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Sharply 3-transitive Groups with Finite Element

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Abstract. In this paper we study sharply 3-transitive groups. The local finiteness of sharply triply transitive permutation groups of characteristic $p > 3$ containing a finite element of order p is proved.

Keywords: group, sharply k-transitive group, sharply 3-transitive group, locally finite group, near-domain, near-field.

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Introduction

We recall that the group G of permutations of the set F ($|F| \geqslant k$) is called *exactly k-transitive* on F if for any two ordered sets $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ elements from F such that $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for $i \neq j$, there is exactly one element of the group G taking α_i to β_i $(i = 1, \ldots, k)$.

In 1872, K. Jordan described the class of finite sharply k-transitive groups for $k \geqslant 4$ ([1, page 215]).

In infinite groups J. Tits and M. Hall established that for $k \geqslant 4$ infinite sharply k-transitive groups do not exist ([1, page 215], [2, page 86–87]).

Unlike the cases $k \geqslant 4$, the sets of finite exactly 2- and 3-transitive groups are countable, and the locally finite sets are continuous.

Sharply 2- and 3-transitive groups are closely related algebraic structures such as near-fields, near-domains, KT-fields (Kerby-Tits fields), etc. (see [1, Ch. V], [2, chap. 20]).

Finite exactly 2- and 3-transitive groups and near-fields were classified by G. Zassenhaus [1, ch. IV and Theorem V.5.2]. Complete description of locally finite sharply 3-transitive groups in 1967 got O. Kegel [3].

The study of the class of infinite exactly 2- and 3-transitive groups is actively continued at the present time. In 2000 V. D. Mazurov in [4] fully described exactly 3 - transitive groups with abelian stabilizers of two points. In 2011, T. Grundhöfer and E. Jabara proved the local finiteness of the binary finite sharply doubly transitive groups [5]. In 2013, in the paper [6], A. I. Sozutov established a similar fact for the periodic groups of Shunkov.

In the paper [7], in the class of sharply triply transitive groups, the local finiteness of permutation groups with a periodic stabilizer of two points was proved and, as a consequence, the local finiteness of the periodic sharply 3-transitive groups.

In the papers [8, 9], examples of sharply doubly transitive groups of characteristic 2 that do not contain regular abelian normal subgroups are constructed, and in [10], there are similar examples of sharply 3-transitive groups. These examples show that there are near-domains of characteristic 2 that are not near-fields and KT-fields, (F, σ), in which near-domains $(F, +, \cdot)$ are not near-fields. This provides a basis for studying these structures with additional restrictions.

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Recall that a nonidentity element k of a group G is called *finite in* G if for any $g \in G$ the subgroup $\langle k, k^g \rangle$ is finite.

Let G be sharply 3-transitive on X, J the set of involutions in G, $J^2 = \{kv | k, v \in J\}$. The characteristic G ($Char(G)$) is defined as follows [1]:

1. $Char(G) = 2$, if elements from J do not fix points from X;
2. $Char(G) = 0$ if each $g \in J^2 \setminus \{1\}$ is of infinite order;
3. $Char(G) = p$, where p is odd prime, if the order of each $g \in J^2 \setminus \{1\}$ is p.

In continuation of the research started in [7] and [11], in this work a special case of Theorem 6 announced in [12] is proved:

Theorem 1. *A sharply triple transitive permutation group of characteristic $p > 3$, containing a finite element of order p, is locally finite.*

Proof of the theorem

Let G be an infinite sharply triply transitive permutation group of the set $X = F \cup \{\infty\}$. By B we denote the stabilizer G_α of the point $\alpha \in X$ and through H — stabilizer $G_{\alpha\,\beta} = G_\alpha \cap G_\beta$ of two points $\alpha = \infty \in X$, $\beta \in F$. Let also J be the set of involutions of the group G, and J_m be the set involutions stabilizing exactly m points, $m = 0, 1, 2$. Let us also formulate the well-known properties of involutions from groups $G = T_3(F, v)$ and $B = T_2(F)$(see, for example, [1, Ch. V]) with comments.

Lemma 1. *The following statements are true:*

1. *The group $B = G_\infty$ is regular on the set F an elementary abelian p-subgroup of U and $B = U \leftthreetimes H$ — Frobenius group.*
2. *U — Sylow p -subgroup of the group G, $B = N_G(U)$, $U^\# = a^H$, $C_G(u) = U$ for any element $u \in U^\#$ and $U \cap U^x = 1$ for any element $x \in G \setminus B$.*
3. *$H = G_\infty \cap G_\alpha$, H contains the only involution z, $z \in J_2$, $C_G(z) = N_G(H)$.*
4. *Each subgroup of order qr in H, where q, r not necessarily different primes, cyclic, and $H \cap H^x = 1$ for any element $x \in G \setminus N_G(H)$.*
5. *$N = N_G(H) = H \leftthreetimes \langle v \rangle$, where v is an involution from J_2, $C_H(v) = \langle z \rangle$.*
6. *If $N \cap N^x \neq 1$ for $x \in G \setminus N$, then $N \cap N^x = \langle t \rangle$, where $t = t(x)$ is an involution.*
7. *$G = B \cup BvU$ and $B \cap B^x = H^b$ for any $x \in G$ setminusB and a suitable $b = b(x) \in B$.*

Proof. 1. The statement follows from [6, Theorem 2].

2. The statement easily follows from the exact 3-transitivity of G (see also [7, Lemma 1], [13], item 1 of the lemma and finiteness of elements from U. Non-trivial element from $U \cap U^x$ must stabilize two points, which is impossible in view of item 1.

3. The statement is well known [1, 6, 14].

4. The statement follows from Burnside's theorem [15, Theorem 1.2], 3-transitivity of G and equality $B \cap B^x = G_\infty \cap G_{\infty^x}$.

5. This statement and statement 6 are obvious.

7. Follows from 2- (and even 3-) transitivity and items $1, 5$ of the lemma.

The lemma is proved. □

The groups H and $N = C_G(z)$ will also be denoted by H_z and N_z, and for $k = z^g$ by H_k and N_k we will denote subgroups H^g and N^g.

Lemma 2. *The following statements are true:*

1. *Either $J = J_2$, or $J = J_0 \cup J_2$, while $J_2 = v^G$.*
2. *For each involution j the set $vN \cap j^G$ is infinite.*
3. *For each involution $j \in J$ the set $J_2 \cap C_G(j)$ is infinite.*
4. *Every Sylow 2-subgroup in H is (locally) cyclic, or (locally) quaternionic; are they conjugate, isomorphic, we do not know yet.*
5. *Every Sylow 2-subgroup of T from N whose order is greater than 4, is a Sylow 2-subgroup of G.*
6. *If a Sylow 2-subgroup T of N is a proper subgroup of a Sylow 2-subgroup R of G, then R is a (locally) dihedral group.*
7. *G contains no elementary abelian subgroups of order 8, containing an involution from J_2. The rank of Sylow 2-subgroups in N is 2. The rank of any Sylow 2-subgroup of G containing an involution from J_2, is equal to 2.*

Proof. 1. The inequalities $0 \leqslant m \leqslant 2$ follow from the sharply 3-transitivity of the group G. Lemma 1 implies that the partitions $J = J_1 \cup J_2$ and $J = J_0 \cup J_1 \cup J_2$ are impossible, and it is obvious that the sets J_1 and J_2 are conjugacy classes. Since $\mathrm{Char}\, G = p > 2$, then either $J = J_2$ or $J = J_0 \cup J_2$.

2. In each such class j^G there is an involution k permuting the points α and β. Further, we apply Ditzmann's lemma [16, Lemma 2.3].

3. The involution j is contained in the subgroup $N_{\gamma\delta}$, if the permutation j contains a cycle $(\gamma\,\delta)$.

4. Follows from Shunkov's theorem [16, Theorem 2.15].

5. The subgroup $\langle z \rangle$ is characteristic in T and $x \in N_G(T)$ implies $x \in N = C_G(z)$.

6. Follows from the fact that $C_R(z) = T$. In particular, potentially R can be an infinite locally dihedral group.

7. If $E_8 \leqslant N$, then $H \cap E_8 = E_4$, which contradicts the uniqueness of the involution z. Further we use item 6 of the lemma. The lemma is proved. □

Lemma 3. *The set of all 2-elements of the group H invertible involution v, is a (locally) cyclic 2-subgroup of S. If $x \in H \setminus S$ and $x^2 \in S$, then the order of the element $x^{-1}vxv$ is infinite.*

Proof. The assertions of the lemma are proved in [13, Lemmas 5, 6]

By the conditions of the theorem, all subgroups $L_x = \langle a, a^x \rangle$ in G are finite, and for $x \in J$, the subgroups K_x are also finite. Let's find out their structure. Let's start with the subgroups $L = \langle a, a^v \rangle$, $K = \langle a, v \rangle$.

Lemma 4. *The subgroup $L = \langle a, a^v \rangle$ is isomorphic to the group $L_2(p^n)$ for some n.*

Proof. It is clear that $|K : L| \leqslant 2$. According to Lemma 1, $P = L \cap U$ and $P_2 = L \cap U^x$ — elementary Abelian Sylow p-subgroups of L, with Silov p-subgroups of L are pairwise coprime, in particular, L is not an abelian group.

It is clear that $B_1 = N_L(P) = L \cap B$. If $B_1 = P$, then $P \cap P^x = 1$ for any $x \in L \setminus P$, and by the Frobenius theorem $L = M \leftthreetimes P$ is the Frobenius group with nilpotent kernel M [15, Thompson's Theorem 1.5] and the cyclic complement $P = \langle a \rangle$ [15, Burnside's Theorem 1.2]. By Lemma 2, the 2-rank of the group K (and the group L) does not exceed 2, and if $2 \in \pi(M)$, then the order of the center of a Sylow 2-subgroup from the Frobenius kernel M is 4. By the conditions $p > 3$ and, therefore, $2 \notin \pi(M)$.

Obviously, $|B \cap K| = 2p$ and by Frattini's argument and Lemma 1 $N_K(P) = \langle a \rangle \leftthreetimes \langle k \rangle = D$ — dihedral group, where $k \in v^K$. Hence, by virtue of the same Burnside theorem [15, Theorem 1.2] $C_Z(k) \neq 1$ for the center Z of each Sylow q-subgroups of M. Obviously, $C_Z(k) < H^x$ for some x, and in view of item 4 of Lemma 1, $|\Omega_1(Z)| = q$. Hence, the dihedral group $B \cap P$ is contained in the group of automorphisms of a cyclic group of order q, a contradiction, therefore, $B \cap P \neq P$.

Note that by Frattini's argument and Lemma 1 the group K contains the group anyway dihedral $D = \langle a \rangle \leftthreetimes \langle k \rangle$, where $k \in v^K$. Let M be the minimal normal subgroup in K from L. Consider the case when M — elementary abelian q-group. As proved above, $q \neq 2$. Since P is strongly isolated in $L = \langle P, P^v \rangle$ as above, we have $q \neq p$, $M \leftthreetimes P$ is a Frobenius group, $P = \langle a \rangle$, $C_M(k) \neq 1$, $|M| = q$ and $D \leqslant \operatorname{Aut} M$, a contradiction. Hence, M is a direct product of non-abelian simple groups, and since the 2-rank of the group M does not exceed 2, then M is a simple group of 2-rank 2.

If $P \nleqslant M$, then by Frattini's lemma $P \cap N_L(S) \neq 1$ for some Sylow 2-subgroup S of M and each element from $P^{\#} \cap N_L(S)$ acts on S regularly, which is impossible, since the 2-rank of G is at most 2 and $p > 3$. Therefore, $P \leqslant M$ and $|L : M| \leqslant 2$, and therefore $M = \langle P, P^v \rangle = L$.

If a Sylow 2-subgroup S in L is dihedral (Lemma 2), then by the Gorenstein-Walter theorem [17, p. 27] $L \simeq L_2(q)$, q is odd, or $L \simeq A_7$.

Let's exclude the group $L \simeq A_7$. For $p = 7$, by Kerby's theorem, H contains a unique subgroup of order 3, and in A_7 is an elementary abelian subgroup E_9, which contradicts Lemma 1. Hence, $p = 5$. The involution k inverting a cyclic subgroup of order 5 is obviously contained in J_2. It is easy to check (see, for example, cite [Proposition 14] LSS), that $C_L(k)$ contains the only subgroup $\langle b \rangle \leqslant E_9$ of order 3, which is contained in H_k. But $E_9 \leqslant C_L(b) \nleqslant H_k$, which contradicts Lemma 1. Therefore, L cannot be isomorphic to A_7.

Let $L \simeq L_2(q)$. If $q \neq p^n$ then $P = \langle a \rangle$ and p divides either $q - 1$ or $q + 1$. Since $C_G(P)$ — $2'$ is a group, then either $q - 1 = 2p$ or $q + 1 = 2p$. Note that then $t \in L \cap J_2$, $C_L(t) \leqslant N_L(P)$, in this case either $|C_L(t)| = q + 1$, or $|C_L(t)| = q - 1$. However, this is not possible. Therefore, $L \simeq L_2(p^n)$. If $v \notin L$, using Lemmas 1–3 and information from [19, p. 8–10], apparently it can be shown that $K \simeq PGL_2(p^n)$.

Let a Sylow 2-subgroup S in L be not dihedral. Since $v \in J_2$, in view of item 6 of Lemma 2, this means that $J \cap L \subset J_2$. As Alperin, Brower and Gorenstein proved [20] finite simple groups of 2-rank 2, up to isomorphism, are the following groups: $L_2(q)$, A_7, $L_3(s)$, $U_3(r)$, M_{11}, $U_3(4)$, where q, s, r are odd and $q > 3$.

First, let's exclude the groups $U_3(4)$ and M_{11} from this list. In $U_3(4)$ all involutions are conjugate and the Sylow 2-subgroup S is of order 64, all its involutions lie in the center of Z, $|Z| = 4$ (see, for example, [18, Proposition 13]). If $v \in L$, then $Z^{\#} \subset J_2$, which contradicts Lemmas 1, 2. If $Z^{\#} \subset J_0$, then $v \notin L$, which contradicts Lemma 2. In M_{11} all involutions are conjugate, the Sylow 2-subgroup S is a semidihedral group of order 16 and the centralizer of the involution is isomorphic to $GL_2(3)$ (see, for example [18, clause 14]). As noted above, $J \cap S \subset J_2$. Therefore, $S < N_k$, where k is the central involution from S.

The group S contains a cyclic subgroup of index 2, suitable for the role intersection of $S \cap H_k$, but each involution from $S \cap H$ centralizes an element of order 4 in $S \cap H_k$, which is impossible by Lemmas 1, 2. Hence, L cannot be isomorphic group M_{11}.

Assume that L is isomorphic to $L_3(s)$, or $U_3(r)$. Then, by [18, Proposition 11], all involutions and quadruple groups in L are conjugate, L contains an element of order 8 and a Sylow 2-subgroup S in L is isomorphic to either a semidihedral group

$$SD_m = \langle s, k \mid s^{2^{m+1}} = k^2 = 1,\ s^k = s^{-1+2^m}\rangle,\ m \geqslant 2, \quad \text{or woven group} \tag{1}$$

$$WR_m = \langle s_1, s_2, k \mid s_1^{2^m} = s_2^{2^m} = k^2 = 1,\ s_1 s_2 = s_2 s_1,\ s_1^k = s_2,\ s_2^k = s_1\rangle,\ m \geqslant 3. \tag{2}$$

Recall that in the case under consideration $S \cap J \subset J_2$ and, therefore, $S \leqslant N_j$ for the involution $j \in Z(S)$. In the group $S = WR_m$ from (2), each subgroup of index 2 contains the subgroup E_4, which is impossible by Lemma 1. And in the cyclic subgroup of order 8 from the group $S = SD_m$ is a subgroup of order 4 commuting with all involutions from S, which again contradicts Lemma 1. Therefore, in all cases $L \simeq L_2(q)$. As proved above, $q = p^n$, and the lemma is proved. □

Lemma 5. *For any element $c \in U^v$ the subgroup $L = \langle a, c\rangle$ is isomorphic to the group $L_2(p^n)$ for some $n = n(a, c)$.*

Proof. By virtue of the finiteness condition for the element a and items 1–2 of Lemma 1 the subgroup L is finite. Further, as in the proof of Lemma 4, $P = L \cap U$ and $P_2 = L \cap U^x$ — elementary Abelian Sylow p-subgroups in L, Sylow p-subgroups in L are pairwise coprime and L is not an abelian group. To continue to follow the logic of the proof of Lemma 4, we prove that the 2-rank of the group L does not exceed 2. If $L \cap J_2$ is nonempty, then the desired follows from Lemma 2. Let $L \cap J_2 = \emptyset$. Note that by claim 3 of Lemma 1 the involution $z \in H$, and by claim 1 of the same lemma, z inverts the elements a and c: $a^z = a^{-1}$, $c^z = c^{-1}$. Therefore, $z \in N_G(L)$, the subgroup $K = \langle a, c, z\rangle$ is finite, $|K : L| \leqslant 2$, $K \cap J_2 \neq 2$ and for K the boundedness of the 2-rank follows from Lemma 2. Hence, the 2-rank of the group L does not exceed 2, and $D = \langle a, z\rangle$ — dihedral group, $D \leqslant K$. Moreover, in the case $L \cap J = \emptyset$, by Lemma 2 the Sylow 2-subgroups in K (and in L) are dihedral. Taking into account these remarks, part of the proof of Lemma 4, on the structure of L groups with dihedral Sylow 2-subgroup, carries over literally to the case under consideration. The lemma is proved. □

Lemma 6. *For any non-permutable elements $x, s \in a^G$ the subgroup $L = \langle s, x\rangle$ is finite and isomorphic to the group $L_2(p^n)$ for suitable $n = n(s, x)$.*

Proof. Due to the arbitrary initial choice of the element a from the class of conjugate elements of a^G it follows that statement of Lemma 5 is true for any $s \in U^\#$ and $x \in U^v \cap a^G = U^{v\#}$. Since G is 3-transitive on the set U^G, we conclude that that the lemma is true. □

Proof of the theorem. According to [19, p. 9] the group $L = \langle a, a^v\rangle$, isomorphic $L_2(q)$ by Lemma 4, has $\dfrac{q(q+1)}{2}$ cyclic subgroups of order $\dfrac{(q-1)}{2}$ (Cartan subgroups), of these, $(B \cap L) \cup (B^v \cap L)$ contains $2q - 1$ such subgroups. Since $\dfrac{q(q+1)}{2} > 2q - 1$ for $q > 3$, then there is a pair of dots $\gamma, \delta \in X \setminus \{\alpha, \beta\}$ for which the intersection $L \cap G_{\alpha\beta}$ is cyclic subgroup conjugate to the Cartan subgroup $L \cap H$ of order $\dfrac{(q-1)}{2}$. The group G acts on the set J_2 twice transitively, since it is twice transitive on the set H^G, and each the subgroup H^g is defined by its unique central involution z^g from J_2 (Lemma 1). Hence we deduce that any pair of involutions from $H \cap J_2$ is contained in an appropriate subgroup conjugate to the subgroup L. This means that the involution v is finite in the group N, and by [16, Corollary 2.30] the subgroup N is locally finite. By Theorem 2 in [21], the group G is locally finite. The theorem is proved. □

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Точно трижды транзитивные группы с конечным элементом

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Аннотация. В настоящей работе исследуются точно трижды транзитивные группы. Доказана локальная конечность точно трижды транзитивных групп подстановок характеристики $p > 3$, содержащих конечный элемент порядка p.

Ключевые слова: группа, точно k-транзитивная группа, точно трижды транзитивная группа, локально конечная группа, почти-область, почти-поле.