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Estimating the Mean of Heavy-tailed Distribution under Random Truncation

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Abstract. Inspired by L.Peng's work on estimating the mean of heavy-tailed distribution in the case of completed data. we propose an alternative estimator and study its asymptotic normality when it comes to the right truncated random variable. A simulation study is executed to evaluate the finite sample behavior on the proposed estimator.

Keywords: random truncation, Hill estimator, Lynden-Bell estimator, heavy-tailed distributions.

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1. Introduction and motivation

Let $(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ be independent copies of a non-negative random variable (rv) $\mathbf{X}$ with cumulative distribution (cdf) $\mathbf{F}$, defined over some probability space $(\Omega, \mathcal{A}, \mathcal{P})$, suppose that $\mathbf{X}$ is right truncated by sequences of independent copies $(\mathbf{Y}_1, \ldots, \mathbf{Y}_N)$ of (rv) $\mathbf{Y}$ with cdf $\mathbf{G}$, throughout the paper, we assume that $\mathbf{F}$ and $\mathbf{G}$ are heavy-tailed in other words that $\overline{\mathbf{F}} = 1 - \mathbf{F}$ and $\overline{\mathbf{G}} = 1 - \mathbf{G}$ are regularly varying ($\mathcal{RV}$) at infinity with respective negative indices $-1/\gamma_1$ and $-1/\gamma_2$; we will use the notation: $\overline{\mathbf{F}} \in \mathcal{RV}(-1/\gamma_1)$ and $\overline{\mathbf{G}} \in \mathcal{RV}(-1/\gamma_2)$ that is for any $x > 0$.

$$\lim_{t\to\infty} \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} = x^{-\frac{1}{\gamma_1}} \quad \text{and} \quad \lim_{t\to\infty} \frac{\overline{\mathbf{G}}(tx)}{\overline{\mathbf{G}}(t)} = x^{-\frac{1}{\gamma_2}} . \tag{1}$$

The statistical literature on such problems of extremes [4] and [13] events is very extensive, one of those problems is for the estimation of the mean $\mathbf{E}(X)$, this problem was already treated by [11] and [3] in the case of complete data, nevertheless in numerous survival practical applications, it happens that one is not able to observe a subject entire lifetime. The subject may leave the study may survive to the closing data, or may enter the study at some time after its lifetime has started, the most current forms of such incomplete data are censorship and truncation. As we mention our aim is to propose an asymptotically normal estimator for the mean of X:

$$\mu = \mathbf{E}(X) = \int_0^\infty \overline{\mathbf{F}}(x)dx. \tag{2}$$

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Whose existence requires that $\gamma_1 < 1$, The sample mean for censored data is obtained and equal to:

$$\widetilde{\mu}_n = \sum_{i=2}^{n} \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{[j:n]}} Z_{i,n}, \tag{3}$$

the asymptotic normality of $\widetilde{\mu}_n$ is established by [14]. The model studied here is based on the random right truncated ($\mathcal{RRT}$) data, in the sense that the rv of interest $\mathbf{X}_i$ and the truncated rv $\mathbf{Y}_i$ are observable only when $\mathbf{X}_i \leqslant \mathbf{Y}_i$, whereas nothing is observed if $\mathbf{X}_i > \mathbf{Y}_i$. We denote (X_i, Y_i), $i = 1; n$ to be observed data as copies of a couple of rv's (X, Y) corresponding to the truncated sample $(\mathbf{X}_i, \mathbf{Y}_i)_{1 \leqslant i \leqslant N}$, where $n = n_N$ is a sequence of discrete rv's by the weak law of large numbers, we have

$$\frac{n}{N} \longrightarrow p = \mathbf{P}(\mathbf{X} \leqslant \mathbf{Y}) \quad \text{as} \quad N \to \infty.$$

We shall assume that $p > 0$, otherwise nothing will be observed. The joint $\mathbf{P}$-distribution of on observed (X, Y) is given by:

$$H(x, y) = \mathbf{P}(X \leqslant x, Y \leqslant y) = \mathbf{P}(\mathbf{X} \leqslant x, \mathbf{Y} \leqslant y \mid \mathbf{X} \leqslant \mathbf{Y} = p^{-1} \int_0^y \mathbf{F}(\min(x, z)) d\mathbf{G}(z).$$

The marginal distributions of the rv's X and Y respectively denoted by F and G are defined by:

$$F(x) = p^{-1} \int_0^x \overline{\mathbf{G}}(z) d\mathbf{F}(z) \quad \text{and} \quad G(y) = p^{-1} \int_0^y \mathbf{F}(z) d\mathbf{G}(z),$$

$$\overline{F}(x) = -p^{-1} \int_x^\infty \overline{\mathbf{G}}(z) d\overline{\mathbf{F}}(z) \quad \text{and} \quad \overline{G}(y) = -p^{-1} \int_y^\infty \mathbf{F}(z) d\overline{\mathbf{G}}(z).$$

For randomly truncated data; the truncation product-limit estimate is the maximum likelihood estimate (MLE) for non-parametric models the well-known non-parametric estimator of F in $\mathcal{RRT}$ model, proposed by [10] :

$$\mathbf{F}_n^{(\mathbf{LB})}(x) = \prod_{i: X_i > x} \exp\left(1 - \frac{1}{nC_n(X_i)}\right). \tag{4}$$

Where $C_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{1}(X_i \leqslant x \leqslant Y_i)$ the empirical counterparts of $C(z) = P(X \leqslant z \leqslant Y)$. Since F and G are heavy-tailed their right endpoints are infinite and thus are equal. As we mentioned this problem has been studied by [11] in the case of sets of complete data from heavy-tailed distributions with a range of $\gamma_1 \in (1/2, 1)$ throughout this paper we restrict ourselves on the case where γ_1 belongs to the following range:

$$\mathcal{R} = \left\{ \gamma_1, \gamma_2 > 0 : \frac{\gamma_2}{1 + 2\gamma_2} < \gamma_1 < 1 \right\}. \tag{5}$$

To ensure that the mean is finite and since we have applied both conditions of [15] paper:

$$I_1 = \int_1^\infty \frac{\varphi^2(x)}{\mathbf{G}(x)} d\mathbf{F}(x), \qquad I_2 = \int_1^\infty \frac{d\mathbf{F}(x)}{\mathbf{G}(x)}. \tag{6}$$

We find those conditions may be infinite when we deal with heavy-tailed distributions. Assumed that both of X and Y are $Pareto(\gamma_1)$ and $Pareto(\gamma_2)$ respectively:

$$1 - \mathbf{F}(x) = \overline{\mathbf{F}}(x) = x^{-\frac{1}{\gamma_1}}, \qquad 1 - \mathbf{G}(x) = \overline{\mathbf{G}}(x) = x^{-\frac{1}{\gamma_2}} \quad \text{with} \quad \gamma_1 > 0, \ \gamma_2 > 0 \ \text{ and } \ x \geqslant 1.$$

We figure out that the central limit theorem (CTL) established by [15] cannot be applied in the previous range when $I_1 = I_2 = \infty$. It is worth to mention that in the case of non truncation we have $\gamma_1 = \gamma$ and $\gamma_2 = \infty$ so $\mathcal{R}$ abbreviate to Peng's range. To define our new estimator we introduce an integer sequences $k = k_n$ representing a fraction of extreme order statistics satisfying the following conditions:

$$1 < k < n, \quad k \longrightarrow \infty \;\text{ and }\; k/n \longrightarrow 0 \;\text{ as }\; n \longrightarrow \infty. \tag{7}$$

So by decomposing μ as the sum of two terms

$$\mu = \int_0^t \overline{\mathbf{F}}(x)dx + \int_t^\infty \overline{\mathbf{F}}(x)dx = \mu_1 + \mu_2. \tag{8}$$

Then we can estimate μ_i, $i = \overline{1,2}$ separately, after integration μ_1 by parts and after changing variables in μ_2 we may write:

$$\mu_1 = t\overline{\mathbf{F}}(t) + \int_0^t x d\mathbf{F}(x) \quad \text{and} \quad \mu_2 = t\overline{\mathbf{F}}(t) \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} dx.$$

By replacing t by $X_{n-k,n}$ where $X_{1,n} < \cdots < X_{n,n}$ denote the order statistics pertaining to $X_1, \ldots, X_n$; and $\mathbf{F}$ by $\mathbf{F}_n^{(\mathbf{LB})}$ we get that:

$$\widehat{\mu}_1 = X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}) + \int_0^{X_{n-k,n}} x d\mathbf{F}_n^{(\mathbf{LB})}(x),$$

hence from [16] we may write:

$$\widehat{\mu}_1 = X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}) + \frac{1}{n}\sum_{i=1}^{n-k} \frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{i,n})}{C_n(X_{i,n})} X_{i,n}. \tag{9}$$

Back to μ_2 building on the Karamata Theorem [9, page 363] we may write:

$$\mu_2 \sim \frac{\gamma_1}{1-\gamma_1} t\overline{\mathbf{F}}(t) \;\text{ as }\; n \longrightarrow \infty, \qquad 0 < \gamma_1 < 1. \tag{10}$$

Notice to estimate (10) it is based on estimator of tail index γ_1, in view of the history of the estimation of γ_1. In [8] introduced an estimator of γ_1 under random truncation. In [1] established the asymptotic normality of this estimator under the tail dependence and the second order conditions of regular variation, throughout this paper we use the estimation of [1]. So that yield us to an estimator to μ_2 :

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}), \tag{11}$$

finally with (9) and (11), we build our estimator $\widehat{\mu}$ for the mean (2) as follow:

$$\hat{\mu} = X_{n-k,n}\; \overline{\mathbf{F}}_n(X_{n-k,n}) \frac{1}{1-\hat{\gamma}_1} + \frac{1}{n}\sum_{i=1}^{n-k} \frac{\mathbf{F}_n^{LB}(X_{i,n})}{C_n(X_{i,n})} X_{i,n}.$$

The rest of this paper is organized as follows. In the second section, we state our main result. This is followed by a simulation study of our proposed estimator where we discuss its behavior with a finite sample.

2. The main results

In extreme value analysis and in the second-order frame work (see, e.g. [9]), weak approximation are achieved. Consequently, it seems quite natural to suppose that df's **F** and **G** satisfy the well-known second-order condition of regular variation we express in terms of the tail quantile functions. That is we assume that for $x > 0$. we have

$$\lim_{t\to\infty} \frac{U_{\mathbf{F}}(tx)/U_{\mathbf{F}}(t) - x^{\gamma_1}}{\mathbf{A}_{\mathbf{F}}(t)} = x^{\gamma_1} \frac{x^{\tau_1}-1}{\tau_1} \tag{12}$$

and

$$\lim_{t\to\infty} \frac{U_{\mathbf{G}}(tx)/U_{\mathbf{G}}(t) - x^{\gamma_2}}{\mathbf{A}_{\mathbf{G}}(t)} = x^{\gamma_2} \frac{x^{\tau_2}-1}{\tau_2}, \tag{13}$$

where τ_1,$\tau_2 < 0$ are the second-order parameters and $\mathbf{A}_{\mathbf{F}}, \mathbf{A}_{\mathbf{G}}$ are functions tending to zero and not changing signs near infinity with regularly varying absolute values at infinity with indices τ_1,τ_2 respectively.

Theorem 2.1. *Assume that (12 and 13) hold and* $\sqrt{k}\mathbf{A}_\circ(n/k) = O(1)$ *for* $\gamma_2/(1+2\gamma_2) < \gamma_1 < 1$. *Let* $k = k_n$ *denote an intermediate integer sequences satisfying (7), then* $\hat{\mu} \to \mu$ *in probability:*

$$\frac{\sqrt{k}(\widehat{\mu}-\mu)}{\overline{\overline{\mathbf{F}}}(X_{n-k,n})X_{n-k,n}} =$$

$$= \mathbf{c}_1\mathbf{W}(1) + \int_0^1 \left\{ \mathbf{c}_2 s^{-\frac{2\gamma_1}{\gamma}+\frac{\gamma}{\gamma_2}+1} + \mathbf{c}_3 s^{-\gamma_1+\frac{\gamma}{\gamma_2}+1} + \mathbf{c}_4\log(s) + \mathbf{c}_5 \right\} s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds +$$

$$+ \frac{(\gamma_1+\tau_1-1)(1-\gamma_1)+(1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}\sqrt{k}\mathbf{A}_\circ(n/k).$$

Corollary 2.1. *Under the assumptions of Theorem 2.1 we suppose that* $\sqrt{k}\mathbf{A}_\circ(n/k) \to \lambda$,

$$\frac{\sqrt{k}(\widehat{\mu}-\mu)}{\overline{\overline{\mathbf{F}}}(X_{n-k,n})X_{n-k,n}} \to \mathcal{N}\left(\lambda \frac{(\gamma_1+\tau_1-1)(1-\gamma_1)+(1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}, \sigma^2\right) \quad as \quad n\to\infty.$$

Where

$$\sigma^2 := \frac{p(1-p)\left[p(1-p)+2\gamma_1^2\right]}{(1-\gamma_1)^2} + \frac{p^3\gamma_1}{1-\gamma_1} + \frac{2p^2(1-p)}{(1-\gamma_1)(-\gamma_1+2)} +$$

$$+ \frac{-2p^4}{(-2+p)(-4+3p)} + \frac{3p^5\gamma_1}{(-2+p)(-2+\gamma_1 p+3p)} + \frac{-2\gamma_1 p^3(1-p)}{(-2+p)(-\gamma_1+2)} +$$

$$+ 3p^5\gamma_1^2\left(\frac{p}{2} - \frac{1}{4-p}\right)^2 - 2p^3\gamma_1^2(1-p)\frac{3p-2}{6}\left(\frac{p}{1+p}\right)^2 +$$

$$+ \frac{p^2\gamma_1(p-1)(1-\gamma_1) - p^2\gamma_1^3}{(-1+p)(-2+p)(1-\gamma_1)}\left[\frac{\gamma_1(-p^3+4-6p)+p^2(\gamma_1-2)+2}{(-1-p)+\gamma_1(-p-2)}\right] +$$

$$+ \frac{1-2p}{p^2} + \frac{-2p^2(1-p)^2(1-\gamma_1)+\gamma_1^2 p}{(1-\gamma_1)(\gamma_1+2)(-\gamma_1+p+1)^2} +$$

$$+ \frac{2p^2(1-p)(1-\gamma_1)+\gamma_1^2 p}{(1-\gamma_1)^2}\left(\left(\frac{p}{p^2-1}\right)^2 + \left(\frac{1}{1-p}\right)^2\right)$$

and

$$p = \frac{\gamma_2}{\gamma_1+\gamma_2}.$$

3. Simulation study

The main purpose of this section is to study the execution of our new estimator $\widehat{\mu}$ for that we generate the data as follows:

- The interset and the truncated variable: we generate two sets of truncated and truncation data both pulled for the first hand from Fréchet model:

$$\overline{\mathbf{F}}(x) = 1 - \exp(-x^{\frac{1}{\gamma_1}}), \quad \overline{\mathbf{G}}(x) = 1 - \exp(-x^{\frac{1}{\gamma_2}}), \qquad x \geqslant 0$$

  and the other hand from Burr model:

$$\overline{\mathbf{F}}(x) = (1 + x^{\frac{1}{\delta}})^{-\frac{\delta}{\gamma_1}}, \quad \overline{\mathbf{G}}(x) = (1 + x^{\frac{1}{\delta}})^{-\frac{\delta}{\gamma_2}}, \qquad x \geqslant 0 \quad \text{and} \quad \delta, \gamma_1, \gamma_2 > 0.$$

- The observed data: for the proportion of observed data is equal to $p = \gamma_2/\gamma_1 + \gamma_2$ we take $p = 70\,\%$, $80\,\%$ and $90\,\%$ we fix $\delta = 1/4$ and choose the values 0.6, 0.7 and 0.8 for γ_1. For each couple (γ_1, p); we solve the equation $p = \gamma_2/\gamma_1 + \gamma_2$ to get the pertaining γ_2-value.
- We vary the common size N of both samples $(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ and $(\mathbf{Y}_1, \ldots, \mathbf{Y}_N)$.
- We apply the algorithm of [12] page 137, to select the optimal numbers of upper order statistics (k^*) used in the computation of $\hat{\gamma}_1$.

The performance of this new estimator named by $\hat{\mu}$ is evaluated in terms of absolute bias (A-bias) root mean squared error (RMSE) which are summarized in tables for Burr model in Tables: 1 for $\gamma_1 = 0.6$, 2 for $\gamma_1 = 0.7$, 3 for $\gamma_1 = 0.8$ and for Fréchet models Tables: 4 for $\gamma_1 = 0.6$, 5 for $\gamma_1 = 0.7$, 6 for $\gamma_1 = 0.8$ adding two forms of graphical representation; we consider two truncated schema of Burr truncated by Burr the first for $\gamma_1 = 0.6$ and the second for $\gamma_1 = 0.8$ we represent the Biases and the RMSE of our estimator as functions of k (number of the longest order statistics).

After examining all tables and figures, and as expected, the sample size affects the estimate in the sense that a larger N gives a better estimate. It is noticeable that the estimation accuracy of estimator decreases when the truncation percentage increase and it is quite expected. Moreover the estimator performs best for the larger value of the tail index larger than 0.5 especially when truncation proportion is high.

4. Appendix

4.1. Proof of Theorem 2.1

We begin by setting $U_i = \overline{F}(X_i)$ and define the corresponding uniform tail process by $\alpha_n(s) = \sqrt{k}(U_n(s) - s)$, for $0 \leqslant s \leqslant 1$ where $U_n(s) = 1/k \sum_{i=1}^{n} \mathbf{1}\left(\mathbf{U}_i \leqslant k\frac{s}{n}\right)$. The weighted weak approximation to $\alpha_n(s)$ given in terms of either a sequence of wiener processes (see, eg., [6] and [5]) or a single Wienner process as in Proposition 3.1 of [7], will be very crucial to our proof procedure.

In the sequel, we use the latter representation which says that: there exists a Wiener process $\mathbf{W}$, such that for every $0 \leqslant \eta \leqslant 1$

$$\sup_{0<s\leqslant 1} \mid \alpha_n(s) - \mathbf{W}(s) \mid \to \mathbf{0}, \text{ as } n \to \infty. \tag{14}$$

Observe that $\widehat{\mu} - \mu = (\widehat{\mu}_1 - \mu_1) + (\widehat{\mu}_2 - \mu_2)$ and starting by:

$$\widehat{\mu}_1 - \mu_1 = \int_0^{X_{n-k;n}} \overline{\mathbf{F}_n}(x)dx - \int_0^t \overline{\mathbf{F}}(x)dx,$$

Table 1. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.6$

$\gamma_1 = 0.6 \longrightarrow \mu = 2.371$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.002	0.130	27	2.374	198
400	0.069	0.858	31	2.440	278
500	0.072	0.257	39	2.300	355
1000	0.001	0.048	40	2.372	681
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.008	0.180	10	2.380	244
400	0.008	0.119	16	2.379	318
500	0.001	0.174	27	2.372	399
1000	0.001	0.106	25	2.372	811
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.005	0.040	4	2.406	268
400	0.006	0.028	7	2.406	361
500	0.003	0.067	8	2.374	445
1000	0.003	0.097	12	2.374	886

Table 2. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.7$

$\gamma_1 = 0.7 \longrightarrow \mu = 3.218$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.016	0.634	25	3.234	215
400	0.008	0.067	34	3.227	290
500	0.008	0.063	58	3.226	3362
1000	0.004	0.023	88	3.222	701
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.021	0.178	18	3.239	246
400	0.002	0.306	23	3.221	319
500	0.002	0.367	39	3.220	403
1000	0.001	0.193	52	3.219	788
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.005	0.028	19	3.223	268
400	0.000	0.134	21	3.218	368
500	0.008	0.246	25	3.226	458
1000	0.002	0.049	37	3.220	896

Table 3. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.8$

$\gamma_1 = 0.8 \longrightarrow \mu = 4.896$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.000	0.152	73	4.896	207
400	0.029	0.070	75	4.925	278
500	0.065	0.631	147	4.961	348
1000	0.013	0.302	228	4.919	697
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.106	0.613	55	5.002	239
400	0.014	0.446	14	4.910	315
500	0.001	0.321	146	4.897	404
1000	0.030	0.039	173	4.926	810
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.094	0.962	67	4.990	275
400	0.058	0.240	86	4.954	359
500	0.029	0.171	67	4.925	451
1000	0.006	0.041	187	4.902	894

Table 4. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.6$

$\gamma_1 = 0.6 \longrightarrow \mu = 2.218$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.155	0.537	28	2.373	170
400	0.153	0.186	25	2.371	217
500	0.004	0.065	32	2.222	284
1000	0.002	0.010	43	2.220	568
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.259	0.263	17	2.475	178
400	0.031	0.598	40	2.249	241
500	0.066	0.222	33	2.284	293
1000	0.074	0.076	31	2.307	569
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.010	0.084	5	2.228	180
400	0.009	0.185	11	2.218	231
500	0.004	0.052	19	2.222	314
1000	0.008	0.106	23	2.227	594

we consider the following decomposition:

$$\widehat{\mu}_1 - \mu_1 = T_{n_1}(x) + T_{n_2}(x).$$

Where:

$$T_{n_1}(x) = \int_0^{X_{n-k;n}} \left(\overline{\mathbf{F}}_n(x) - \overline{\mathbf{F}}(x)\right) dx \quad \text{and} \quad T_{n_2}(x) = \int_{X_{n-k;n}}^{t} \overline{\mathbf{F}}(x) dx.$$

Table 5. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.7$

$\gamma_1 = 0.7 \longrightarrow \mu = 2.992$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.085	0.213	23	3.076	168
400	0.080	0.356	57	3.072	227
500	0.025	0.365	49	3.016	278
1000	0.020	0.385	58	3.011	564
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.031	0.171	30	3.022	169
400	0.000	0.063	26	2.992	250
500	0.016	0.352	44	3.007	274
1000	0.001	0.122	48	2.993	598
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.001	0.213	22	2.993	193
400	0.082	0.206	25	3.074	225
500	0.086	0.189	29	3.078	306
1000	0.000	0.257	40	2.992	584

Table 6. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.8$

$\gamma_1 = 0.8 \longrightarrow \mu = 4.591$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.084	0.720	15	4.675	164
400	0.185	0.604	42	4.776	225
500	0.001	0.037	52	4.591	297
1000	0.063	0.674	109	4.654	540
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.267	0.282	12	4.857	173
400	0.131	0.147	29	4.722	222
500	0.044	0.045	41	4.635	306
1000	0.011	0.331	68	4.690	597
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.222	0.301	37	4.813	172
400	0.128	0.283	72	4.719	256
500	0.057	0.576	70	4.648	302
1000	0.001	0.382	133	4.592	604

It follows after changing variables that:

$$T_{n_1}(x) = X_{n-k,n} \int_0^1 \frac{\overline{\mathbf{F}}(a_k x)}{\overline{\mathbf{F}}(a_k x)} \overline{\mathbf{F}}_n(xX_{n-k,n}) - \overline{\mathbf{F}}(xX_{n-k,n})dx,$$

$$T_{n_2}(x) = -X_{n-k,n} \int_1^{\frac{t}{X_{n-k,n}}} \overline{\mathbf{F}}(xX_{n-k,n})dx.$$

In order to established the result of theorem we apply the results of [2], we have:

$$\sqrt{k}\frac{\overline{\mathbf{F}}_n(xX_{n-k,n}) - \overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(a_k x)} = x^{\frac{1}{\gamma}} \frac{\gamma}{\gamma_1} W(x^{-\frac{1}{\gamma_1}}) + \frac{\gamma}{\gamma_1 + \gamma_2} x^{\frac{1}{\gamma_1}} \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1} \mathbf{W}(x^{-\frac{1}{\gamma_1}} s)ds.$$

After some elementary but tedious manipulations of integral calculus (change of variables and integration by parts) and by making use of the uniform inequality of the second-order regularly varying functions $\overline{\mathbf{F}}$, to $T_{n_1}(x)$ becomes:

$$\sqrt{k}\frac{T_{n_1}(x)}{X_{n-k,n}\overline{\mathbf{F}}(a_k)} = \int_0^1 (-\gamma s^{-\frac{2\gamma_1}{\gamma}} + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\frac{\gamma}{\gamma_2}-1} + \\ + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\gamma_1})\mathbf{W}(s)ds + o_{\mathbf{p}}(1). \tag{15}$$

Next we move to$T_{n_2}(x)$ which we may write it as follow after changing variables:

$$\frac{\sqrt{k}T_{n_2}(x)}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = \int_1^{\frac{t}{X_{n-k,n}}} \sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}}dx + \int_1^{\frac{t}{X_{n-k,n}}} x^{-\frac{1}{\gamma_1}}dx = \mathbf{I}_1 + \mathbf{I}_2.$$

For $\mathbf{I}_1$ we apply the results of [2]

$$\sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}} = x^{-\frac{1}{\gamma_1}} \frac{x^{-\frac{\tau_1}{\gamma_1}} - 1}{\gamma_1 \tau_1} \sqrt{k}\mathbf{A}_\circ(n/k) + o_p\left(x^{-\frac{1}{\gamma_1}+(1-\eta)/\gamma \pm \varepsilon}\right).$$

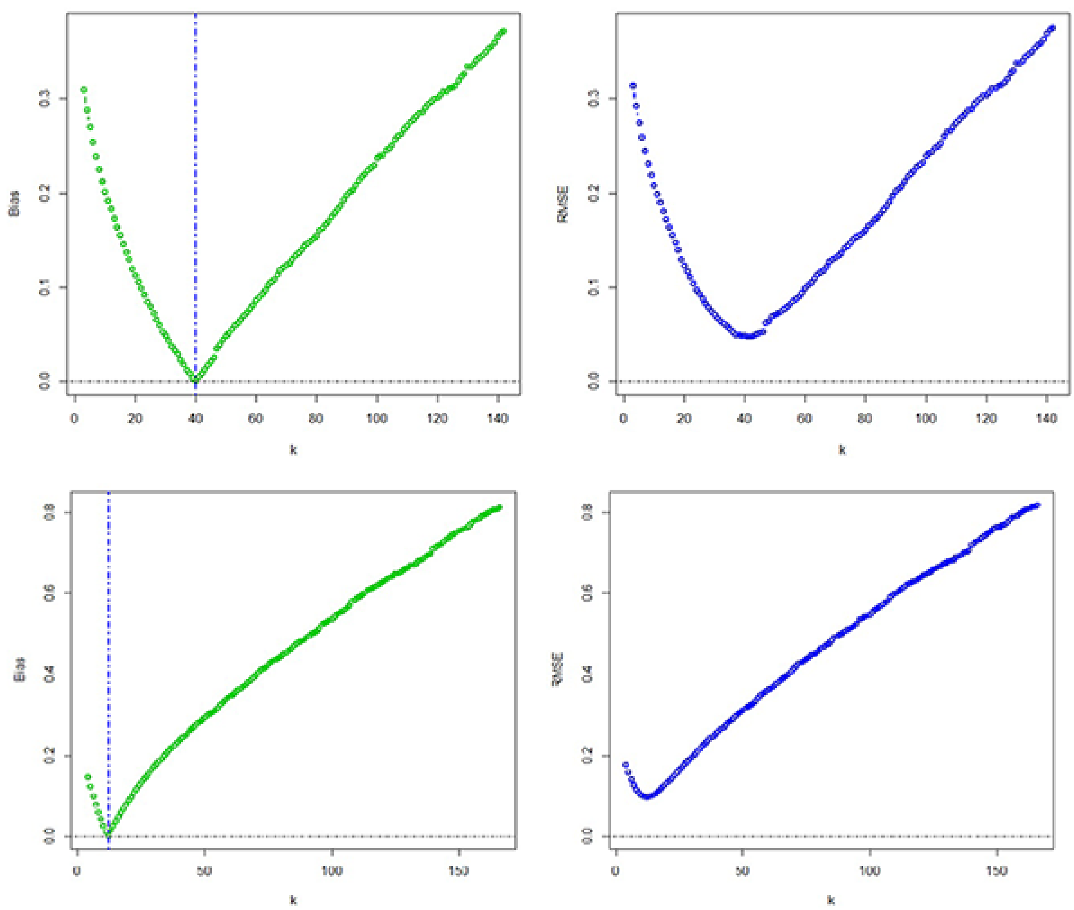

Fig. 1. Absolute Bias (left panel) and RMSE (right panel) of $\widehat{\mu}$ based on samples of size 1000 from Burr distribution truncated by another Burr model with $p = 0.7$ (top) and $p = 0.9$ (bottom) and $\gamma_1 = 0.6$

This implies, almost surely, that

$$\int_1^{\frac{t}{X_{n-k,n}}} \sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}}\,dx = \int_1^{\frac{t}{X_{n-k,n}}} x^{-\frac{1}{\gamma_1}}\frac{x^{-\frac{\tau_1}{\gamma_1}}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_\circ(n/k)dx.$$

Which is equal after simple calculus and by using the mean value theorem we get $\mathbf{I}_1 = o_{\mathbf{p}}(1)$, for the second step by similar argument and using the fact that from Theorem 2.1 of [1] we have $\sqrt{k}\left(\frac{X_{n-k,n}}{t}-1\right) - \gamma\mathbf{W}(1) = o_{\mathbf{P}}(1)$ we get $\mathbf{I}_2 = -\gamma\mathbf{W}(1) + o_{\mathbf{p}}(1)$, that yield to:

$$\frac{\sqrt{k}T_{n_2}(x)}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = -\gamma\mathbf{W}(1) + o_{\mathbf{p}}(1). \tag{16}$$

The two approximation 15 and 16 together give:

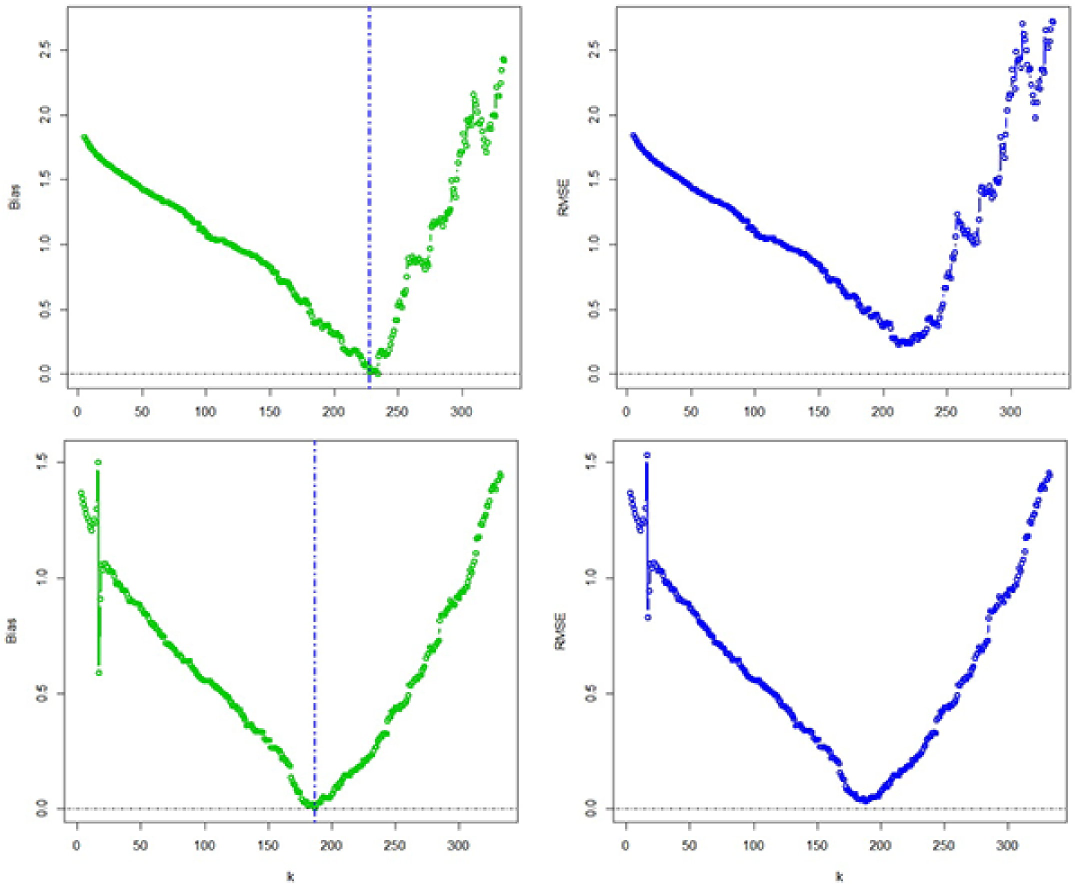

Fig. 2. Absolute Bias (left panel) and RMSE (right panel) of $\widehat{\mu}$ based on samples of size 1000 from Burr distribution truncated by another Burr model with $p = 0.7$ (top) and $p = 0.9$ (bottom) and $\gamma_1 = 0.8$

$$\sqrt{k}\frac{\widehat{\mu}_1 - \mu_1}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = \int_0^1 \Big(-\gamma s^{-\frac{2\gamma_1}{\gamma}} + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\frac{\gamma}{\gamma_2}-1} + \\ + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\gamma_1}\Big)\mathbf{W}(s)ds - \gamma\mathbf{W}(1) + o_{\mathbf{p}}(1). \tag{17}$$

Let us now treat term $\dfrac{\sqrt{k}(\widehat{\mu}_2 - \mu_2)}{t\,\overline{\mathbf{F}}(t)}$. Consider the following forms of μ_2 and $\widehat{\mu}_2$:

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n}) \quad \text{and} \quad \mu_2 = \int_t^\infty \overline{\mathbf{F}}(x)dx,$$

$$\widehat{\mu}_2 - \mu_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n}) - \int_t^\infty \overline{\mathbf{F}}(x)dx.$$

After changing variables we can obtain:

$$\mu_2 = \int_1^\infty t\overline{\mathbf{F}}(tx)dx = t\overline{\mathbf{F}}(t)\int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx$$

and

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n})\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}$$

so the previous equation leads to

$$\widehat{\mu}_2 - \mu_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n})\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - t\overline{\mathbf{F}}(t)\int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx$$

if we devise this equation by $t\overline{\mathbf{F}}(t)$ we can get:

$$\frac{\sqrt{k}\widehat{\mu}_2 - \mu_2}{t\overline{\mathbf{F}}(t)} = \sqrt{k}\frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{t\overline{\mathbf{F}}(t)}\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - \sqrt{k}\int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx.$$

So after adding and Subtract some terms we can decompose $\dfrac{\sqrt{k}(\widehat{\mu}_2 - \mu_2)}{t\overline{\mathbf{F}}(t)}$ into the sum of:

$$\mathbf{I}_1 := \sqrt{k}\frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1}\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(t)}\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}\left[\frac{X_{n-k,n}}{t} - 1\right]$$

$$\mathbf{I}_2 := \sqrt{k}\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(t)}\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}\left[\frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} - \frac{\gamma_1}{1-\gamma_1}\right]$$

$$\mathbf{I}_3 := \sqrt{k}\frac{\gamma_1}{1-\gamma_1}\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(t)}\left[\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - 1\right]$$

$$\mathbf{I}_4 := \sqrt{k}\frac{\gamma_1}{1-\gamma_1}\left[\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(t)} - \left(\frac{X_{n-k,n}}{t}\right)^{-\frac{1}{\gamma_1}}\right]$$

$$\mathbf{I}_5 := \sqrt{k}\frac{\gamma_1}{1-\gamma_1}\left[\left(\frac{X_{n-k,n}}{t}\right)^{-\frac{1}{\gamma_1}} - 1\right]$$

$$\mathbf{I}_6 := \sqrt{k}\left[\frac{\gamma_1}{1-\gamma_1} - \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx\right].$$

For $\mathbf{I}_1$, we have, $\widehat{\gamma}_1 \to \gamma_1$ and $X_{n-k,n}/t \to 1$. Since $\overline{\mathbf{F}}$ is regular variation we obtain $\overline{\mathbf{F}}(X_{n-k,n}) =$ $= (1+o_{\mathbf{P}}(1))\overline{\mathbf{F}}(t)$. From remark 4.1 of [1], we have $\overline{\mathbf{F}}_n(X_{n-k,n})/\overline{\mathbf{F}}(X_{n-k,n}) \to 1$. So,

$$\sqrt{k}\mathbf{I}_1 = (1+o_{\mathbf{P}}(1))\sqrt{k}\left(\frac{X_{n-k,n}}{t} - 1\right).$$

From Theorem 2.1 of [1] we have

$$\sqrt{k}\left(\frac{X_{n-k,n}}{t} - 1\right) - \gamma\mathbf{W}(1) = o_{\mathbf{P}}(1),$$

then

$$\sqrt{k}\mathbf{I}_1 = (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma}{1-\gamma_1}\mathbf{W}(1). \tag{18}$$

For $\mathbf{I}_2$, by using a similar way of $\mathbf{I}_1$, we prove that:

$$\sqrt{k}\mathbf{I}_2 = (1+o_{\mathbf{P}}(1))\frac{1}{(1-\gamma_1)^2}\sqrt{k}(\widehat{\gamma}_1 - \gamma_1). \tag{19}$$

From Theorem 3.1 of [2] we have

$$\sqrt{k}(\widehat{\gamma}_1 - \gamma_1) = \frac{\sqrt{k}\mathbf{A}_\circ(n/k)}{1-\tau_1} - \gamma\mathbf{W}(1) + \frac{\gamma}{\gamma_1+\gamma_2}\int_0^1 (\gamma_2 - \gamma_1 - \gamma\log s)s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds + o_{\mathbf{P}}(1).$$

For $\mathbf{I}_3$ we have

$$\sqrt{k}\mathbf{I}_3 = (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma}{1-\gamma_1}\sqrt{k}\left(\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}-1\right).$$

From Theorem 4.1 of [1] we have

$$\sqrt{k}\left(\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}-1\right) = \frac{\gamma_2}{\gamma_1+\gamma_2}\mathbf{W}(1)+\frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds+o_{\mathbf{P}}(1).$$

So,

$$\begin{aligned}\sqrt{k}\mathbf{I}_3 =&(1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)}\mathbf{W}(1)+\\ &+\ (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma_2^2}{(\gamma_1+\gamma_2)^2(1-\gamma_1)}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds+o_{\mathbf{P}}(1).\end{aligned} \tag{20}$$

For $\mathbf{I}_4$, after the second-order condition of regular variation

$$\sqrt{k}\mathbf{I}_4 = o_{\mathbf{P}}(1). \tag{21}$$

For $\mathbf{I}_5$, using the mean value theorem with $X_{n-k,n}/t \to 1$, we get

$$\sqrt{k}\mathbf{I}_5 = -(1+o_{\mathbf{P}}(1))\frac{1}{1-\gamma_1}\sqrt{k}\left(\frac{X_{n-k,n}}{t}-1\right). \tag{22}$$

From Theorem 2.1 of [1] we have

$$\sqrt{k}\left(\frac{X_{n-k,n}}{t}-1\right)-\gamma\mathbf{W}(1) = o_{\mathbf{P}}(1),$$

then

$$\sqrt{k}\mathbf{I}_5 = -(1+o_{\mathbf{P}}(1))\frac{\gamma}{1-\gamma_1}\mathbf{W}(1).$$

For $\mathbf{I}_6$, we have

$$\int_1^\infty x^{-1/\gamma_1}dx = \frac{\gamma_1}{1-\gamma_1},$$

then

$$\mathbf{I}_6 = \int_1^\infty x^{-1/\gamma_1}dx - \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx.$$

Then, by applying the uniform inequality of regularly varying functions (see, e.g., Theorem 2.3.9 in [9, page 48]) together with the regular variation of $|\mathbf{A}_\circ|$, we show that

$$\sqrt{k}\mathbf{I}_6 \sim \frac{\sqrt{k}\mathbf{A}_\circ(t)}{(\gamma_1+\tau_1-1)(1-\gamma_1)}. \tag{23}$$

Summing up above equations, we get

$$\begin{aligned}\frac{\sqrt{k}\left(\widehat{\mu}_2-\mu_2\right)}{t\overline{\mathbf{F}}(t)} =& \left(\frac{\gamma_1\gamma_2-2\gamma(\gamma_1+\gamma_2)}{(1-\gamma_1)(\gamma_1+\gamma_2)}\right)\mathbf{W}(1)-\frac{\gamma^2}{\gamma_1+\gamma_2}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}W(s)\log s ds+\\ &+\ \frac{\gamma_1^2\gamma_2(\gamma_2-\gamma_1)}{(\gamma_1+\gamma_2)^2(1-\gamma_1)}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds+\frac{\sqrt{k}\mathbf{A}_\circ(n/k)}{1-\tau_1}+\\ &+\ \frac{\sqrt{k}\mathbf{A}_\circ(t)}{(\gamma_1+\tau_1-1)(1-\gamma_1)}.\end{aligned} \tag{24}$$

Finally, Summing up equations 17 and 24 achieves the proof.

4.2. Proof of Corollary 2.1

We set:

$$\frac{\sqrt{k}(\widehat{\mu}-\mu)}{\overline{\mathbf{F}}(X_{n-k,n})X_{n-k,n}} = \Delta + \frac{(\gamma_1+\tau_1-1)(1-\gamma_1)+(1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}\sqrt{k}\mathbf{A}_\circ(n/k),$$

where $\Delta = c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3 + c_4\Delta_4 + c_5\Delta_5$ with

$$\Delta_1 = \mathbf{W}(1), \quad \Delta_2 = \int_0^1 s^{-\frac{2\gamma_1}{\gamma}}\mathbf{W}(s)ds, \quad \Delta_3 = \int_0^1 s^{-\gamma_1}\mathbf{W}(s)ds,$$
$$\Delta_4 = \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\log(s)\mathbf{W}(s)ds, \quad \Delta_5 = \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1\cdot}\mathbf{W}(s)ds.$$

After elementary but tedious computations, we find the following covariance as asymptotic variance: $\mathbf{\Gamma\Sigma\Gamma}^t$, where

$$\mathbf{\Gamma} = \left(\frac{p(1-p)}{1-\gamma_1}, -p\gamma_1, p(1-p), \gamma_1 p^2(1-p), p(1-p)+\frac{\gamma_1^2 p}{1-\gamma_1}\right)$$

and $\mathbf{\Gamma}^t$ is the transpose of $\mathbf{\Gamma}$, Σ is the variance-covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} \\ \alpha_{1,2} & \alpha_2 & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} \\ \alpha_{1,3} & \alpha_{2,3} & \alpha_3 & \alpha_{3,4} & \alpha_{3,5} \\ \alpha_{1,4} & \alpha_{2,4} & \alpha_{3,4} & \alpha_4 & \alpha_{4,5} \\ \alpha_{1,5} & \alpha_{2,5} & \alpha_{3,5} & \alpha_{4,5} & \alpha_5 \end{bmatrix},$$

$$\mathbf{E}(\Delta_1^2) = 1, \quad \alpha_2 := \mathbf{E}(\Delta_2^2) = \frac{2p^2}{(-2+p)(-4+3p)},$$

$$\alpha_3 := \mathbf{E}(\Delta_3^2) = \frac{(1-2p)}{p^4(1-p)},$$

$$\alpha_4 := \mathbf{E}(\Delta_4^2) = \frac{1-2p}{p^4(1-p)^2} - \frac{2\gamma_1 p}{(1-p)^3} - \frac{2(1-p)^{-2}}{(-1-p)} + \frac{1}{(1-p)^2(2p-1)^2},$$

$$\alpha_5 := \mathbf{E}(\Delta_5^2) = \frac{4p-3}{-p(1-p)^2(2p-1)},$$

$$\alpha_{1,2} := \mathbf{E}(\Delta_1\Delta_2) = \frac{p}{-2(1-p)},$$

$$\alpha_{1,3} := \mathbf{E}(\Delta_1\Delta_3) = \frac{1}{-\gamma_1+2},$$

$$\alpha_{1,4} := \mathbf{E}(\Delta_1\Delta_4) = -\frac{1}{p^2},$$

$$\alpha_{1,5} := \mathbf{E}(\Delta_1\Delta_5) = \frac{1}{p},$$

$$\alpha_{2,3} := \mathbf{E}(\Delta_2\Delta_3) = \frac{3p^3}{2(-2+p)(p-1)(-2+\gamma_1 p+3p)} + \frac{p}{(-2+p)(-\gamma_1+2)},$$

$$\alpha_{2,4} := \mathbf{E}(\Delta_2\Delta_4) = \frac{3p^2}{2(p-1)}\left(\frac{p}{2}-\frac{1}{4-p}\right)^2 + \frac{3p-2}{6}\left(\frac{p}{1+p}\right)^2,$$

$$\alpha_{2,5} := \mathbf{E}(\Delta_2\Delta_5) = \frac{-p^3\gamma_1}{2(-1+p)(-2+p)(-1-p+\gamma_1(-2+p))} + \frac{1}{-2+p},$$

$$\alpha_{3,4} := \mathbf{E}(\Delta_3\Delta_4) = \frac{-1}{(\gamma_1+2)(-\gamma_1+p+1)^2} + \frac{1}{(-\gamma_1+1)}\left[\left(\frac{p}{-1+p^2}\right)^2 + \left(\frac{1}{1-p}\right)^2\right],$$

$$\alpha_{3,5} := \mathbf{E}(\Delta_3\Delta_5) = \frac{1}{(-\gamma_1+2)\,(-\gamma_1+p+1)} + + \frac{p^3\gamma_1^3}{(-\gamma_1+1)(-p\gamma_1-p\gamma_1^2-p^2\gamma_1^2-p+1)},$$

$$\alpha_{4,5} := \mathbf{E}(\Delta_4\Delta_5) = \frac{(1-p)^2}{p\gamma_1(-\gamma_1-1)(2p-1)} + \frac{1-p}{p^2}.$$

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Оценка среднего распределения с тяжелыми хвостами при случайном усечении

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Аннотация. Вдохновленные работой Л. Пэна по оценке среднего значения распределения с тяжелыми хвостами в случае полных данных, мы предлагаем альтернативную оценку и изучаем ее асимптотическую нормальность, когда дело касается усеченной справа случайной величины. Имитационное исследование выполняется для анализа поведения конечной выборки на предлагаемой оценке.

Ключевые слова: случайное усечение, оценка Хилла, оценка Линдена-Белла, распределения с тяжелыми хвостами.