ \varphi$.

It is well known that a circle homeomorphisms f with irrational rotation number is strictly ergodic, i.e. it has a unique f-invariant probability measure ν_f. A remarkable fact is that the conjugacy φ can be defined by $\varphi(x) = \nu_f([0, x])$, which shows, that the regularity properties of conjugacy φ and the absolute continuity of invariant measure ν_f are closely related. The problem of smoothness of the conjugacy φ for diffeomorphisms is one of the important problems of circle dynamics. The fundamental results were obtained by V. I. Arnold [1] , J. Moser [15], M. Herman [9], J. Yoccoz [17], Ya. G. Sinai and K. Khanin [12], Y. Katsnelson and D. Ornstein [13]. Notice that for sufficiently smooth circle deffeomorphisms f with a typical irrational rotation number the conjugacy φ is C^1-diffeomorphism. Consequently, the invariant measure ν_f is absolutely continuous with respect to Lebesgue measure μ on S^1.

Since the works of Mostow, Margulis, Sullivan, and others, rigidity problems occupy a central place in the theory of holomorphic dynamical systems. This type of problems is classical in dynamics: a rigidity theorem postulates that in a certain class of dynamical systems equivalence (combinatorial, continuous, smooth, etc.) automatically has a higher regularity. The dynamical systems considered in this paper are critical circle maps, that is smooth homeomorphisms of the circle with a single critical point having an odd type. These maps have been a subject of intensive study since the early 1980's as one of the two main examples of universality in transition to chaos. Yoccoz in [17] generalized Denjoy's classical result, a critical circle homeomorphism with irrational rotation number is topologically conjugate to an irrational rotation.

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Definition 1.1. *The point $x_{cr} \in S^1$ is called non-flat critical point of a homeomorphism f with order $(2m+1)$, $m \in N$, if for a some δ-neighborhood $U_\delta(x_{cr})$, the function f belongs to the class of $C^{2m+1}(U_\delta(x_{cr}))$ and*

$$f'(x_{cr}) = f''(x_{cr}) = \cdots = f^{(2m)}(x_{cr}) = 0, \quad f^{(2m+1)}(x_{cr}) \neq 0.$$

The order of the critical point x_{cr} is $2m+1$. By a *critical circle map* we define an orientation preserving circle homeomorphism with exactly one non-flat critical point of odd type.

An important one-parameter family of examples of critical circle maps are the Arnold's maps defined by

$$f_\theta(x) := x + \theta + \frac{1}{2\pi} \sin 2\pi x \ (mod 1), \quad x \in S^1.$$

For every $\theta \in R^1$ the map f_θ is a critical map with critical point 0 of cubic type.

Graczyk and Swiatek in [7] proved that if f is C^3 smooth circle homeomorphism with finitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map φ is singular function on S^1 i.e. $\varphi'(x) = 0$ a.e. on S^1. Consequently, the invariant measure of critical circle homeomorphisms is singular w.r.t. Lebesque measure on S^1. Hence the problem of regularity of the conjugacy between two critical maps with identical irrational rotation number arises naturally. This is called the rigidity problem for critical circle homeomorphisms. For the critical circle maps the rigidity problem is developed by de Faria, de Melo, Yampolsky, Khanin and Teplinsky, Guarino among others.

The first result concerning on rigidity for critical maps was proven by de Melo and de Faria [6].

Theorem 1.1 (see [6])**.** *If f_1, f_2 are C^3 critical circle mappings with the same irrational rotation number of bounded type and the same power-law at the critical point, then there exists a $C^{1+\alpha}$ conjugacy h between f_1 and f_2 for some universal $\alpha > 0$.*

The following result of D. Khmelev and M. Yampolski [14] seemed to indicate that the analytic case could be different.

Theorem 1.2 ([14])**.** *There exists a universal constant $\alpha > 0$ such that the following holds. Let f_1 and f_2 be two analytic critical circle maps with the same irrational rotation number. Denote $h : S^1 \to S^1$ conjugacies between f_1 and f_2 fixing the critical points. Then h is $C^{1+\alpha}$ at the critical point.*

K. Khanin and A. Teplinskii [11] proved that any two f_1 and f_2 analytic critical circle maps with the same order of critical points and the same irrational rotation number are C^1-smoothly conjugate to each other. Later, A. Avila [2] showed, that there exist f_1 and f_2 analytic homeomorphisms with the same irrational rotation number such that h is not $C^{1+\alpha}$ for any $\alpha > 0$.

Next we formulate the result of P. Guarino, M. Martens, and W. de Melo [8].

Theorem 1.3 ([8])**.** *Let f_1 and f_2 be two analytic C^4-circle homeomorphisms with the same irrational rotation number and with a unique critical point of the same odd type. Then they are C^1-smoothly conjugate to each other. The conjugacy is $C^{1+\alpha}$ for Lebesgue almost every rotation number.*

The present work continuous and completes the above results. Namely we show that if the rotation numbers of two critical homeomorphisms coincide but the orders of critical points are different then the conjugacy h is a singular function. Now we formulate our main result.

Theorem 1.4. *Let f_1 and f_2 be C^3 critical circle maps with the same irrational rotation number. Suppose that the orders of critical points of f_1 and f_2 are different i.e. $2m_1 + 1 \neq 2m_2 + 1$. Then the conjugacy h between f_1 and f_2 is a singular function on S^1.*

2. Notations, terminology, background

Let f be a circle homeomorphism that preserves orientation, i.e. $f(x) = F(x)(mod\,1)$, $x \in S^1 \simeq [0,1)$, where F is continuous, strictly increasing on R^1 and $F(x+1) = F(x)+1$ for any $x \in R$. F is called *lift* of homeomorphism f. The important characteristic of the circle homeomorphism f is it's *rotation number* (see for instance [6]) ρ_f which defined by $\rho_f = \lim\limits_{n\to\infty} \dfrac{F^n(x)}{n}(mod\,1)$, here and later F^n denotes the n-th iteration of F. The rotation number ρ_f is rational if and only if f has periodic orbits.

2.1. Dynamical partition. Let f be an orientation preserving homeomorphism of the circle with lift F and irrational rotation number $\rho = \rho_f$. We denote by $\{a_n, n \in \mathbb{N}\}$ the sequence of entries in the continued fraction expansion of ρ, i.e. $\rho = [a_1, a_2, \ldots, a_n, \ldots]$. Denote by $p_n/q_n = [a_1, a_2, \ldots, a_n]$ the convergents of ρ. Their denominators q_n satisfy the recurrence relation, that is $q_{n+1} = a_{n+1}q_n + q_{n-1},\ n \geqslant 1,\ q_0 = 1,\ q_1 = a_1$.

For an arbitrary point $x_0 \in S^1$ we define $\Delta_0^{(n)}(x_0)$ the closed interval on S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$. Note that for odd n the point x_{q_n} lies to the left of x_0 and for even n to the right. Denote by $\Delta_i^{(n)}(x_0)$ the iterates of the interval $\Delta_0^{(n)}(x_0)$ under f:$\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0)),\ i \geqslant 1$.

Lemma 2.1 (see [12]). *Consider an arbitrary point $x_0 \in S^1$. A finite piece $\{x_i, 0 \leqslant i < q_n + q_{n-1}\}$ of the trajectory of this point divides the circle into the following disjoint (except for the endpoints) intervals: $\Delta_i^{(n-1)}(x_0),\ 0 \leqslant i < q_n$, $\Delta_j^{(n)}(x_0),\ 0 \leqslant j < q_{n-1}$.*

We denote the obtained partition by $\xi_n(x_0)$ and call it n-th *dynamical partition* of the circle. We now briefly describe the process of transition from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$. All intervals $\Delta_j^{(n)}(x_0),\ 0 \leqslant j < q_{n-1}$, are preserved, and each of the intervals $\Delta_i^{(n-1)}(x_0)$ is divided into $a_{n+1} + 1$ sub intervals:

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

Obviously one has $\xi_1(x_0) \leqslant \xi_2(x_0) \leqslant \ldots \leqslant \xi_n(x_0) \leqslant \ldots$.

Definition 2.1. *Let $K > 1$ be a constant. We call two intervals I_1 and I_2 of S^1 are K-comparable, if the inequalities $K^{-1}\mu(I_2) \leqslant \mu(I_1) \leqslant K\mu(I_2)$ hold.*

Next we formulate the lemma, that is proved in the similar way as in [16].

Let $x_{cr} \in S^1$ be a critical point of homeomorphism f. For any $x_0 \in S^1$, consider the dynamical partition $\xi_n(x_0)$. For definiteness we assume that n is odd. Then $x_{q_n} \prec x_0 \prec x_{q_{n-1}}$. The structure of the dynamical partition implies that $\overline{x}_{cr} = f^{-p}(x_{cr}) \in [x_{q_n}, x_{q_{n-1}}]$, for some p, $0 < p < q_n$. Let I_1 and I_2 be any elements of a dynamical partition $\xi_m(\overline{x}_{cr})$, $m \geqslant n$ having a common endpoints.

Lemma 2.2. *Let $f \in C^3(S^1)$ be a critical circle homeomorphism with irrational rotation number. Then there exists a constant $K > 1$ depending only on f such that the intervals I_1 and I_2 are K-comparable.*

It follows from the Lemma 2.2 that the trajectory of each point is dense in S^1. Hence it follows that there exists conjugation map φ between f and f_ρ, i.e. $\varphi(f(x)) = f_\rho(\varphi(x))$ for any $x \in S^1$.

We assume that $\Delta^{(m+k)}$ is element of partitioning $\xi_{m+k}(\overline{x}_{cr})$, while $\Delta^{(m)}$ is an element of partitioning $\xi_m(\overline{x}_{cr})$ that contains $\Delta^{(m+k)}$.

Lemma 2.3 (see [10])**.** *There exist constants $\lambda_1(f) < \lambda_2(f) < 1$ such that*

$$\ell(\Delta^{(m+k)}) \leqslant const\, \lambda_2^k(f)\ell(\Delta^{(m)}), \quad \ell(\Delta_0^{(m)}) \geqslant const\, \lambda_1^m(f).$$

2.2. Cross-ratio tools. In the proof of our main theorem the tool of cross-ratio plays a key role.

Definition 2.2. *The cross-ratio of four points (z_1, z_2, z_3, z_4), $z_1 < z_2 < z_3 < z_4$ is the number*

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

Definition 2.3. *Given four real numbers (z_1, z_2, z_3, z_4) with $z_1 < z_2 < z_3 < z_4$ and a strictly increasing function $F : \mathbb{R}^1 \to \mathbb{R}^1$. The distortion of their cross-ratio under F is given by*

$$Dist(z_1, z_2, z_3, z_4; F) = \frac{Cr(F(z_1), F(z_2), F(z_3), F(z_4))}{Cr(z_1, z_2, z_3, z_4)}.$$

For $m \geqslant 3$ and $z_i \in S^1$, $1 \leqslant i \leqslant m$, suppose that $z_1 \prec z_2 \prec \cdots \prec z_m \prec z_1$ (in the sense of the ordering on the circle). Then we set $\hat{z}_1 := z_1$ and

$$\hat{z}_i := \begin{cases} z_i & \text{if } z_1 < z_i < 1, \\ 1 + z_i & \text{if } 0 < z_i < z_1. \end{cases}$$

for $2 \leqslant i \leqslant m$.

Obviously, $\hat{z}_1 < \hat{z}_2 < \ldots < \hat{z}_m$. The vector $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m)$ is called the lifted vector of $(z_1, z_2, \ldots, z_m) \in (S^1)^m$.

Let f be a circle homeomorphism with lift F. We define the cross-ratio distortion of (z_1, z_2, z_3, z_4), $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ with respect to f by $Dist(z_1, z_2, z_3, z_4; f) = Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; F)$, where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4). We need the following lemma.

Lemma 2.4 ([5])**.** *Let $z_i \in S^1$, $i = 1, 2, 3, 4$, $z_1 \prec z_2 \prec z_3 \prec z_4$. Consider a circle homeomorphism f with $f \in C^{2+\varepsilon}([z_1, z_4])$, $\varepsilon > 0$, and $f'(x) \geqslant const > 0$ for $x \in [z_1, z_4]$. Then there is a positive constant $C_1 = C_1(f)$ such that*

$$|\, Dist(z_1, z_2, z_3, z_4; f) - 1 \,| \leqslant C_1 |\hat{z}_4 - \hat{z}_1|^{1+\varepsilon},$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

We now consider the case when the interval $[z_1, z_4]$ contains a critical point x_{cr} of the homeomorphism f. More precisely, suppose that $z_2 = x_{cr}$. We define numbers α, β, γ, ξ and η as follows:

$$\alpha := \hat{z}_2 - \hat{z}_1, \quad \beta := \hat{z}_3 - \hat{z}_2, \quad \gamma := \hat{z}_4 - \hat{z}_3, \quad \xi := \frac{\beta}{\alpha}, \quad \eta := \frac{\beta}{\gamma},$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

Thus we need the following lemma.

Lemma 2.5. *Suppose that a homeomorphism f with lift F has a critical point x_{cr} with order $2m+1$, $m \in N$. Then for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$, such that for all $z_i \in U_\delta(x_{cr})$, $i = \overline{1,n}$, $z_1 \prec z_2 = x_{cr} \prec z_3 \prec z_4$ one has*

$$\left| Dist(z_1, z_2, z_3, z_4; f) - \frac{1}{1-\xi+\xi^2-\cdots+\xi^{2m}} \times \frac{e_{2m}\eta^{2m} + e_{2m-1}\eta^{2m-1} + \cdots + e_1\eta + 1}{\eta^{2m} + C_{2m}^1\eta^{2m-1} + \cdots + C_{2m}^{2m-1}\eta + 1} \right| < R_0\varepsilon,$$

where the constants $e_{2m} = 2m+1$, $e_i = C_{2m}^i + C_{2m-1}^{i-1} + \cdots + C_{2m-i}^0$ and R_0 depends only on function f.

Proof. Fix a number ε. It is easy to check that for any $z_i \in S^1$, $i = \overline{1,n}$, $z_1 \prec z_2 \prec z_3 \prec z_4$ one has

$$F(z_1) = F(\hat{z}_2) - F'(\hat{z}_2)(\hat{z}_2 - \hat{z}_1) + \cdots + \frac{F^{(2m)}(\hat{z}_2)}{2m!}(\hat{z}_2 - \hat{z}_1)^{2m} - \frac{1}{2m!}\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y-\hat{z}_1)^{2m}dy,$$

$$\begin{aligned} F(\hat{z}_s) =& F(\hat{z}_2) + F'(\hat{z}_2)(\hat{z}_s - \hat{z}_2) + \cdots + \frac{F^{(2m)}(\hat{z}_2)}{2m!}(\hat{z}_s - \hat{z}_2)^{2m} + \\ &+ \frac{1}{2m!}\int_{\hat{z}_2}^{\hat{z}_s} F^{(2m+1)}(y)(\hat{z}_s - y)^{2m}dy, \quad s = 3, 4. \end{aligned} \tag{2.1}$$

By the assumption of the lemma, $z_2 = x_{cr}$, and using the (2.1) we write $Cr(f(z_1), f(z_2), f(z_3), f(z_4))$ as follows

$$\begin{aligned} Cr(f(z_1), f(z_2), f(z_3), f(z_4)) &= \frac{(F(\hat{z}_2) - F(\hat{z}_1))(F(\hat{z}_4) - F(\hat{z}_3))}{(F(\hat{z}_3) - F(\hat{z}_1))(F(\hat{z}_4) - F(\hat{z}_2))} = \\ = \frac{\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y-\hat{z}_1)^{2m}dy}{\int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(y)(\hat{z}_3-y)^{2m}dy + \int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y-\hat{z}_1)^{2m}dy} &\times \\ \times \frac{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(y)(\hat{z}_4-y)^{2m}dy - \int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(y)(\hat{z}_3-y)^{2m}dy}{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(y)(\hat{z}_4-y)^{2m}dy}&, \end{aligned} \tag{2.2}$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4). Since $F^{(2l+1)} \in C(U_\omega(x_{cr}))$, there exist $\delta(\varepsilon) > 0$, such that for any $x, y \in (x_{cr} - \omega, x_{cr} + \omega)$ the inequality $|F^{(2m+1)}(x) - F^{(2m+1)}(y)| < \varepsilon$ is true.

Hence from (2.2) we have

$$\begin{aligned} & Cr(f(z_1), f(z_2), f(z_3), f(z_4)) = \\ = & \frac{\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(x_{cr})(y-\hat{z}_1)^{2m}dy(1+O(\varepsilon))}{\left(\int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(x_{cr})(\hat{z}_3-y)^{2m}dy + \int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(x_{cr})(y-\hat{z}_1)^{2m}dy\right)(1+O(\varepsilon))} \times \\ \times & \frac{\left(\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(x_{cr})(\hat{z}_4-y)^{2m}dy - \int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(x_{cr})(\hat{z}_3-y)^{2m}dy\right)(1+O(\varepsilon))}{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(x_{cr})(\hat{z}_4-y)^{2m}dy(1+O(\varepsilon))} = \end{aligned}$$

$$= \frac{\alpha^{2m+1}}{\alpha^{2m+1}+\beta^{2m+1}} \cdot \frac{(\gamma+\beta)^{2m+1}-\beta^{2m+1}}{(\gamma+\beta)^{2m+1}}(1+O(\varepsilon)).$$

From the last equality it follows that

$$Dist(z_1, z_2, z_3, z_4; f) = \frac{1}{1-\xi+\xi^2-\cdots+\xi^{2m}} \times$$

$$\times \frac{(1+\eta)^{2m}+(1+\eta)^{2m-1}\eta+\cdots+(1+\eta)\eta^{2m-1}+\eta^{2m}}{(1+\eta)^{2m}}(1+O(\varepsilon)) =$$

$$= \frac{1}{1-\xi+\xi^2-\cdots+\xi^{2m}} \times \frac{e_{2m}\eta^{2m}+e_{2m-1}\eta^{2m-1}+\cdots+e_1\eta+1}{\eta^{2m}+C_{2m}^1\eta^{2m-1}+\cdots+C_{2m}^{2m-1}\eta+1}(1+O(\varepsilon)).$$

Thus Lemma 2.5 is proved. □

Next suppose the interval $[z_1, z_4]$ is a subset of the interval $U_\omega(x_{cr})$ but does not contain a critical point x_{cr} of the homeomorphism f. Let $d = \min_{1\leqslant s\leqslant 4} \ell([z_s, x_{cr}])$. We now state an assertion from [10].

Lemma 2.6 (see [10])**.** *Suppose that a homeomorphism f satisfies the conditions of Lemma 2.5. Then the following equality holds*

$$Dist(z_1, z_2, z_3, z_4; f) = 1 + O\left(\left(\frac{\alpha+\beta+\gamma}{d}\right)^2\right).$$

3. Proof of Theorem 1.4

In order to prove Theorem 1.4 we need several lemmas which we formulate next. Their proofs will be given later. We consider two copies of the unit circle S^1. The homeomorphism f_1 acts on the first circle and f_2 acts on the second one. Assume that f_i, $i = 1, 2$ satisfies the conditions of Theorem 1.4.

Let φ_1 and φ_2 be conjugations of f_1 and f_2 to linear rotation f_ρ, i.e. $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$ and $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$. It is easy to check that the homeomorphisms f_1 and f_2 are conjugated by $h = \varphi_2 \circ \varphi_1^{-1}$, i. e. $h \circ f_1(x) = f_2 \circ h(x), \forall x \in S^1$. Recall that every φ_i, $i = 1, 2$ is unique up to an additional constant. This gives us a possibility to choose h with initial condition $h(x_{cr}^{(1)}) = x_{cr}^{(2)}$.

Notice the conjugation $h(x)$ is continuous function on S^1. It suffices to show that $h'(x) = 0$ for almost all x with respect to the Lebesgue measure. The derivative $h'(x) = 0$ exists for almost all x with respect to the Lebesgue measure because the function h is monotonic. Let us show that $h'(x) = 0$ at all points where the derivative is defined.

Lemma 3.1 (see [5])**.** *Assume, that the conjugating homeomorphism $h(x)$ has a positive derivative $h'(x_0) = \omega_0$ at some point $x_0 \in S^1$, and that the following conditions hold for the points $z_i \in S^1$, $i = 1, \ldots, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4$, and some constant $R_1 > 1$:*

- *(a) the intervals $[z_1, z_2], [z_2, z_3], [z_3, z_4]$ are pairwise R_1-comparable;*
- *(b) $\max_{1\leqslant i\leqslant 4} \ell([z_i, x_0]) \leqslant R_1 \ell([z_1, z_2])$.*

Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|Dist(z_1, z_2, z_3, z_4; h) - 1| \leqslant C_2\varepsilon, \tag{3.1}$$

if $z_i \in (x_0 - \delta,\ x_0 + \delta)$ for all $i = 1, 2, 3, 4$, where the constant $C_2 > 0$ depends only on R_1, ω_0 and not on ε.

Suppose that $h'(x_0) = \omega_0$, where $x_0 \in S^1$. Let $\xi_n(x_0)$ be its n-th dynamical partition. Put $t_0 := h(x_0)$ and consider the dynamical partition $\tau_n(t_0)$ of t_0 on the second circle determined by the homeomorphism f_2, i.e.

$$\tau_n(t_0) = \{I_i^{(n-1)}(t_0),\quad 0 \leqslant i \leqslant q_n - 1\} \cup \{I_j^{(n)}(t_0),\quad 0 \leqslant j \leqslant q_{n-1} - 1\}$$

with $I_0^{(n)}(t_0)$ the closed interval with endpoints t_0 and $f_2^{q_n}(t_0)$. Choose an odd natural number $n_1 = n(f_1, f_2)$ such that the n_1-th renormalization neighborhoods $[x_{q_{n_1}}, x_{q_{n_1-1}}]$ and $[t_{q_{n_1}}, t_{q_{n_1-1}}]$ do not contain critical point of f_1 and f_2 respectively. Since the identical rotation number ρ of f_1 and f_2 is irrational, the order of the points on the orbit $\{f_1^k(x_0),\ k \in \mathbb{Z}\}$ on the first circle will be precisely the same as the one for the orbit $\{f_2^k(t_0),\ k \in \mathbb{Z}\}$ on the second one. This together with the relation $h(f_1(x)) = f_2(h(x))$ for $x \in S^1$ implies that

$$h(\Delta_i^{(n_1-1)}) = I_i^{(n_1-1)},\quad 0 \leqslant i \leqslant q_{n_1} - 1,\quad h(\Delta_j^{(n_1)}) = I_j^{(n_1)},\quad 0 \leqslant j \leqslant q_{n_1-1} - 1. \tag{3.2}$$

The structure of the dynamical partitions implies that $\overline{x}_{cr}^{(1)}(n_1) = f_1^{-l}(x_{cr}^{(1)}) \in [x_{q_{n_1}}, x_{q_{n_1-1}}]$, where $l \in (0, q_{n_1-1})$ if $\overline{x}_{cr}^{(1)}(n_1) \in [x_{q_{n_1}}, x_0]$, and $l \in (0, q_{n_1})$ if $\overline{x}_{cr}^{(1)}(n_1) \in [x_0, x_{q_{n_1-1}}]$. Since h conjugation between f_1 and f_2, we get

$$f_2^l(h(\overline{x}_{cr}^{(1)})) = f_2^{l-1}(f_2(h(\overline{x}_{cr}^{(1)}))) = f_2^{l-1}(h(f_1(\overline{x}_{cr}^{(1)}))) = \cdots = h(f_1^l(\overline{x}_{cr}^{(1)})) = h(x_{cr}^{(1)}) = x_{cr}^{(2)}.$$

Hence $\overline{x}_{cr}^{(2)}(n_1) = f_2^{-l}(x_{cr}^{(2)}) \in [t_{q_{n_1}}, t_{q_{n_1-1}}]$. The points $\overline{x}_{cr}^{(1)}(n_1)$ and $\overline{x}_{cr}^{(2)}(n_1)$ are called the q_{n_1}-*pre-images* of the critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$, respectively.

Next we introduce the concept of a "regular" cover of the critical point. Let $z_i \in S^1,\ i = \overline{1,4}$, $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. Define for each j, $0 < j < q_n$

$$\xi_{f_1}(j) = \frac{\ell\big([f_1^j(z_2), f_1^j(z_3)]\big)}{\ell\big([f_1^j(z_1), f_1^j(z_2)]\big)},\quad \eta_{f_1}(j) = \frac{\ell\big([f_1^j(z_2), f_1^j(z_3)]\big)}{\ell\big([f_1^j(z_3), f_1^j(z_4)]\big)}.$$

Definition 3.1. *Let $M > 1$, $\zeta \in (0,1)$, $\delta > 0$ be constant numbers, n is a positive integer and $x_0 \in S^1$. We say that a triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$, $z_i \in S^1$, $i = 1,2,3,4$, covers the critical point of $x_{cr}^{(1)}$ "$(M, \zeta, \theta, \delta; x_0)$-regularly", if the following conditions hold:*

1) $[z_1, z_4] \subset (x_0 - \delta, x_0 + \delta)$, and the system of intervals $\{f_1^j([z_1, z_4]),\ 0 \leqslant j \leqslant q_n - 1\}$ cover critical point $x_{cr}^{(1)}$ only once;

2) $z_2 = f_1^{-l}(x_{cr}^{(1)})$ for some l, $0 < l < q_n$;

3) $\xi_{f_1}(l) < \zeta$ and $\eta_{f_1}(l) \geqslant M$.

Denote

$$L = \min\{2m_1 + 1,\ 2m_2 + 1,\ 2|m_1 - m_2|\}.$$

Lemma 3.2. *Suppose that the homeomorphisms f_i, $i = 1,2$ satisfy the conditions of Theorem 1.4. Then for any $x_0 \in S^1$ and $\delta > 0$ there exist constant $M_0 > 1$ and $\zeta_0 \in (0,1)$, such that for all triples of intervals $[z_s, z_{s+1}] \subset (x_0 - \delta,\ x_0 + \delta)$, $s = 1,2,3$, and $[h(z_s), h(z_{s+1})]$, $s = 1,2,3$, covering the critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$ regularly with constants M_0 and ζ_0 the following inequalities hold:*

$$\Big|\frac{1}{1 - \xi_{f_1}(l) + \cdots + \xi_{f_1}^{2m_1}(l)} \times \frac{e_{2m_1}\eta_{f_1}^{2m_1}(l) + e_{2m_1-1}\eta_{f_1}^{2m_1-1}(l) + \cdots + 1}{\eta_{f_1}^{2m_1}(l) + C_{2m_1}^1\eta_{f_1}^{2m_1-1}(l) + \cdots + 1} - (2m_1 + 1)\Big| < \frac{L}{16},$$

$$\Big|\frac{1}{1-\xi_{f_2}(l)+\cdots+\xi_{f_2}^{2m_2}(l)}\times\frac{e_{2m_2}\eta_{f_2}^{2m_2}(l)+e_{2m_2-1}\eta_{f_2}^{2m_2-1}(l)+\cdots+1}{\eta_{f_2}^{2m_2}(l)+C_{2m_2}^{1}\eta_{f_2}^{2m_2-1}(l)+\cdots+1}-(2m_2+1)\Big|<\frac{L}{16},$$

where m_1 and m_2 are orders of critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$ respectively.

Assume that the homeomorphism f_1 satisfies the conditions of Theorem 1.4. Let $\xi_n(x_{cr}^{(1)})$ be a dynamical partition of the circle by f_1. We take a natural number r, such that $\Delta_0^{(r)}(x_{cr}^{(1)})\cup\Delta_0^{(r-1)}(x_{cr}^{(1)})\subset U_{\omega_1}(x_{cr}^{(1)})$. Suppose that $h'(x_0)=p_0>0$ for some $x_0\in S^1$. Consider the dynamical partition $\xi_n(x_0)$ of the point x_0 under f_1. Suppose that $n>r$ an odd natural number. Let $\overline{x}_{cr}^{(1)}=f^{-l}(x_{cr}^{(1)})\in[x_{q_n},x_{q_{n-1}}]$.

Let $\{\xi_{n+k}(\overline{x}_{cr}^{(1)})\}_{k=0}^{\infty}$ be a sequence of dynamical partitions of the point $\overline{x}_{cr}$. We define the points z_i, $i=1,2,3,4$ as follows

$$z_1=f^{q_{n+k_0}}(\overline{x}_{cr}^{(1)}),\quad z_2=\overline{x}_{cr}^{(1)},\quad z_3=f^{q_{n+k_0+k_1}}(\overline{x}_{cr}^{(1)}),\quad z_4=f^{q_{n+k_0+k_1}+q_{n+k_2}}(\overline{x}_{cr}^{(1)}).$$

Lemma 3.3. *Suppose that the homeomorphisms f_1 and f_2 satisfies the conditions of Theorem 1.4. Let $h'(x_0)=p_0>0$ for some $x_0\in S^1$, $\delta\in(0,1)$ and $k_0\in N$. Then there exist natural numbers k_1, k_2 such that for sufficiently large n, the triple of intervals $[z_s,z_{s+1}]\subset(x_0-\delta,\ x_0+\delta)$, $s=1,2,3$ satisfies the following properties:*

(1) the intervals $\{[f_1^j(z_1),f_1^j(z_4)],\ 0\leqslant j\leqslant q_n\}$ cover each point at most once;

(2) the intervals $[z_s,z_{s+1}]$ and $[f_1^{q_n}(z_s),f_1^{q_n}(z_{s+1})]$, $s=1,2,3$ satisfy conditions (a) and (b) of Lemma 3.1 with some constant $R_1>1$ depending on k_0, k_1, k_2;

(3) the triples of intervals ($[z_s,z_{s+1}]$, $s=1,2,3$) and ($[h(z_s),h(z_{s+1})]$, $s=1,2,3$) cover the critical points $x_{cr}^{(1)}$, $x_{cr}^{(2)}$, "$(M_0,\zeta_0,\delta;x_0)$-regularly" and "$(M_0,\zeta_0,\delta;h(x_0))$-regularly", respectively.

Lemma 3.4. *Suppose the circle homeomorphisms f_1 and f_2 satisfy the conditions of Theorem 1.4. Then there exists natural number k_0 such that for intervals $[z_s,z_{s+1}]$, $s=1,2,3$ satisfying conditions (1)–(3) of Lemma 3.3, and for sufficiently large n the following inequality holds*

$$\Big|\frac{Dist(z_1,z_2,z_3,z_4;f_1^{q_n})}{Dist(h(z_1),h(z_2),h(z_3),h(z_4);f_2^{q_n})}-1\Big|\geqslant R_2>0,\tag{3.3}$$

where the constant R_2 depends only on f_1 and f_2.

Proof of Theorem 1.4. Let f_1 and f_2 be circle homeomorphisms satisfying the conditions of Theorem 1.4. The lift $H(x)$ of the conjugating map $h(x)$ is a continuous and monotone increasing function on R^1. Hence $H(x)$ has a finite derivative $H'(x)$ for almost all x with respect to Lebesgue measure. We claim that $h'(x)=0$ at all points x where the finite derivative exists. Suppose $h'(x_0)>0$ for some point $x_0\in S^1$. Fix $\varepsilon>0$. We take a triple of intervals $[z_s,z_{s+1}]\subset(x_0-\delta,\ x_0+\delta)$, $s=1,2,3$, satisfying the conditions of Lemma 3.4.

Using the assertion of Lemma 3.1 we obtain

$$\Big|Dist(z_1,z_2,z_3,z_4;h)-1\Big|\leqslant C_3\varepsilon,\tag{3.4}$$

$$\Big|Dist(f_1^{q_n}(z_1),f_1^{q_n}(z_2),f_1^{q_n}(z_3),f_1^{q_n}(z_4);h)-1\Big|\leqslant C_3\varepsilon.\tag{3.5}$$

Hence

$$\Big|\frac{Dist(z_1,z_2,z_3,z_4;h)}{Dist(f_1^{q_n}(z_1),f_1^{q_n}(z_2),f_1^{q_n}(z_3),f_1^{q_n}(z_4);h)}-1\Big|\leqslant C_4\varepsilon,\tag{3.6}$$

where the constant $C_4>0$ does not depend on ε and n.

Since h is conjugating f_1 and f_2 we can readily see that

$$Cr(h(f_1^{q_n}(z_1)), h(f_1^{q_n}(z_2)), h(f_1^{q_n}(z_3)), h(f_1^{q_n}(z_4))) =$$

$$= Cr(f_2^{q_n}(h(z_1)), f_2^{q_n}(h(z_2)), f_2^{q_n}(h(z_3)), f_2^{q_n}(h(z_4))).$$

Hence we obtain

$$\frac{Dist(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); h)}{Dist(z_1, z_2, z_3, z_4; h)} =$$

$$= \frac{Cr(h(f_1^{q_n}(z_1)), h(f_1^{q_n}(z_2)), h(f_1^{q_n}(z_3)), h(f_1^{q_n}(z_4)))}{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))} \times$$

$$\times \frac{Cr(z_1, z_2, z_3, z_4)}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))} = \frac{Cr(f_2^{q_n}(h(z_1)), f_2^{q_n}(h(z_2)), f_2^{q_n}(h(z_3)), f_2^{q_n}(h(z_4)))}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))} :$$

$$: \frac{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))}{Cr(z_1, z_2, z_3, z_4)} = \frac{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})}{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}.$$

This, together with (3.6) obviously implies that

$$\Big|\frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1\Big| \leqslant C_5\varepsilon,$$

where the constant $C_5 > 0$ does not depend on ε and n. This contradicts equation (3.3). Therefore Theorem 1.4 is completely proved. □

4. The proofs of Lemmas 3.2–3.4

Proof of Lemma 3.2. Denote

$$\psi_1(\xi_{f_1}(l)) = \frac{1}{1 - \xi_{f_1}(l) + \cdots + \xi_{f_1}^{2m_1}(l)},$$

and

$$\psi_2(\eta_{f_1}(l)) = \frac{e_{2m_1}\eta_{f_1}^{2m_1}(l) + e_{2m_1-1}\eta_{f_1}^{2m_1-1}(l) + \cdots + 1}{\eta_{f_1}^{2m_1}(l) + C_{2m_1}^1\eta_{f_1}^{2m_1-1}(l) + \cdots + 1}.$$

It is easy to check that for $\eta_{f_1}(l) > 0$ the function $\psi_2(\eta_{f_1}(l))$ is monotone increasing and $1 < \psi_2(\eta_{f_1}(l)) < 2m_1 + 1$. Obviously

$$\lim_{\xi_{f_1}(l)\to 0} \psi_1(\xi_{f_1}(l)) = 1, \quad \lim_{\eta_{f_1}(l)\to\infty} \psi_2(\eta_{f_1}(l)) = 2m_1 + 1.$$

Taking these remarks into account and using the explicit form of the functions $\psi_1(\xi_{f_1}(l))$ and $\psi_2(\eta_{f_1}(l))$ we can now estimate $\mid \psi_1 \cdot \psi_2 - (2m_1 + 1) \mid$. Firstly, we estimate ψ_2 for large value of $\eta_{f_1}(l)$. Using the explicit form of the function $\psi_2(\eta_{f_1}(l))$, we see that the inequality

$$|\psi_2 - (2m_1 + 1)| = O\left(\frac{1}{\eta_{f_1}(l)}\right) \leqslant R_3\left(\frac{1}{\eta_{f_1}(l)}\right), \tag{4.1}$$

where the constant $R_3 > 0$ depends only on f_1. If we choose $\eta_{f_1}(l)$ satisfying the inequality $R_2\left(\frac{1}{\eta_{f_1}(l)}\right) < \frac{L}{32}$, then

$$|\psi_2(\eta_{f_1}(l)) - (2m_1 + 1)| < \frac{L}{32},$$

for $\eta_{f_1}(l) > \dfrac{32R_3}{L}$.

We next estimate $|\psi_1 - 1|$ for small value of $\xi_{f_1}(l)$. Using the explicit form of the function $\psi_1(\xi_{f_1}(l))$, we see that $|\psi_1(\xi_{f_1}(l)) - 1| = O(\xi_{f_1}(l)) \leqslant R_4\xi_{f_1}(l)$. It follows from this together with (4.1) that $|\psi_1 \cdot \psi_2 - (2m_1+1)| \leqslant |\psi_2 - (2m_1+1)| + |\psi_2| \cdot |\psi_1 - 1| \leqslant \dfrac{L}{32} + (2m_1+1)R_4\xi_{f_1}(l)$. If we take

$$\zeta_1 := \min\Big\{\frac{L}{32(2m_1+1)R_5}, 1\Big\}, \quad M_1 := \max\Big\{\frac{32R_5}{L}, 1\Big\},$$

where $R_5 = \max\{R_3, R_4\}$, then for all $\xi_{f_1}(l) < \zeta_1$ and $\eta_{f_1}(l) > M_1$ the following inequality holds

$$|\psi_1 \cdot \psi_2 - (2m_1+1)| \leqslant \frac{L}{16}.$$

Similarly it can be shown that with

$$\zeta_2 := \min\Big\{\frac{L}{32(2m_2+1)R_6}, 1\Big\}, \quad M_2 := \max\Big\{\frac{32R_6}{L}, 1\Big\}, \tag{4.2}$$

and $\xi_{f_2}(l) < \zeta_2$ and $\eta_{f_2}(l) > M_2$, the second assertion of Lemma 3.2 holds. In (4.2) the constants $R_6 > 0$ depends only on f_2. Finally, if we set $\zeta_0 := \min\{\zeta_1, \zeta_2\}$ and $M_0 := \max\{M_1, M_2\}$, then Lemma 3.2 holds for $\xi_{f_1}(l)$, $\xi_{f_2}(l) \in [0, \zeta_0)$ and $\eta_{f_1}(l)$, $\eta_{f_2}(l) \geqslant M_0$. Lemma 3.2 is proved. □

Proof of Lemma 3.3. Firstly, we prove the third assertion of the lemma. By the construction of the points z_i, $i = 1, 2, 3, 4$, it implies that the intervals $[z_s, z_{s+1}]$ and $[h(z_s), h(z_{s+1})]$, $s = 1, 2, 3$ satisfy the 1) and 2) conditions of definition of "regularly" covering. We consider dynamical partition $\xi_n(x_{cr}^{(1)})$. According to Lemma 2.2 the intervals $\Delta_0^{(n)}(x_{cr}^{(1)})$ and $\Delta_0^{(n-1)}(x_{cr}^{(1)})$ are K-comparable, i.e. there exist constant $K > 1$ such that $K^{-1}\ell(\Delta_0^{(n-1)}(x_{cr}^{(1)})) \leqslant \ell(\Delta_0^{(n)}(x_{cr}^{(1)})) \leqslant K\ell(\Delta_0^{(n-1)}(x_{cr}^{(1)}))$. Thus it follows that there exists $k_1^{(1)} \in N$ such that the following inequality holds

$$\frac{\ell\big([x_{cr}^{(1)}, f_1^{q_{n+k_0+k_1^{(1)}}}(x_{cr}^{(1)})]\big)}{\ell\big([f_1^{q_{n+k_0}}(x_{cr}^{(1)}), x_{cr}^{(1)}]\big)} < \zeta_0. \tag{4.3}$$

Indeed, it is clear that

$$\frac{\ell\big(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\big)}{\ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big)} = \frac{1}{\frac{\ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big)}{\ell\big(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\big)}} \leqslant \frac{1}{1+\frac{1}{K}} = \frac{K}{K+1}.$$

Hence $\ell\big(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\big) \leqslant \frac{K}{K+1}\ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big)$. Using the last inequality we obtain that for any k

$$\ell\big(\Delta_0^{(q_{n+k_0+k})}(x_{cr}^{(1)})\big) \leqslant \Big(\frac{K}{K+1}\Big)^k \ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big).$$

Since $\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})$ and $\Delta_0^{(q_{n+k_0})}(x_{cr}^{(1)})$ are K-comparable, there exists a $k_1^{(1)} \in N$ such that the inequality (4.3) is true. Similarly, we can show that there exists a $k_2^{(1)} \in N$ such that the following inequality holds

$$\frac{\ell\big([x_{cr}^{(1)}, f_1^{q_{n+k_0+k_1}}(x_{cr}^{(1)})]\big)}{\ell\Big([f_1^{q_{n+k_0+k_1}}(x_{cr}^{(1)}), f_1^{q_{n+k_0+k_1^{(1)}}+q_{n+k_2^{(1)}}}(x_{cr}^{(1)})]\Big)} > M_0.$$

Similarly, it can be shown that with natural numbers $k_1^{(2)}$ and $k_1^{(2)}$ the inequalities

$$\frac{\ell\big([x_{cr}^{(2)}, f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)})]\big)}{\ell\big([f_2^{q_{n+k_0}}(x_{cr}^{(2)}), x_{cr}^{(2)}]\big)} < \zeta_0, \quad \frac{\ell\big([x_{cr}^{(2)}, f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)})]\big)}{\ell\Big([f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)}), f_2^{q_{n+k_0+k_1}+q_{n+k_2^{(2)}}}(x_{cr}^{(2)})]\Big)} > M_0$$

hold. If we take $k_1 = \max\{k_1^{(1)}, k_1^{(2)}\}$ and $k_2 = \max\{k_2^{(1)}, k_2^{(2)}\}$ then the third assertion of Lemma 3.3 holds for k_1 and k_2. By the definition of the points z_i, $i = 1,2,3$ it implies the first assertion of the lemma.

Let $\xi_n\big(\overline{x}_{cr}^{(1)}\big)$ be a dynamical partition of the point $\overline{x}_{cr}^{(1)}$. According to Lemma 2.2 the intervals $\Delta_0^{(n)}\big(\overline{x}_{cr}^{(1)}\big)$ and $\Delta_0^{(n-1)}\big(\overline{x}_{cr}^{(1)}\big)$ are K-comparable. Hence, it implies that the intervals $[z_s, z_{s+1}]$, $s = 1,2,3$ are pairwise $K^{k_1+k_2}$- comparable. It is easy to see that the intervals $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$, $s = 1,2,3$ are pairwise $K^{k_1+k_2}$-comparable. Obviously,

$$\frac{1}{K^{k_0+1}} \leqslant \frac{\ell\big(\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})\big)}{\ell\big([z_1, z_2]\big)} \leqslant K^{k_0+1}, \quad \frac{1}{K^{k_0+1}} \leqslant \frac{\ell\big(\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})\big)}{\ell\big([f_1^{q_n}(z_1), f_1^{q_n}(z_2)]\big)} \leqslant K^{k_0+1}.$$

Since the intervals $\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})$ and $\Delta_0^{(n-1)}\big(f_1^{-q_{n-1}}(\overline{x}_{cr}^{(1)})\big)$ are K-comparable and $x_0 \in \Delta_0^{(n-1)}\big(f_1^{-q_{n-1}}(\overline{x}_{cr}^{(1)})\big) \cup \Delta_0^{(n-1)}\big(\overline{x}_{cr}^{(1)}\big)$ we get

$$\max_{1\leqslant i\leqslant 4}\{\ell\big([f^{q_n}(z_i), x_0]\big),\, \ell([z_i, x_0])\} \leqslant (K+1)K^{k_0+1}\ell\big([z_1, z_2]\big).$$

If we take $R_1 = (K+1)K^{k_0+k_1+k_2}$, then we obtain the proof of the second assertion of Lemma 3.3 with constant R_1. Lemma 3.3 is proved. □

Proof of Lemma 3.4. Suppose, the triples of intervals ($[z_s, z_{s+1}]$, $s=1,2,3$) and ($[h(z_s), h(z_{s+1})]$, $s = 1,2,3$) satisfy the conditions of Lemma 3.3. We want to compare the distortion $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})$. We estimate only the first distortion, the second one can be estimated analogously. Obviously

$$Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) = \prod_{i=0}^{q_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1).$$

We denote

$$J_r(x_{cr}^{(1)}) = \Delta_0^{(r)}(x_{cr}^{(1)}) \cup \Delta_0^{(r-1)}(x_{cr}^{(1)}), \quad A = \{i : (f_1^i(z_1), f_1^i(z_4)) \cap J_r(x_{cr}^{(1)}) = \emptyset\},$$
$$B = \{i : (f_1^i(z_1), f_1^i(z_4)) \cap J_r(x_{cr}^{(1)}) \neq \emptyset\}.$$

It is clear that $A \cup B = \{0, 1, \ldots, q_n\}$.

Next we rewrite $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ in the form

$$Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) = \prod_{i\in A} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1)\times$$
$$\times \prod_{i\in B} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1). \tag{4.4}$$

We estimate the first factor in (4.4). Using the Lemmas 2.4 we obtain

$$\Big|\prod_{i\in A} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1\Big| = \Big|\prod_{i\in A}\Big(1 + O\big(\ell([f_1^i(z_1), f_1^i(z_4)])\big)^{1+\nu}\Big) - 1\Big| =$$
$$= \max_i \big(\ell([f_1^i(z_1), f_1^i(z_4)])\big)^{\nu}\, O\Big(\sum_{i\in A}\ell\big([f_1^i(z_1), f_1^i(z_4)]\big)\Big) = O(\lambda_{f_1}^{n\nu}),$$

where $\nu > 0$ and $0 < \lambda_{f_1} < 1$. We fix $\varepsilon > 0$. There exists $N_0 = N_0(\varepsilon) \geqslant 1$ such that for any $n \geqslant N_0$ the estimate

$$\Big|\prod_{i\in A} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) - 1\Big| < C_6\varepsilon \tag{4.5}$$

holds. We now estimate the second factor in (4.4). We rewrite the second factor in the following form

$$\prod_{i\in B} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) =$$

$$= \prod_{i\in B, i\neq l} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) \times Dist\big(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1\big). \tag{4.6}$$

By applying Lemmas 2.5 and 3.2 we obtain

$$|Dist\big(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1\big) - (2m_1+1)| < \frac{L}{8}. \tag{4.7}$$

Using Lemma 2.6 for the first factor in (4.6), we get

$$\Big|\prod_{i\in B, i\neq l} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) - 1\Big| = \Big|\prod_{i\in B, i\neq l}\bigg(1 + O\Big(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\Big)^2\bigg) - 1\Big| =$$

$$= \bigg|\exp\bigg\{\sum_{i\in B, i\neq l}\log\bigg(1 + O\Big(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\Big)^2\bigg)\bigg\} - 1\bigg| \leqslant const \sum_{i\in B, i\neq l}\bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg)^2 =$$

$$= const \sum_{q=0}^{n-r} \sum_{i:[f_1^i(z_1), f_1^i(z_4)]\subset (J_{n-q}(x_{cr}^{(1)})\setminus J_{n-q+1}(x_{cr}^{(1)})), i\neq l} \bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg)^2.$$

Obviously,

$$\sum_{i:[f_1^i(z_1), f_1^i(z_4)]\subset (J_{n-q}(x_{cr}^{(1)})\setminus J_{n-q+1}(x_{cr}^{(1)})), i\neq l} \bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg) = const$$

and it follows from Lemma 2.3 that $\dfrac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i} \leqslant const\, \lambda_{f_1}^{k_0+1+q}$. Consequently

$$\Big|\prod_{i\in B, i\neq l} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1\Big| \leqslant C_7\lambda_{f_1}^{k_0}, \tag{4.8}$$

where $C_7 > 0$ depends only on f_1.

Similarly one can show that for the triple of intervals $([h(z_s), h(z_{s+1})],\ s = 1, 2, 3)$ the following inequality

$$\Big|\prod_{i\in B, i\neq l} Dist(f_2^i(h(z_1)), f_2^i(h(z_2)), f_2^i(h(z_3)), f_2^i(h(z_4)); f_2) - 1\Big| \leqslant C_8\lambda_{f_2}^{k_0}, \tag{4.9}$$

where $C_8 > 0$ depends only on f_2 and $0 \leqslant \lambda_{f_2} \leqslant 1$ is defined in Lemma 2.3.

If we choose

$$k_0 = \max\bigg\{\bigg[\log_{\lambda_{f_1}} \frac{L}{(16m_1+8+L)C_7}\bigg] + 1, \bigg[\log_{\lambda_{f_2}} \frac{L}{(16m_2+8+L)C_8}\bigg] + 1\bigg\},$$

where constants $0 \leqslant \lambda_{f_1}, \lambda_{f_2} \leqslant 1$ are defined in Lemma 2.3, then from the relations (4.4)–(4.8) it implies that for sufficiently large n

$$|Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) - (2m_1 + 1)| < \frac{L}{4}. \tag{4.10}$$

Similarly

$$|Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n}) - (2m_2 + 1)| < \frac{L}{4}. \tag{4.11}$$

The inequalities (4.10) and (4.11) implies

$$\frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1 \geqslant \frac{8(m_1 - m_2) - 2L}{8m_2 + L + 4} > 0, \tag{4.12}$$

if $m_1 > m_2$, and

$$\frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1 \leqslant \frac{8(m_1 - m_2) + 2L}{8m_2 - L + 4} < 0, \tag{4.13}$$

if $m_1 < m_2$. If we set

$$R_2 := \min\left\{\frac{|8(m_1 - m_2) - 2L|}{8m_2 - L + 4}, \frac{|8(m_1 - m_2) + 2L|}{8m_2 + L + 4}\right\}, \tag{4.14}$$

then it follows from (4.12)–(4.14) that the assertion of the lemma holds.

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О сопряжение между двумя критическими отображениями окружности

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Аннотация. В статье изучается сопряжение между двумя критическими гомеоморфизмами окружности с иррациональным числом вращения. Пусть f_i, $i = 1, 2$ являются C^3-гомеоморфизмы окружности с критической точкой $x_{cr}^{(i)}$ порядка $2m_i + 1$. Доказано, что если $2m_1 + 1 \neq 2m_2 + 1$, то сопряжение между f_1 и f_2 — сингулярная функция.

Ключевые слова: гомеоморфизм окружности, критическая точка, сопрягащий гомеоморфизм, число вращения, сингулярная функция.