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On Transcendental Systems of Equations

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Abstract. Several types of transcendental systems of equations are considered: the simplest ones, special, and general. Since the number of roots of such systems, as a rule, is infinite, it is necessary to study power sums of the roots of negative degree. Formulas for finding residue integrals, their relation to power sums of a negative degree of roots and their relation to residue integrals (multidimensional analogs of Waring's formulas) are obtained. Various examples of transcendental systems of equations and calculation of multidimensional numerical series are given.

Keywords: transcendental systems of equations, power sums of roots, residue integral.

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Introduction

Based on the multidimensional logarithmic residue, for systems of non-linear algebraic equations in $\mathbb{C}^n$ formulas for finding power sums of the roots of a system without calculating the roots themselves were earlier obtained (see [1–3]). For different types of systems such formulas have different forms. Based on this, a new method for the study of systems of algebraic equations in $\mathbb{C}^n$ have been constructed. It arose in the work of L. A. Aizenberg [1], its development was continued in monographs [2–4]. The main idea is to find power sums of roots of systems (for positive powers) and then, to use one-dimensional or multidimensional recurrent Newton formulas (see [5]). Unlike the classical method of elimination, it is less labor-intensive and does not increase the multiplicity of roots. It is based on the formula (see [1]) for a sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations without finding the roots themselves.

For systems of transcendental equations, formulas for the sum of the values of the roots of the system, as a rule, cannot be obtained, since the number of roots of a system can be infinite and a series of coordinates of such roots can be diverging. Nevertheless, such transcendental systems of equations may very well arise, for example, in the problems of chemical kinetics [6, 7]. Thus, this is an important task to consider such systems.

In the works [8–21] power sums of roots are considered for a negative power for different systems of non-algebraic (transcendental) equations. To compute these power sums, a residue integral is used, the integration is carried out over skeletons of polycircles centered at the origin. Note that this residue integral is not, generally speaking, a multidimensional logarithmic residue or a Grothendieck residue. For various types of lower homogeneous systems of functions included in the system, formulas are given for finding residue integrals, their relationship with power sums of the roots of the system to a negative degree are established.

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The paper [12] investigated more complex systems in which the lower homogeneous parts are decomposed into linear factors and integration cycles in residue integrals are constructed from these factors. In [11], a system is studied that arises in the Zel'dovich–Semenov model (see [6,7]) in chemical kinetics.

The object of this study is transcendental systems of equations in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations: formulas are found for calculating the residue integrals, power sums of roots for a negative power, their relationship with the residue integrals are established. See [21].

1. The simplest transcendental systems of equations

Consider a system of functions of the form

$$f_1(z),\ f_2(z), \dots, f_n(z),$$

holomorphic in a neighborhood of the point $0 \in \mathbb{C}^n$, $z = (z_1, z_2, \dots, z_n)$ and having the following form:

$$f_j(z) = z^{\beta^j} + Q_j(z), \quad j = 1, 2, \dots, n, \tag{1}$$

where $\beta^j = (\beta_1^j, \beta_2^j, \dots, \beta_n^j)$ is a multi-index with integer non-negative coordinates, $z^{\beta^j} = z_1^{\beta_1^j} \cdot z_2^{\beta_2^j} \cdots z_n^{\beta_n^j}$ and $\|\beta^j\| = \beta_1^j + \beta_2^j + \ldots + \beta_n^j = k_j$, $j = 1, 2, \dots, n$. The functions Q_j can be expanded in absolutely and uniformly converging Taylor series in a neighborhood of the origin of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a_\alpha^j z^\alpha, \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \geqslant 0$, $\alpha_j \in \mathbb{Z}$, a $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

Consider the cycles $\gamma(r) = \gamma(r_1, r_2, \dots, r_n)$, which are skeletons of polydisks:

$$\gamma(r) = \{z \in \mathbb{C}^n : |z_s| = r_s,\ s = 1, 2, \dots, n\}, \quad r_1 > 0, \dots, r_n > 0.$$

For sufficiently small r_j, the cycles $\gamma(r)$ lie in the domain of holomorphy of functions f_j, therefore the series

$$\sum_{\|\alpha\| > k_j} |a_\alpha^j| r_1^{\alpha_1} \cdots r_n^{\alpha_n}, \quad j = 1, \dots n,$$

converge. Then on the cycle $\gamma(tr) = \gamma(tr_1, tr_2, \dots, tr_n)$ for sufficiently small $t > 0$ we have

$$|z|^{\beta^j} = t^{k_j} \cdot r_1^{\beta_1^j} \cdot r_2^{\beta_2^j} \cdots r_n^{\beta_n^j} = t^{k_j} \cdot r^{\beta^j},$$

and

$$|Q_j(z)| = \left| \sum_{\|\alpha\| > k_j} a_\alpha^j z^\alpha \right| \leqslant$$

$$\leqslant \sum_{\|\alpha\| > k_j} t^{\|\alpha\|} |a_\alpha^j| r^\alpha \leqslant t^{k_j + 1} \sum_{\|\alpha\| \geqslant 0} |a_\alpha^j| r^\alpha, \quad j = 1, \dots, n.$$

Therefore, for such t on the cycle $\gamma(tr)$ the inequalities hold

$$|z|^{\beta^j} > |Q_j(z)|, \quad j = 1, 2, \dots, n. \tag{3}$$

Thus

$$f_j(z) \neq 0 \quad \text{on} \quad \gamma(tr), \quad j = 1, 2, \dots, n.$$

In what follows, we will assume that $t = 1$. Consider a system of equations of the form

$$\begin{cases} f_1(z) = 0, \\ f_2(z) = 0, \\ \ldots\ldots\ldots \\ f_n(z) = 0. \end{cases} \tag{4}$$

From (3) it follows that for sufficiently small r_j the following integrals are defined

$$\int\limits_{\gamma(r)} \frac{1}{z^{\beta+I}} \cdot \frac{df}{f} = \int\limits_{\gamma(r_1,r_2,\ldots,r_n)} \frac{1}{z_1^{\beta_1+1} \cdot z_2^{\beta_2+1} \cdots z_n^{\beta_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n},$$

where $\beta_1 \geqslant 0,\ \beta_2 \geqslant 0, \ldots, \beta_n \geqslant 0,\ \beta_j \in \mathbb{Z},\ I = (1, 1, \ldots, 1)$. We call them *residue integrals* ([22]).

The logarithmic residue theorem does not apply to these integrals, and they are not standard Grothendieck residues.

Since condition (3) is satisfied on the cycles $\gamma(r)$, by the Cauchy–Poincaré theorem, these integrals are independent of $(r_1, \ldots, r_n)$. Let us denote

$$J_\beta = \frac{1}{(2\pi i)^n} \int\limits_{\gamma(r)} \frac{1}{z^{\beta+I}} \cdot \frac{df}{f}.$$

Theorem 1. *Under the assumptions made, for a function f_j of the form (1), (2) the next formulas are valid*

$$J_\beta = \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1+\ldots+k_n)} \frac{(-1)^{\|\alpha\|}}{(\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n)!} \times$$

$$\times \frac{\partial^k (\Delta \cdot Q^\alpha)}{\partial z^{\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n}} \Bigg|_{z=0} =$$

$$= \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1+\ldots+k_n)} (-1)^{\|\alpha\|} \mathfrak{M} \left[\frac{\Delta \cdot Q^\alpha}{z^{\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n}} \right],$$

where $k = \|\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n\|$, $\beta! = \beta_1! \cdot \beta_2! \cdots \beta_n!$, $Q^\alpha = Q_1^{\alpha_1} \cdot Q_2^{\alpha_2} \cdots Q_n^{\alpha_n}$, $\dfrac{\partial^{\|\beta\|}}{\partial z^\beta} = \dfrac{\partial^{\|\beta\|}}{\partial z_1^{\beta_1} \partial z_2^{\beta_2} \cdots \partial z_n^{\beta_n}}$, *$\Delta$ is the Jacobian of the system of functions (1) and, finally, $\mathfrak{M}$ is a linear functional assigning to the Laurent series (under the sign of the functional $\mathfrak{M}$) its free term.*

Corollary 1. *If all $\beta^j = (0, 0, \ldots, 0)$, $j = 1, \ldots, n$, then the integral*

$$J_\beta = \sum_{\|\alpha\| \leqslant \|\beta\|} (-1)^{\|\alpha\|} \mathfrak{M} \left[\frac{\Delta \cdot Q^\alpha}{z^\beta} \right] = \sum_{\|\alpha\| \leqslant \|\beta\|} \frac{(-1)^{\|\alpha\|}}{\beta!} \frac{\partial^{\|\beta\|}}{\partial z^\beta} (\Delta \cdot Q^\alpha) \Bigg|_{z=0}.$$

Our further goal is to relate the considered integrals to power sums of roots of the system (4). To do this, we will narrow the function class f_j. First, we take as functions Q_j $(j = 1, 2, \ldots, n)$ polynomials of the form

$$Q_j(z) = \sum_{\alpha \in M_j} a_\alpha^j z^\alpha, \tag{5}$$

where M_j is a finite set of multi-indices such that for $\alpha \in M_j$ the coordinates $\alpha_k \leqslant \beta^j_k$, $k = 1, 2, \dots, n,\ k \neq j$. (But it is still assumed that $\|\alpha\| > k_j$ for all $\alpha \in M_j$).

Denote

$$\sigma_{\beta+I} = \sigma_{(\beta_1+1,\beta_2+1,\dots,\beta_n+1)} = \sum_{k=1}^{M} \frac{1}{z_{1(k)}^{\beta_1+1} \cdot z_{2(k)}^{\beta_2+1} \cdots z_{n(k)}^{\beta_n+1}},$$

where $\beta = (\beta_1, \dots, \beta_n)$ is some multi-index. This expression is a power sum of roots that do not lie on the coordinate planes of the system (4), but in negative power (or a power sum of the reciprocal of the roots).

Theorem 2. *For the system (4) with functions f_j of the form (1) and polynomials Q_j of the form (5) the next formulas are valid*

$$J_\beta = (-1)^n \sigma_{\beta+I},$$

i.e.

$$\sigma_{\beta+I} = \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1+\ldots+k_n)} (-1)^{\|\alpha\|+n} \mathfrak{M}\left[\frac{\Delta \cdot Q^\alpha}{z^{\beta+(\alpha_1+1)\beta^1+\ldots+(\alpha_n+1)\beta^n}}\right].$$

Consider a system of equations in three complex variables

$$\begin{cases} f_1(z_1, z_2, z_3) = 1 + a_1 z_1 = 0, \\ f_2(z_1, z_2, z_3) = 1 + b_1 z_1 + b_2 z_2 = 0, \\ f_3(z_1, z_2, z_3) = 1 + c_1 z_1 + c_2 z_2 + c_3 z_3 = 0. \end{cases} \tag{6}$$

Here the functions do not satisfy the conditions of Theorem 2, but they satisfy the conditions of Theorem 1. We find the integral

$$J_{(\beta,0,0)} = \frac{1}{(2\pi i)^3} \int\limits_{\gamma(r)} \frac{1}{z_1^{\beta_1+1} z_2 z_3} \cdot \frac{df_1 \wedge df_2 \wedge df_3}{f_1 \cdot f_2 \cdot f_3} =$$

$$= \frac{1}{(2\pi i)^3} \int\limits_{\gamma(r)} \frac{1}{z_1^{\beta_1+1} z_2 z_3} \cdot \frac{a_1 b_2 c_3 dz_1 \wedge dz_2 \wedge dz_3}{(1 + a_1 z_1)(1 + b_1 z_1 + b_2 z_2)(1 + c_1 z_1 + c_2 z_2 + c_3 z_3)} =$$

$$= \frac{a_1 b_2 c_3}{\beta!} \cdot \frac{\partial^\beta}{\partial z_1^\beta} \cdot \left[\frac{1}{(1 + a_1 z_1)(1 + b_1 z_1)(1 + c_1 z_1)}\right]\Bigg|_{z_1=0}.$$

To calculate the last derivative, we transform the expression

$$\frac{1}{(1 + a_1 z_1)(1 + b_1 z_1)(1 + c_1 z_1)} = \frac{A}{1 + a_1 z_1} + \frac{B}{1 + b_1 z_1} + \frac{C}{1 + c_1 z_1},$$

$$\begin{cases} A = \dfrac{a_1^2}{(a_1 - b_1)(a_1 - c_1)}, \\ B = -\dfrac{b_1^2}{(a_1 - b_1)(b_1 - c_1)}, \\ C = \dfrac{c_1^2}{(a_1 - c_1)(b_1 - c_1)}, \end{cases} \tag{7}$$

assuming that $a_1 \neq b_1,\ a_1 \neq c_1,\ b_1 \neq c_1$, then

$$J_{(\beta,0,0)} = (-1)^\beta a_1 b_2 c_3 \times$$

$$\times\left[\frac{a_1^{\beta+2}}{(a_1-b_1)(a_1-c_1)}-\frac{b_1^{\beta+2}}{(a_1-b_1)(b_1-c_1)}+\frac{c_1^{\beta+2}}{(a_1-c_1)(b_1-c_1)}\right].$$

The roots of the system (6)

$$z_1=-\frac{1}{a_1},\quad z_2=\frac{b_1-a_1}{a_1b_2},\quad z_3=\frac{b_2c_1-b_1c_2+a_1c_2-a_1b_2}{a_1b_2c_3}.$$

If the numerator in the formula for z_3 is 0, then this root lies on a coordinate plane, and we should not take it into consideration.

Therefore, the power sum

$$\sigma_{(\beta_1+1,1)}=\frac{(-1)^{\beta+1}a_1^{\beta+3}b_2^2c_3}{(b_1-a_1)(b_2c_1-b_1c_2+a_1c_2-a_1b_2)},$$

i.e.

$$J_{(\beta,0,0)}=-\sigma_{(\beta_1+1,1)}-\frac{(-1)^{\beta}a_1^2b_2^2c_3b_1^{\beta+1}}{(b_1-a_1)(b_2c_1-b_1c_2+a_1c_2-a_1b_2)}+$$
$$+\frac{(-1)^{\beta+1}a_1b_2c_2c_3}{(b_2c_1-b_1c_2+a_1c_2-a_1b_2)}\times$$
$$\times\left[-a_1c_2\cdot\frac{a_1^{\beta+1}-c_1^{\beta+1}}{a_1-c_1}+(b_1c_2-b_2c_1)\cdot\frac{b_1^{\beta+1}-c_1^{\beta+1}}{b_1-c_1}\right]. \tag{8}$$

We recall the well-known expansions of the sine into an infinite product and the power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}}=\prod_{k=1}^{\infty}\left(1-\frac{z}{k^2\pi^2}\right)=\sum_{k=0}^{\infty}\frac{(-1)^kz^k}{(2k+1)!},$$

which converge uniformly and absolutely on the complex plane.

Consider the system of equations

$$\begin{cases} f_1(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_1-a^2}}{\sqrt{z_1-a^2}}=\prod\limits_{k=1}^{\infty}\left(1-\dfrac{z_1-a^2}{k^2\pi^2}\right)=0,\\ f_2(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_2-z_1-a^2}}{\sqrt{z_2-z_1-a^2}}=\prod\limits_{m=1}^{\infty}\left(1-\dfrac{z_2-z_1-a^2}{m^2\pi^2}\right)=0,\\ f_3(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_3-z_2-a^2}}{\sqrt{z_3-z_2-a^2}}=\prod\limits_{s=1}^{\infty}\left(1-\dfrac{z_3-z_2-a^2}{s^2\pi^2}\right)=0. \end{cases} \tag{9}$$

Each of the functions of this system can be expanded into an infinite product of functions from system (6).

The roots of the system (9) are the points $(\pi^2k^2+a^2,\pi^2(k^2+m^2)+2a^2,\pi^2(k^2+m^2+s^2)+3a^2)$, $k,m,s\in\mathbb{N}$. Therefore, the power sum $\sigma_{(\beta+1,1,1)}$ is equal to the sum of the series

$$\sigma_{(\beta+1,1,1)}=\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2k^2+a^2)^{(\beta+1)}(\pi^2(k^2+m^2)+2a^2)(\pi^2(k^2+m^2+s^2)+3a^2)},$$

which converges as $a\neq\pi ki$.

For the system (9)

$$f_1=\sum_{k=0}^{\infty}\frac{(-1)^k(z_1-a^2)^k}{(2k+1)!},$$

$$f_2 = \sum_{k=0}^{\infty} \frac{(-1)^k (z_2 - z_1 - a^2)^k}{(2k+1)!},$$
$$f_3 = \sum_{k=0}^{\infty} \frac{(-1)^k (z_3 - z_2 - a^2)^k}{(2k+1)!},$$

therefore

$$f_1(0,0,0) = f_2(0,0,0) = f_3(0,0,0) = \sum_{k=0}^{\infty} \frac{a^2 k}{(2k+1)!} = \frac{\mathrm{sha}}{a}.$$

Therefore, to apply the formula from Theorem 1, we need to divide the functions f_1, f_2, f_3 by these constants (normalize).

Consider the integral $J_{(\eta,0,0)}$ for the system (9). Using the form of the roots of the system (9), we obtain that

$$a_1 = -\frac{1}{\pi^2 k^2 + a^2}, \quad b_1 = \frac{1}{\pi^2 m^2 + a^2}, \quad b_2 = -\frac{1}{\pi^2 m^2 + a^2}, \quad c_2 = \frac{1}{\pi^2 s^2 + a^2}, \quad c_3 = -\frac{1}{\pi^2 s^2 + a^2}.$$

$$J_{(\beta,0,0)} = -\sigma_{(\beta+1,1,1)} + (-1)^{\beta+1} \times$$
$$\times \sum_{k,m,s=1}^{\infty} \frac{1}{(m^2\pi^2 + a^2)^{\beta+1}(\pi^2(k^2+m^2) + 2a^2)(\pi^2(k^2+m^2+s^2) + 3a^2)} +$$
$$+(-1)^{\beta+1} \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2 s^2 + a^2)(\pi^2(k^2+m^2+s^2)+3a^2)} \times \left[\frac{1}{(m^2\pi^2+a^2)^{\beta+1}} + \frac{(-1)^\beta}{(k^2\pi^2+a^2)^{\beta+1}}\right],$$

$$J_{(\beta,0,0)} = \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2(k^2+m^2)+2a^2)(\pi^2(k^2+m^2+s^2)+3a^2)} \times$$
$$\times \left[\frac{-1}{(k^2\pi^2+a^2)^{\beta+1}} + \frac{(-1)^{\beta+1}}{(m^2\pi^2+a^2)^{\beta+1}}\right] +$$
$$+(-1)^{\beta+1} \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2 s^2 + a^2)(\pi^2(k^2+m^2+s^2)+3a^2)} \times \left[\frac{1}{(m^2\pi^2+a^2)^{\beta+1}} + \frac{(-1)^\beta}{(k^2\pi^2+a^2)^{\beta+1}}\right].$$

$\times$

For odd β the integral $J_{(\beta,0,0)} = 0$, and for even $\beta = 2n$ we obtain the following formula for finding the sum of the series

$$J_{(2n,0,0)} = \sum_{\|\alpha\| \leqslant 2n} \mathfrak{M}\left[\frac{\Delta \cdot Q^\alpha}{z_1^{2n}}\right] = -2\sigma_{(2n+1,1,1)} -$$
$$-2 \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2 k^2 + a^2)^{(2n+1)} \cdot (\pi^2 s^2 + a^2) \cdot (\pi^2(k^2+m^2+s^2)+3a^2)}.$$

Let us calculate, for example:

$$J_{(0,0,0)} = \mathfrak{M}[\Delta] = \mathfrak{M}\left[\frac{\partial f_1}{\partial z_1} \cdot \frac{\partial f_2}{\partial z_2} \cdot \frac{\partial f_3}{\partial z_3}\right] = \left(\frac{1}{2a^2} - \frac{1}{2a}\mathrm{ctha}\right)^3.$$

Applying the identity

$$\frac{1}{(\pi^2 k^2 + a^2)(\pi^2(k^2+m^2)+2a^2)} + \frac{1}{(\pi^2 m^2 + a^2)(\pi^2(k^2+m^2)+2a^2)} =$$

$$= \frac{1}{(\pi^2k^2 + a^2)(\pi^2m^2 + a^2)},$$

We get that

$$2\sigma_{(1,1,1)} = \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2k^2 + a^2)\cdot(\pi^2s^2 + a^2)\cdot(\pi^2(k^2 + m^2 + s^2) + 3a^2)}.$$

Thus, we get

$$\sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2k^2 + a^2)\cdot(\pi^2(k^2 + m^2) + 2a^2)\cdot(\pi^2(k^2 + m^2 + s^2) + 3a^2)}) =$$

$$= \frac{(a\text{ctha} - 1)^3}{48a^6}.$$

2. Special systems of equations

Consider a system of functions $f_1(z),\ f_2(z), \dots, f_n(z)$ of the form

$$\begin{cases} f_1(z) = (1 - a_{11}z_1)^{m_{11}} \cdots (1 - a_{1n}z_n)^{m_{1n}} + Q_1(z), \\ f_2(z) = (1 - a_{21}z_1)^{m_{21}} \cdots (1 - a_{2n}z_n)^{m_{2n}} + Q_2(z), \\ \dots \\ f_n(z) = (1 - a_{n1}z_1)^{m_{n1}} \cdots (1 - a_{nn}z_n)^{m_{nn}} + Q_n(z), \end{cases} \tag{10}$$

where m_{ij} are natural numbers, a_{ij} are complex numbers that are different for fixed j, $Q_i(z)$ are entire functions, $i = 1, \dots, n$. Let $J = (j_1, \dots, j_n)$ be a multi-index, where $(j_1 \dots j_n)$ is a permutation of $(1, \dots, n)$. Let us define $a_J = (a_{1j_1}, \dots, a_{nj_n})$ for a multi-index J. We denote

$$q_i(z_1, \dots, z_n) = (1 - a_{i1}z_1)^{m_{i1}} \cdots (1 - a_{in}z_n)^{m_{in}}, \quad i = 1, \dots, n, \tag{11}$$

then the system (10) can be rewritten as

$$f_i(z_1, \dots, z_n) = q_i(z_1, \dots, z_n) + Q_i(z_1, \dots, z_n), \quad i = 1, \dots, n. \tag{12}$$

For each m we define the function

$$h_m(z) = \begin{cases} q_m(z) & \text{if } a_{mj} \neq 0 \ \text{ for all } j; \\ q_m(z)\cdot\dfrac{1}{z_{j_1}}\cdot\ldots\cdot\dfrac{1}{z_{j_k}} & \text{if } a_{mj_1} = \ldots = a_{ij_k} = 0. \end{cases} \tag{13}$$

A system

$$h_m(z) = 0, \quad i = 1, \dots, n, \tag{14}$$

has $n!$ isolated roots in $\overline{\mathbb{C}}^n$, where $\overline{\mathbb{C}}^n = \overline{\mathbb{C}} \times \cdots \times \overline{\mathbb{C}}$. Recall that $\overline{\mathbb{C}}$ is a compactification of the complex plane $\mathbb{C}$ (the Riemann sphere). Then $\overline{\mathbb{C}}^n$ is one of the well-known compactifications of $\mathbb{C}^n$ (the function theory space). The roots of the system (14) are equal

$$\tilde{a}_J = \begin{cases} \left(1/a_{1j_1}, \dots, 1/a_{nj_n}\right) & \text{if } a_{kj_k} \neq 0 \text{ for } \ k = 1, \dots, n, \\ \left(1/a_{1j_1}, \dots, \infty_{[i_1]}, \dots, \infty_{[i_k]}, \dots, 1/a_{nj_n}\right) & \text{if } a_{i_1j_{i_1}} = \ldots = a_{i_kj_{i_k}} = 0, \end{cases}$$

where $k, j = 1, \dots, n$. If $a_{j_1,i_1} = 0$, then in $\tilde{a}_J$ we write ∞, this is the point at infinity in $\overline{\mathbb{C}}$.

By Γ_h we denote the (global) cycle:

$$\Gamma_h = \{z \in \mathbb{C}^n \colon |h_m| = r_m,\ r_i > 0,\ m = 1, \ldots, n\}. \tag{15}$$

In the case when all $a_{k,j_k} \neq 0$, we define the (local) cycle $\Gamma_{h,\tilde{a}_J}$ as follows

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ |1 - a_{2j_2} z_2| = r_2, \\ \ldots \\ |1 - a_{nj_n} z_n| = r_n. \end{cases} \tag{16}$$

If $a_{i_1 j_{i_1}} = \ldots = a_{i_k j_{i_k}} = 0$ for some $i_1, \ldots, i_k$, then $\Gamma_{h,\tilde{a}_J}$ is defined as

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ \ldots \\ |1/z_{i_1}| = r_{i_1}, \\ \ldots \\ |1/z_{i_k}| = r_{i_k}, \\ \ldots \\ |1 - a_{nj_n} z_n| = r_n. \end{cases} \tag{17}$$

Lemma 1. *For sufficiently small r_m, the global cycle Γ_h has connected components (local cycles) in the neighborhood of the roots a_J. Moreover, Γ_h is homologous to the sum of local cycles $\Gamma_{h,\tilde{a}_J}$.*

Consider the system of equations

$$F_m(z, t) = q_m(z) + t \cdot Q_m(z) = 0, \quad m = 1, \ldots, n, \tag{18}$$

depending on the real parameter $t \geqslant 0$. Let $r_1 > 0, \ldots, r_n > 0$ be fixed real numbers. Then, for sufficiently small $t > 0$, the inequalities

$$\big|q_m(z)\big| > \big|t \cdot Q_m(z)\big|, \quad m = 1, \ldots, n$$

on cycles

$$\Gamma_h = \{z \in \mathbb{C}^n \colon |h_m| = r_m, \quad m = 1, \ldots, n\}$$

because Γ_h is compact.

We denote by $J_\gamma(t)$ the residue integral

$$J_\gamma(t) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_h} \frac{1}{z^{\gamma+I}} \cdot \frac{dF}{F} = \tag{19}$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_h} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{dF_1}{F_1} \wedge \frac{dF_2}{F_2} \wedge \ldots \wedge \frac{dF_n}{F_n},$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, and $I = (1, 1, \ldots, 1)$.

We denote by $\Delta = \Delta(t)$ the Jacobian of the system of functions $F_1(z, t), \ldots, F_n(z, t)$ in the variables $z_1, \ldots, z_n$.

Theorem 3. *Under the assumptions made on the functions F_i defined by formulas (18), the following expressions for $J_\gamma(t)$ are absolutely convergent (for sufficiently small t) series:*

$$J_\gamma(t) = \sum_J{}' \sum_\alpha \frac{(-1)^{s(J)}(-t)^{||\alpha||+\|\beta(\alpha,J)|+n}}{\beta(\alpha,J)!\cdot a_J^{\beta+I}} \times$$

$$\times \frac{\partial^{||\beta(\alpha(J)||}}{\partial z^{\beta(\alpha,J)}} \left[\frac{\Delta(t)}{z_1^{\gamma_1+1}\cdot\ldots\cdot z_n^{\gamma_n+1}} \cdot \frac{Q^\alpha}{q^{\alpha+I}(J)} \right]\Bigg|_{z=\tilde{a}_J},$$

where $(-1)^{s(J)}$ is the parity of the permutation J, $\alpha=(\alpha_1,\ldots,\alpha_n)$ is a multi-index of length n, $q^{\alpha+I}(J)=q_1^{\alpha_1+1}[j_1]\cdots q_n^{\alpha_n+1}[j_n]$, and $q_s[j_s]$ is the product of all $(1-a_{j1}z_1)^{m_{j1}}\cdots(1-a_{jn}z_n)^{m_{jn}}$, except $(1-a_{sj_s}z_s)^{m_{sj_s}}$,

$$\beta(\alpha,J)=\big(m_{1j_1}(\alpha_{j_1}+1)-1,\ldots,m_{nj_n}(\alpha_{j_n}+1)-1\big),$$

$$\beta(\alpha,J)! = \prod_p \big(m_{pj_p}(\alpha_{j_p}+1)-1\big)!,$$

$$a_j^{\beta+I} = a_{1j_1}^{m_{1j_1}(\alpha_{j_1}+1)}\cdots a_{nj_n}^{m_{nj_n}(\alpha_{j_n}+1)},$$

$$\frac{\partial^{||\beta(\alpha(J)||}}{\partial z^{\beta(\alpha,J)}} = \frac{\partial^{m_{1j_1}(\alpha_{j_1}+1)-1+\ldots+m_{nj_n}(\alpha_{j_n}+1)-1}}{\partial z_1^{m_{1j_1}(\alpha_{j_1}+1)-1}\ldots\partial z_n^{m_{nj_n}(\alpha_{j_n}+1)-1}}.$$

The dash at the summation sign means that the summation is performed over all multi-indices J for which there are no zero coordinates in a_J.

Suppose $Q_s(z)$ are polynomials:

$$Q_s(z) = z_1\cdots z_n \sum_{|\alpha\|\geqslant 0} C_\alpha^s z^\alpha \quad s=1,\ldots,n, \tag{20}$$

where α is a multi-index, $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, and $\deg_{z_j} Q_s \leqslant m_{sj}$, $s,j=1,\ldots,n$ for all nonzero a_{sj}. If $a_{sj}=0$, then there are no restrictions on the degree $\deg_{z_j} Q_s$.

Assuming that all $w_j\neq 0$, we make the change $z_j=\dfrac{1}{w_j}$, $j=1,\ldots,n$ in the functions

$$F_s(z,t) = \big(q_s(z)+t\cdot Q_s(z)\big), \quad s=1,\ldots,n.$$

Hence, for $s=1,\ldots,n$ we have

$$F_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right) = q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)+t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)=$$
$$=\left(1-a_{s1}\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(1-a_{sn}\frac{1}{w_n}\right)^{m_{sn}}+t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right)=$$
$$=\left(\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(\frac{1}{w_n}\right)^{m_{sn}}\cdot(w_1-a_{s1})^{m_{s1}}\cdots(w_n-a_{sn})^{m_{sn}}+t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right).$$

Then we arrive at the formula

$$F_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right) = \left(\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(\frac{1}{w_n}\right)^{m_{sn}}\cdot\left(\widetilde{q}_s(w)+t\cdot\widetilde{Q}_s(w)\right), \tag{21}$$

where $\widetilde{q}_s$ are functions

$$\widetilde{q}_s = (w_1-a_{s1})^{m_{s1}}\cdots(w_n-a_{sn})^{m_{sn}},$$

and $\widetilde{Q}_i$ are polynomials

$$\widetilde{Q}_s = w_1^{m_{s1}} \cdots w_n^{m_{sn}} \cdot Q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right).$$

From the formula (20) we obtain

$$\deg_{w_j} \widetilde{Q}_s < m_{sj}, \quad s, j = 1, \dots, n.$$

We denote

$$\widetilde{F_s} = \widetilde{F_s}(w, t) = \widetilde{q}_s(w) + t \cdot \widetilde{Q}_s(w), \quad s = 1, \dots, n. \tag{22}$$

If $0 \leqslant t \leqslant 1$, then the system (22) has a finite number of roots in $\mathbb{C}^n$ that depend on t. Moreover, (22) has no infinite roots in $\overline{\mathbb{C}}^n$.

Consider the cycle

$$\widetilde{\Gamma}_h = \left\{ w \in \mathbb{C}^n : \left| h_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right| = \varepsilon_s, \quad s = 1, \dots, n \right\},$$

for t close enough to zero. The compactness of the cycle $\widetilde{\Gamma}_h$ implies

$$\left| q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right| > \left| t \cdot Q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right|, \quad s = 1, \dots, n.$$

Therefore, $\widetilde{\Gamma}_h$ is homologous to the sum of cycles $\widetilde{\Gamma}_{h,\tilde{a}_J}$:

$$\begin{cases} \left|1 - a_{1j_1}\dfrac{1}{w_1}\right| = \varepsilon_1, \\ \left|1 - a_{2j_2}\dfrac{1}{w_2}\right| = \varepsilon_2, \\ \dots\dots\dots\dots \\ \left|1 - a_{nj_n}\dfrac{1}{w_n}\right| = \varepsilon_n, \end{cases} \tag{23}$$

obtained from the cycles $\Gamma_{h,\tilde{a}_J}$ by replacing $z_j = \dfrac{1}{w_j}$.

The equation

$$\left|1 - a_{js_j}\frac{1}{w_j}\right| = \varepsilon$$

defines a circle. Indeed, we rewrite it as

$$|w_j - a_{js_j}| = \varepsilon |w_j| \quad \text{or} \quad |w_j - a_{js_j}|^2 = \varepsilon^2 |w_j|^2,$$

then

$$(1 - \varepsilon^2)\left| w_j - \frac{a_{js_j}}{1 - \varepsilon^2}\right|^2 = \frac{\varepsilon^2 \cdot |a_{js_j}|^2}{(1 - \varepsilon^2)}$$

or

$$\left| w_j - \frac{a_{js_j}}{1 - \varepsilon^2}\right|^2 = \frac{\varepsilon^2 \cdot |a_{js_j}|^2}{(1 - \varepsilon^2)^2}, \quad j = 1, \dots, n,$$

for sufficiently small ε the point a_{js_j} lies outside the circle and, therefore, $\widetilde{\Gamma}_{h,\tilde{a}_J}$ is homologous to the cycle $\widetilde{\Gamma}_{h,a_J}$:

$$\begin{cases} |w_1 - a_{1j_1}| = \varepsilon_1, \\ |w_2 - a_{2j_2}| = \varepsilon_2, \\ \dots\dots\dots\dots \\ |w_n - a_{nj_n}| = \varepsilon_n. \end{cases}$$

Here some a_{ij} can be zero.

Lemma 2. *The residue integral (19) is*

$$J_\gamma(t) = \frac{(-1)^n}{(2\pi i)^n} \int\limits_{\widetilde{\Gamma}_h} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\widetilde{F_1}}{\widetilde{F_1}} \wedge \frac{d\widetilde{F_2}}{\widetilde{F_2}} \wedge \ldots \wedge \frac{d\widetilde{F}_n}{\widetilde{F}_n}. \tag{24}$$

Theorem 4. *The following equalities are valid*

$$\sum_{j=1}^{p} \frac{1}{z_{j1}(t)^{\gamma_1+1} \cdot z_{j2}(t)^{\gamma_2+1} \cdots z_{jn}(t)^{\gamma_n+1}} =$$

$$= \sum_{K \in \Re} (-t)^{||K||+n} \sum_{J} \frac{(-1)^{s(J)}}{\beta(K,J)!} \cdot \frac{\partial^{||\beta(K,J)||}}{\partial w^{\beta}(K,J)} \times \left[\widetilde{\Delta}(t) \cdot w_1^{\gamma_1+1} \cdots w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)} \right] \Bigg|_{w=a_J}.$$

Since zeros of (22) are polynomials in t, the equality (4) also holds for $t = 1$. We denote

$$\sigma_{\gamma+I} = \sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}},$$

where $z^{(j)} = (z_{j1}, \ldots, z_{jn}) = (z_{j1}(1), \ldots, z_{jn}(1))$, $j = 1, \ldots, n$.

Theorem 5. *For the system (10) with functions f_j defined in (12) and Q_i defined in (20), the following formulas hold:*

$$\sigma_{\gamma+I} = \sum_{j=1}^{p} \frac{1}{z_{j1}^{\gamma_1+1} \cdot z_{j2}^{\gamma_2+1} \cdots z_{jn}^{\gamma_n+1}} =$$

$$= \frac{1}{(2\pi i)^n} \sum_{\|K\| \geqslant 0} (-1)^{\|K\|+n} \sum_{J} (-1)^{s(J)} \times$$

$$\times \int\limits_{\widetilde{\Gamma}_{h,a_J}} \widetilde{\Delta} \cdot w_1^{\gamma_1+1} \ldots w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}_1^{k_1} \cdot \ldots \cdot \widetilde{Q}_n^{k_n}}{\widetilde{q}_1^{k_1+1} \cdot \ldots \cdot \widetilde{q}_n^{k_n+1}} dw =$$

$$= \sum_{K \in \Re} (-1)^{||K||+n} \sum_{J} \frac{(-1)^{s(J)}}{\beta(K,J)!} \cdot \frac{\partial^{||\beta(K,J)||}}{\partial w^{\beta}(K,J)} \left[\widetilde{\Delta} \cdot w_1^{\gamma_1+1} \cdots w_n^{\gamma_n+1} \cdot \frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)} \right] \Bigg|_{w=a_J}.$$

Consider the following system of equations in two complex variables:

$$\begin{cases} f_1(z_1, z_2) = (1 - a_2 z_2)^2 + a_3 z_1 z_2^2 = 0, \\ f_2(z_1, z_2) = (1 - b_1 z_1)^2 (1 - b_2 z_2) + b_3 z_1^2 z_2 = 0. \end{cases} \tag{25}$$

Then Q_m, $m = 1, 2$ have the form (20). The system (25) has, as is easy to verify, 5 roots (z_{j1}, z_{j2}), $j = 1, 2, 3, 4, 5$. If $a_2 \neq b_2$, then these roots do not lie on the coordinate planes.

Let us change the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1 (w_2 - a_2)^2 + a_3 = 0, \\ \widetilde{f}_2 = (w_1 - b_1)^2 (w_2 - b_2) + b_3 = 0. \end{cases} . \tag{26}$$

Jacobian of the system (26)

$$\widetilde{\Delta} = \begin{vmatrix} (w_2 - a_2)^2 & 2 w_1 (w_2 - a_2) \\ 2(w_1 - b_1)(w_2 - b_2) & (w_1 - b_1)^2 \end{vmatrix} =$$

$$= (w_1 - b_1)^2 (w_2 - a_2)^2 - 4w_1(w_1 - b_1)(w_2 - a_2)(w_2 - b_2).$$

Then, by Theorem 5, we obtain

$$\sigma_\gamma = \sum_{j=1}^{5} \frac{1}{z_{j1}^{\gamma_1+1}} \cdot \frac{1}{z_{j2}^{\gamma_2+1}} = \sum_J \sum_{K\in\Re} \frac{(-1)^{\|K\|+s(j)}}{(2\pi i)^2} \times$$

$$\times \int\limits_{\widetilde{\Gamma}_{h,a_J}} \frac{w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot a_3^{k_1} \cdot b_3^{k_2} \cdot \widetilde{\Delta} \cdot dw_1 \wedge dw_2}{w_1^{k_1+1}(w_2 - a_2)^{2(k_1+1)} \cdot (w_1 - b_1)^{2(k_2+1)} (w_2 - b_2)^{k_2+1}}. \qquad (27)$$

Here the multi-indices $\Re = \{K = (k_1, k_2)| \quad \exists m : \gamma_m + 2 > k_1 + k_2, \quad m = 1, 2\}$. The cycles $\widetilde{\Gamma}_{h,a_J}$ are cycles of the form $\{|w_1| = r_{11}, |w_2 - b_2| = r_{22}\}$, taken with positive orientation, and $\{|w_2 - a_2| = r_{12}, |w_1 - b_1| = r_{21}\}$ are with negative orientation.

In particular, calculating $J_{(0,0)}$, after some transformations we obtain

$$\sigma_{(1,1)} = 4a_2 b_1 - \frac{a_3 b_2}{(b_2 - a_2)^2} \qquad (28)$$

without finding the roots.

Consider a system of equations in three complex variables:

$$\begin{cases} f_1(z_1, z_2, z_3) = 1 - a_1 z_1 - a_2 z_2 - a_3 z_3 + a_1 a_2 z_1 z_2 + a_1 a_3 z_1 z_3 + a_2 a_3 z_2 z_3 = \\ = (1 - a_1 z_1)(1 - a_2 z_2)(1 - a_3 z_3) + a_1 a_2 a_3 z_1 z_2 z_3 = 0, \\ f_2(z_1, z_2, z_3) = 1 - b_1 z_1 - b_2 z_2 - b_3 z_3 + b_1 b_2 z_1 z_2 + b_1 b_3 z_1 z_3 + b_2 b_3 z_2 z_3 = \\ = (1 - b_1 z_1)(1 - b_2 z_2)(1 - b_3 z_3) + b_1 b_2 b_3 z_1 z_2 z_3 = 0, \\ f_3(z_1, z_2, z_3) = 1 - c_1 z_1 - c_2 z_2 - c_3 z_3 + c_1 c_2 z_1 z_2 + c_1 c_3 z_1 z_3 + c_2 c_3 z_2 z_3 = \\ = (1 - c_1 z_1)(1 - c_2 z_2)(1 - c_3 z_3) + c_1 c_2 c_3 z_1 z_2 z_3 = 0. \end{cases} \qquad (29)$$

The roots of the system (29) are (z_{j1}, z_{j2}, z_{j3}), $j = 1, \ldots, 12$.

Change the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$ and $z_3 = \dfrac{1}{w_3}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 w_3 - a_1 w_2 w_3 - a_2 w_1 w_3 - a_3 w_1 w_2 + a_1 a_2 w_3 + a_1 a_3 w_2 + a_2 a_3 w_1 = \\ = (w_1 - a_1)(w_2 - a_2)(w_3 - a_3) + a_1 a_2 a_3 = 0, \\ \widetilde{f}_2 = w_1 w_2 w_3 - b_1 w_2 w_3 - b_2 w_1 w_3 - b_3 w_1 w_2 + b_1 b_2 w_3 + b_1 b_3 w_2 + b_2 b_3 w_1 = \\ = (w_1 - b_1)(w_2 - b_2)(w_3 - b_3) + b_1 b_2 b_3 = 0, \\ \widetilde{f}_3 = w_1 w_2 w_3 - c_1 w_2 w_3 - c_2 w_1 w_3 - c_3 w_1 w_2 + c_1 c_2 w_3 + c_1 c_3 w_2 + c_2 c_3 w_1 = \\ = (w_1 - c_1)(w_2 - c_2)(w_3 - c_3) + c_1 c_2 c_3 = 0. \end{cases} \qquad (30)$$

The Jacobian of the system (30)

$$\widetilde{\Delta} = (w_2 - a_2)(w_3 - a_3)[(w_1 - b_1)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_1 - c_1)(w_3 - c_3)] -$$

$$-(w_1 - a_1)(w_3 - a_3)[(w_2 - b_2)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_2 - c_2)(w_3 - c_3)] +$$

$$+(w_1 - a_1)(w_2 - a_2)[(w_2 - b_2)(w_3 - b_3)(w_1 - c_1)(w_3 - c_3) - (w_1 - b_1)(w_3 - b_3)(w_2 - c_2)(w_3 - c_3)].$$

Then, by Theorem 5, we obtain $J_{(0,0,0)} = \sum_J (-1)^s(J)$

$$\sum_{\|k\|<2} \frac{(-1)^{\|k\|}}{(2\pi i)^2} \int\limits_{\widetilde{\Gamma}_{q,a_J}} \frac{w_1 w_2 w_3 \cdot (a_1 a_2 a_3)^{k_1} (b_1 b_2 b_3)^{k_2} (c_1 c_2 c_3)^{k_3} \cdot \widetilde{\Delta}}{(w_1 - a_1)^{k_1+1}(w_2 - a_2)^{k_1+1}(w_3 - a_3)^{k_1+1}} \times$$

$$\times \frac{dw_1 \wedge dw_2 \wedge dw_3}{(w_1-b_1)^{k_2+1}(w_2-b_2)^{k_2+1}(w_3-b_3)^{k_2+1} \cdot (w_1-c_1)^{k_3+1}(w_2-c_2)^{k_3+1}(w_3-c_3)^{k_3+1}}, \tag{31}$$

where $\widetilde{\Gamma}_{q,a_J}$ are cycles of the form $\{|w_1-a_1|=r_{11}, |w_2-b_2|=r_{22}, |w_3-c_3|=r_{33}\}$; $\{|w_3-a_3|=r_{13}, |w_1-b_1|=r_{21}, |w_2-c_2|=r_{32}\}$; $\{|w_2-a_2|=r_{12}, |w_3-b_3|=r_{23}, |w_1-c_1|=r_{31}\}$, taken with a positive orientation, and $\{|w_1-a_1|=r_{11}, |w_3-b_3|=r_{23}, |w_2-c_2|=r_{32}\}$; $\{|w_2-a_2|=r_{12}, |w_1-b_1|=r_{21}, |w_3-c_3|=r_{33}\}$; $\{|w_3-a_3|=r_{13}, |w_2-b_2|=r_{22}, |w_1-c_1|=r_{31}\}$ with negative orientation.

Calculating these integrals, we get

$$-\sigma_{(1,1,1)} = J_{(0,0,0)} = a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1 + \tag{32}$$
$$+\frac{a_3c_1c_2c_3}{a_3-c_3}\cdot\left[\frac{b_1}{b_1-c_1}+\frac{b_2}{b_2-c_2}\right]+\frac{a_1b_1b_2b_3}{a_1-b_1}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_2}{c_2-b_2}\right]+$$
$$+\frac{a_2b_1b_2b_3}{a_2-b_2}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_1}{c_1-b_1}\right]+\frac{a_3b_1b_2b_3}{a_3-b_3}\cdot\left[\frac{c_2}{c_2-b_2}+\frac{c_1}{c_1-b_1}\right]+$$
$$+\frac{a_1c_1}{a_1-c_1}\cdot\left[\frac{b_2c_2c_3}{b_2-c_2}+\frac{b_3c_2c_3}{b_3-c_3}+\frac{a_2a_3b_2}{a_2-b_2}+\frac{a_2a_3b_3}{a_3-b_3}\right]+$$
$$+\frac{a_2c_2}{a_2-c_2}\cdot\left[\frac{b_1c_1c_3}{b_1-c_1}+\frac{b_3c_1c_3}{b_3-c_3}+\frac{a_1a_3b_3}{a_3-b_3}+\frac{a_1a_3b_1}{a_1-b_1}\right].$$

So, we found the sums of the roots $\sigma_{(1,1,1)}$ without calculating the roots of the system themselves.

3. General systems of transcendental equations

Let $f_1(z),\ldots,f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in the multidimensional complex space $\mathbb{C}^n$, $z=(z_1,\ldots,z_n)$.

We expand the functions $f_1(z),\ldots,f_n(z)$ in Taylor series in the vicinity of the origin and consider a system of equations of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad i = 1,\ldots,n, \tag{33}$$

where P_j is the lowest homogeneous part of the Taylor expansion of the function $f_j(z)$. The degree of all monomials (with respect to the totality of variables) included in P_j, is equal to m_j, $j=1,\ldots,n$. In the functions Q_jl, the degrees of all monomials are strictly greater than m_j.

The expansion of the functions Q_j, P_j, $j=1,\ldots,n$ in a neighborhood of zero in Taylor series converging absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\|>m_j} a^j_\alpha z^\alpha, \tag{34}$$

$$P_j(z) = \sum_{\|\beta\|=m_j} b^j_\beta z^\beta, \tag{35}$$

$$j = 1,\ldots,n,$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\beta=(\beta_1,\ldots,\beta_n)$ are multi-indexes, i.e. α_j and β_j are non-negative integers, $j=1,\ldots,n$, $\|\alpha\|=\alpha_1+\ldots+\alpha_n$, $\|\beta\|=\beta_1+\ldots+\beta_n$, and monomials $z^\alpha = z_1^{\alpha_1}\cdot z_2^{\alpha_2}\cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1}\cdot z_2^{\beta_2}\cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z),\ldots,P_n(z)$ is *nondegenerate*, that is, its common zero is only point 0, the origin.

Consider an open set (a special analytic polyhedron) of the form

$$D_P(r_1,\dots,r_n)=\{z:\ |P_j(z)|<r_j,\ i=j,\dots,n\},$$

where $r_1,\dots,r_n$ are positive numbers. Its *skeleton* has the form

$$\Gamma_P(r_1,\dots,r_n)=\Gamma_P(r)=\{z:\ |P_j(z)|=r_j,\ j=1,\dots,n\}.$$

Let us start with a statement.

Lemma 3. *The next equality is true*

$$J_\gamma=\frac{1}{(2\pi i)^n}\int\limits_{\Gamma_P}\frac{1}{z_1^{\gamma_1+1}\cdot z_2^{\gamma_2+1}\cdots z_n^{\gamma_n+1}}\cdot\frac{df_1}{f_1}\wedge\frac{df_2}{f_2}\wedge\dots\wedge\frac{df_n}{f_n}=$$

$$=\frac{(-1)^n}{(2\pi i)^n}\int\limits_{\Gamma_{\tilde P}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde f_1}{\tilde f_1}\wedge\frac{d\tilde f_2}{\tilde f_2}\wedge\dots\wedge\frac{d\tilde f_n}{\tilde f_n}=(-1)^n\tilde J_\gamma.$$

For what follows, we need a generalized formula for transforming the Grothendieck residue.

Theorem 6. *Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$ and $g_j(w)$, $j,k=1,\dots,n$, are related by the relations*

$$g_j=\sum_{k=1}^n a_{jk}f_k,\quad j=1,2,\dots,n,$$

the matrix $A=\|a_{jk}\|_{j,k=1}^n$ consists of polynomials. Consider the cycles

$$\Gamma_f=\{w:\ |f_j(w)|=r_j,\ j=1,\dots,n\},$$

$$\Gamma_g=\{w:\ |g_j(z)|=r_j,\ j=1,\dots,n\},$$

where all $r_j>0$. Then the equality

$$\int\limits_{\Gamma_f}h(w)\frac{dw}{f^\alpha}=\sum_{K,\ \sum\limits_{s=1}^n k_{sj}=\beta_s}\frac{\beta!}{\prod\limits_{s,j=1}^n(k_{sj})!}\int\limits_{\Gamma_g}h(w)\frac{\det A\prod\limits_{s,j=1}^n a_{sj}^{k_{sj}}\,dw}{g^\beta},\tag{36}$$

holds. Here $\beta!=\beta_1!\beta_2!\dots\beta_n$, $\beta=(\beta_1,\beta_2,\dots,\beta_n)$, the summation in the formula is over all non-negative integer matrices $K=\|k_{sj}\|_{s,j=1}^n$ with the conditions that the sum $\sum\limits_{s=1}^n k_{sj}=\alpha_j$, then $\beta_j=\sum\limits_{j=1}^n k_{js}$. Here$f^\alpha=f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^\beta=g_1^{\beta_1}\cdots g_n^{\beta_n}$.

Theorem 7. *The next formulas are valid*

$$\sum_{j=1}^p\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}}=$$

$$=(2\pi i)^n)\int\limits_{\Gamma_{\tilde P}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde f_1}{\tilde f_1}\wedge\frac{d\tilde f_2}{\tilde f_2}\wedge\dots\wedge\frac{d\tilde f_n}{\tilde f_n}=$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n} \frac{(-1)^{n+\|\alpha\|}}{(2\pi i)^n} \int_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\times$$

$$\times\frac{\tilde{\Delta}\cdot\tilde{Q}_1^{\alpha_1}\cdot\tilde{Q}_2^{\alpha_2}\cdots\tilde{Q}_n^{\alpha_n}dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\tilde{P}_1^{\alpha_1+1}\cdot\tilde{P}_2^{\alpha_2+1}\cdots\tilde{P}_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\|\leqslant\|\gamma\|+n} \frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n}k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\tilde{\Delta}\cdot\det A\cdot Q^{\alpha}\prod\limits_{s,j=1}^{n}a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n}w_j^{\beta_jN_j+\beta_j+N_j}}\right],$$

where $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$, *and the functional* $\mathfrak{M}$ *assigns its free term to the Laurent polynomial.*

In fact, in Theorem 7, analogs of the classical Waring formulas for finding power sums of roots of a system of algebraic equations are obtained.

Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1,z_2) = a_1z_1 - a_2z_2 + z_1^2 = 0,\\ f_2(z_1,z_2) = b_1z_1 + b_2z_2 + z_2^2 = 0. \end{cases} \tag{37}$$

It satisfies the conditions on $Q_j(z)$ We will assume that $a_1b_2 + a_2b_1 \neq 0$, i.e. the system of lower homogeneous polynomials is nondegenerate.

Let us change variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \tilde{f}_1 = -a_2w_1^2 + a_1w_1w_2 + w_2 = 0,\\ \tilde{f}_2 = b_2w_1w_2 + b_1w_2^2 + w_1 = 0. \end{cases} . \tag{38}$$

This system has 4 roots, on the coordinate planes there is one root — (0,0).

The Jacobian of the system (38)

$$\tilde{\Delta} = \begin{vmatrix} -2a_2w_1 + a_1w_2 & a_1w_1 + 1\\ b_2w_2 + 1 & 2b_1w_2 + b_2w_1 \end{vmatrix} =$$

$$= -2a_2b_2w_1^2 - 4a_2b_1w_1w_2 + 2a_1b_1w_2^2 - a_1w_1 - b_2w_2 - 1.$$

Notice that

$$\tilde{Q}_1 = w_2, \quad \tilde{Q}_2 = w_1. \tag{39}$$

$$\tilde{P}_1 = -a_2w_1^2 + a_1w_1w_2, \quad \tilde{P}_2 = b_2w_1w_2 + b_1w_2^2. \tag{40}$$

To find the matrix A we use Example 8.3 from [4].

We introduce the matrix

$$\mathrm{Res} = \begin{pmatrix} -a_2 & a_1 & 0 & 0\\ 0 & -a_2 & a_1 & 0\\ 0 & b_2 & b_1 & 0\\ 0 & 0 & b_2 & b_1 \end{pmatrix}.$$

The determinant Δ of the matrix Res is $\Delta = a_2b_1(a_2b_1 + a_1b_2)$. Let us calculate some minors according to example 8.3 from [4]:

$$\tilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_2b_1^2 - a_1b_1b_2, \quad \tilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_1b_1^2,$$

$$\widetilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2b_1, \qquad \tilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \end{vmatrix} = 0.$$

$$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = 0, \qquad \Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2b_2^2,$$

$$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2b_2, \qquad \Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2b_1 + a_1a_2b_2.$$

Therefore, the elements a_{ij} of the matrix A are

$$a_{11} = \frac{1}{\Delta}\left(\tilde{\Delta}_1 w_1 + \tilde{\Delta}_2 w_2\right) = \frac{1}{\Delta}\left((-a_2b_1^2 - a_1b_1b_2)w_1 - a_1b_1^2w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\widetilde{\Delta}_3 w_1 + \widetilde{\Delta}_4 w_2\right) = \frac{a_1^2b_1w_1}{\Delta}, \qquad a_{21} = \frac{1}{\Delta}\left(\Delta_1 w_1 + \Delta_2 w_2\right) = \frac{-a_2b_2^2w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3 w_1 + \Delta_4 w_2\right) = \frac{1}{\Delta}\left(-a_2^2b_2w_1 + (a_2^2b_1 + a_1a_2b_2)w_2\right).$$

Then it is easy to check that

$$w_1^3 = a_{11}\tilde{P}_1 + a_{12}\tilde{P}_2, \quad w_2^3 = a_{21}\tilde{P}_1 + a_{22}\tilde{P}_2.$$

We calculate $\det A$:

$$\det A = \frac{1}{\Delta}\left(a_2b_2w_1^2 - a_2b_1w_1w_2 - a_1b_1w_2^2\right).$$

By Theorem 7

$$J_{(0,0)} = \sum_{\|K\| \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11} + k_{12})! \cdot (k_{21} + k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times \mathfrak{M}\left[\frac{\tilde{\Delta} \cdot \det A \cdot \tilde{Q}_1^{k_{11}+k_{21}} \cdot \tilde{Q}_2^{k_{12}+k_{22}} \cdot a_{11}^{k_{11}} \cdot a_{12}^{k_{12}} \cdot a_{21}^{k_{21}} \cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1} \cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

We denote $\bar{\Delta} = a_2b_1 + a_1b_2$. Cumbersome but simple calculations (using the definition of the functional $\mathfrak{M}$) give that

$$J_{(0,0)} = \frac{1}{\bar{\Delta}} - \frac{2a_1b_2}{a_2b_1\bar{\Delta}} + \frac{6a_1^2b_2^2}{a_2b_1\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2} + \frac{a_1^3}{a_2\bar{\Delta}^2} + \frac{8a_1b_2}{\bar{\Delta}^2} - \frac{4}{a_2b_1} =$$

$$= \frac{a_1^3}{a_2\bar{\Delta}^2} - \frac{a_1b_2}{\bar{\Delta}^2} - \frac{3a_2b_1}{\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2}.$$

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References

[1] L.A.Aizenberg, On a formula of the gereralized multidimensional logarithmic residue and the solution of system of nonlinear equations, *Sov. Math. Doc.*, **18**(1977), 691–695.

[2] L.A.Aizenberg, A.P.Yuzhakov, Integral representations and residues in multidimensional complex analysis, Amer. Math. Monographs, AMS, Providence, 1983.

[3] A.K.Tsikh, Multidimensional residues and their applications. Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1992.

[4] V.I.Bykov, A.M.Kytmanov, M.Z.Lazman, Elimination Methods in Polynomial Computer Algebra, Springer, New–York, 1998.

[5] L.A.Aizenberg, A.M.Kytmanov, Multidimensional analogues of Newton's formulas for systems of nonlinear algebraic equations and some of their applications, *Siberian Mathematical Journal*, **22**(1981), 180–189.

[6] V.I.Bykov, Modeling of the critical phenomena in chemical kinetics, Komkniga, Moscow, 2006 (in Russian).

[7] V.I.Bykov, S.B.Tsybenova, Non-linear models of chemical kinetics, KRASAND, Moscow, 2011 (in Russian).

[8] A.M.Kytmanov, Z.E.Potapova Formulas for determining power sums of roots of systems of meromorphic functions, *Izvestiya VUZ. Matematika*, **49**(2005), no. 8, 36–45 (in Russian).

[9] V.I.Bykov, A.M.Kytmanov, S.G.Myslivets, Power sums of nonlinear systems of equations, *Dokl. Math.*, **76**(2007), 641–644. DOI: 10.1134/S1064562407050018

[10] A.M.Kytmanov, E.K.Myshkina, *Russian Mathematics*, **57**(2013), no. 12, 31–43. DOI: 10.3103/S1066369X13120049

[11] O.V.Khodos, On Some Systems of Non-algebraic Equations in $\mathbb{C}^n$, *Journal of Siberian Federal University. Mathematics &Physics*, **7**(2014), no. 4, 455–465.

[12] A.M.Kytmanov, E.K.Myshkina, On calculation of power sums of roots for one class of systems of non-algebraic equations, *Sib. Elektron. Mat. Rep.*, **12**(2015), 190–209 (in Russian). DOI: 10.17377/semi.2015.12.016

[13] A.A.Kytmanov, A.M.Kytmanov, E.K.Myshkina, Finding residue integrals for systems of non-algebraic equations in $\mathbb{C}^n$, *Journal of Symbolic Computation*, **66**(2015), 98–110. DOI: 10.1016/j.jsc.2014.01.007

[14] A.V.Kytmanov, E.R.Myshkina, *J. Math. Sciences*, **213**(2016), no. 6, 868–886. DOI: 10.1007/s10958-016-2748-7

[15] A.M.Kytmanov, O.V.Khodos, On systems of non-algebraic equation in $\mathbb{C}^n$, *Contemporary Mathematics*, **662**(2016), 77–88.

[16] A.M.Kytmanov, O.V.Khodos, On localization of the zeros of an entire function of finite order of growth, *Journal Complex Analysis and Operator Theory*, **11**(2017), no. 2, 393–416.

[17] A.M.Kytmanov, Ya.M.Naprienko, One approach to finding the resultant of two entire functions, *Complex variables and elliptic equations* **62** (2) (2017) , 269-286.

[18] A.M.Kytmanov, O.V.Khodos, On one approach to determining the resultant of two entire functions, *Russian Math.*, no. 4, 2018, 49–59.

[19] A.A.Kytmanov, A.M.Kytmanov, E.K.Myshkina, *Residue Integrals and Waring's Formulas for a Class of Systems of Transcendental Equations in* $\mathbb{C}^n$, Journal of Complex variables and Elliptic Equations, **64**(2018), no. 1. 93–111. DOI: 10.1080/17476933.2017.1419210

[20] A.M.Kytmanov, E.K.Myshkina, Residue integrals and Waring formulas for algebraic and transcendental systems of equations, *Russian Math.*, 2019, no. 5, 40–55. DOI: 10.26907/0021-3446-2019-5-40-55

[21] A.M.Kytmanov, Algebraic and transcendental systems of equations, SFU, Krasnoyarsk, 2019.

[22] M.Passare, A.Tsikh, Residue integrals and their Mellin transforms, *Can. J. Math.*, **47**(1995), no. 5, 1037–1050.

О трансцендентных системах уравнений

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Аннотация. Рассмотрены различные типы систем трансцендентных уравнений: простейшие, специальные и общие. Поскольку число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Получены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга. Приведены различные примеры трансцендентных систем уравнений и вычислены суммы многомерных числовых рядов.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.