DOI: 10.17516/1997-1397-2021-14-2-230-241
УДК 519.21

Joint Distribution of the Number of Vertices and the Area of Convex Hulls Generated by a Uniform Distribution in a Convex Polygon

Isakjan M. Khamdamov*
National University of Uzbekistan named after Mirzo Ulugbek
Tashkent, Uzbekistan
Zoya S. Chay†
Tashkent University of Information Technologies named after M. al-Khwarizmi
Tashkent, Uzbekistan

Received 05.07.2020, received in revised form 13.12.2020, accepted 20.01.2021

Abstract. A convex hull generated by a sample uniformly distributed on the plane is considered in the case when the support of a distribution is a convex polygon. A central limit theorem is proved for the joint distribution of the number of vertices and the area of a convex hull using the Poisson approximation of binomial point processes near the boundary of the support of distribution. Here we apply the results on the joint distribution of the number of vertices and the area of convex hulls generated by the Poisson distribution given in [6]. From the result obtained in the present paper, in particular, follow the results given in [3, 7], when the support is a convex polygon and the convex hull is generated by a homogeneous Poisson point process.

Keywords: convex hull, convex polygon, Poisson point process, binomial point process, central limit theorem.

Citation: I.M. Khamdamov, Z.S. Chay, Joint Distribution of the Number of Vertices and the Area of Convex Hulls Generated by a Uniform Distribution in a Convex Polygon, J. Sib. Fed. Univ. Math. Phys., 2021, 14(2), 230–241. DOI: 10.17516/1997-1397-2021-14-2-230-241.

Introduction

This paper is devoted to the study of properties of convex hulls generated by independent observations over a random vector that has a uniform distribution in a convex polygon. Convex hulls are very complex objects from the analytic point of view. Therefore, studying the properties of the simplest functionals of convex hulls, such as, the number of vertices or the area, is not an easy task. This explains the fact that, prior to obtaining the central limit theorem for the number of vertices of a convex hull by P. Groeneboom, the main achievement was considered to be the study of asymptotic expressions for the mean values of similar functionals (see, for example, [4, 5, 16]); the problems on asymptotic expressions for the variance remained unsolved until the appearance of the studies by C. Buchta [1, 2] and J. Pardon [14, 15].

It should be noted that P. Groeneboom, using the well-known property of homogeneous binomial point processes, which is that near the boundary of the support, it is almost indistinguishable

*khamdamov.isakjan@gmail.com
†chay1526@mail.ru

from a homogeneous Poisson point process, and using such powerful techniques as strongly mixing stationary processes and martingales, has proved the central limit theorems for the number of vertices of a convex hull in the case when the support of the original uniform distribution is either a convex polygon or a unit disk. The modified P. Groeneboom technique was applied in [3] to prove limit theorems for the area and perimeter of a convex hull in a polygon, and in [9], to prove a limit theorem for an area outside a convex hull in a disk.

Similar results were obtained later by J. Pardon [16, 17] without imposing any regularity conditions on the support boundary. In the present work, there is no need for using martingales, strongly mixing stationary processes, etc.; the approach used is a modification of the methods presented in [7, 10–13]. The results obtained by Sh. K. Formanov, I. M. Khamdamov in [6], are applied here; a joint limit distribution for the number of vertices and the area of the convex hull generated by a Poisson point process in a cone was obtained by elementary analytical and direct probabilistic methods.

1. Statement of problem and results

Let $\vec{x}_j,\ j = 1, 2, \ldots, n$ be the independent observations over a random vector having a uniform distribution in a convex polygon A with r sides. A matrix X_n is called a sample, the j-row of which is formed by the components of the vector $\vec{x}_j$. Let us denote the convex hull generated by vectors $\vec{x}_j,\ j = 1, 2, \ldots, n$ by $C_n = C_n(X_n)$.

We are interested in the joint limit distribution of the following functionals of C_n: the total number of vertices ν_n and the area S_n. It is clear that C_n, and, consequently, the indicated functionals, are uniquely determined by the set of vertices W_n. If the principle of vertex labeling is chosen, then it can be represented as a $\nu_n \times 2$ matrix. It is easy to show that this matrix has the property of sufficiency with respect to the boundary of the set A — the support of original distribution. The latter circumstance is of interest from the point of view of statistics of uniform distributions.

Before formulating the main results, we introduce some notation. Let S be the area of the polygon A. Then we assume that

$$D_n = S - S_n$$

and let

$$a_n = \frac{2r\log n}{3},\quad a'_n = \frac{Sa_n}{n},\quad b_n = \sqrt{\frac{27}{10r\log n}},\quad b'_n = n\sqrt{\frac{27}{28rS^2\log n}} = \frac{\sqrt{5}nb_n}{\sqrt{14}S}.$$

We denote by ω a vector having a two-dimensional normal distribution with a zero vector of mean values, unit variances and a correlation coefficient $\sqrt{5/14}$.

Let us state the main theorem.

Theorem 1. *Under our assumptions, a random vector with components $b_n(\nu_n - a_n)$ and $b'_n(D_n - a'_n)$ converges in distribution to ω.*

Let us make the necessary explanations of the notation. The symbols $\xrightarrow{d}$, $\xrightarrow{p}$, $\to a.s.$ denote convergence in distribution, in probability, and almost sure, respectively. $f(\varepsilon) \asymp g(\varepsilon)$ means that there are positive constants c_1, c_2, ε_0 such that $c_1 f(\varepsilon) \leqslant g(\varepsilon) \leqslant c_2 f(\varepsilon)$ for any $0 < \varepsilon < \varepsilon_0$. Generally $o_p(1)$ is used for a sequence of random variables converging in probability to zero. The notation $\xi_n = O_p(1)$ means that $\sup_{n\geqslant 1} P(|\xi_n| > t) \to 0$ as $t \to \infty$. Everywhere $c,\ c_1,\ c_2, \ldots$ are the positive constants whose values might be changed from line to line and $c(\beta),\ c_1(\beta),\ c_2(\beta)$

are the positive constants, depending on the specified arguments. Further, $\xi \overset{dis}{=} \zeta$ means that the random variables ξ and ζ have a common law of probability distribution.

2. The Poisson approximation

In this section, we present the key idea of [7] about the Poisson approximation of a homogeneous binomial point process (h.b.p.p) $B_n(A)$ generated by n independent observations of a random variable having a uniform distribution with support A in a slightly different way. Here we consider the more general case, assuming that A is an arbitrary bounded convex set in R^2.

Let Γ_A be the boundary of the set A. For each $z \in \Gamma_A$, consider an open sphere $S(z, \varepsilon)$ of radius ε centered in z. It is easy to see that the set $A_\varepsilon = A - \bigcup_{z \in \Gamma_A} S(z, \varepsilon)$ is a strip along the border Γ_A. Let us denote $B_\varepsilon = A - A_\varepsilon$ and assume that $\lambda(A) = 1$, where $\lambda(\cdot)$ is the Lebesgue measure.

Let W_n, as before, be the set of vertices of the convex hull C_n generated by $B_n(A)$. The next lemma is a simple modification of Lemma 2.1 and its Corollary 2.1 given in [7].

Lemma 1. *There is a sequence of positive numbers ε_n converging to zero such that the probability that at least one of the vertices C_n laying in B_ε, converges to zero in $\varepsilon > \varepsilon_n$.*

Proof. It is easy to see that the event $E = \{W_n \bigcap B_\varepsilon \neq \varnothing\}$ coincides with the event "there is a pair of neighboring vertices w_1 and w_2 such that $w_1 \in B_\varepsilon$ ". Let the straight line $(p, z - w_1) = 0$ pass through the point w_2. Since $w_1 \in B_\varepsilon$, then this line divides A into two parts, the measure of each is no less than some value of $c(\varepsilon) > 0$ such that $\lim_{\varepsilon \to 0} c(\varepsilon) = 0$. Therefore at $n > 2$

$$P(E) = \frac{n(n-1)}{2} \iint\limits_{w_1 \in B_\varepsilon, w_2 \in A} P^{n-2} \{n-2 \text{ the sample points } X_n \text{ lie on one side}$$

$$\text{of the straight line } (p, z - w_1) = 0\} \, dw_1 dw_2 \leqslant n^2 (1 - c(\varepsilon))^n .$$

It remains to assume that

$$\varepsilon_n = \inf \left\{ \varepsilon : c(\varepsilon) \geqslant \frac{3 \log n}{n} \right\}. \tag{1}$$

The lemma is proved. □

Note that the rate of decrease $c(\varepsilon)$ at $\varepsilon \to 0$ depends on the smoothness Γ_A. In particular, if A is a sphere, then $c(\varepsilon) \asymp \varepsilon^{\frac{3}{2}}$; if A is a polygon, then $c(\varepsilon) \asymp \varepsilon^2$ and etc.

Since we are not interested in the estimates of the rate of convergence in the theorems given below, we will not worry about optimizing the choice of the strip containing W_n.

Let now $\Pi_n(\cdot)$ be a homogeneous Poisson point process (h.p.p.p.), the intensity of which is equal to $n\lambda(\cdot)$.

Consider the narrowing $\Pi_n(A)$ of this process to the set A. We denote by C'_n the convex hull generated by it, and the set of its vertices we denote by W'_n.

Lemma 2. *The probability that at least one of the vertices C'_n laying in B_ε, converges to zero, as $n \to \infty$ uniformly in $\varepsilon > \varepsilon_n$, where ε_n, is determined by relation (1).*

Proof. We assume that

$$E' = \left\{ W'_n \bigcap B_\varepsilon \neq \varnothing \right\}$$

and let $\mu_n(\cdot)$ be the random counting measure corresponding to $\Pi_n(A)$. By the formula of total probability we have

$$P(E') = \sum_{k=0}^{\infty} P\left(\mu_n(A) = k\right) P\left(E' / \mu_n(A) = k\right). \tag{2}$$

Since the conditional distribution $\Pi_n(A)$ under the condition $\mu_n(A) = k$ coincides with $B_n(A)$, according to Lemma 1 for $k \geqslant 3$ we have

$$P\left(E' / \mu_n(A) = k\right) \leqslant k^2 \left(1 - c(\varepsilon)\right)^{k-2}. \tag{3}$$

Taking into account (2) and (3), we write

$$P(E') \leqslant \sum_{|k-n|<\frac{n}{4}} k^2 \left(1 - c(\varepsilon)\right)^{k-2} P\left(\mu_n(A) = k\right) + P\left(|\mu_n(A) - n| \geqslant \frac{n}{4}\right) = \Sigma_1 + \Sigma_2. \tag{4}$$

Using the Chebyshev inequality, we have

$$\Sigma_2 \leqslant 16 n^{-1}. \tag{5}$$

Further on, for sufficiently small $\varepsilon > 0$

$$\Sigma_1 \leqslant \max_{|k-n|<\frac{n}{4}} k^2 \left(1 - c(\varepsilon)\right)^{k-2} \leqslant \left(\frac{3n}{4}\right)^2 \left(1 - c(\varepsilon)\right)^{\frac{3n}{4}-2}.$$

It is easy to see that

$$\sup_{\varepsilon \geqslant \varepsilon_n} \Sigma_1 = o(1). \tag{6}$$

Combining (4)–(6), we arrive at the assertion of the lemma being proved. The lemma is proved. □

Let C_ε be the convex hull constructed from the part of the sample X_n in A_ε.

Lemma 1 implies that

$$\sup_{\varepsilon \geqslant \varepsilon_n} P\left(C_n \neq C_\varepsilon\right) \to 0 \text{ as } n \to \infty. \tag{7}$$

Let $B_n(A_\varepsilon)$ be the narrowing of the h.b.p.p. $B_n(\cdot)$ on A_ε. According to Lemma 2.2 in [7], $\Pi_n(A_\varepsilon)$ and $B_n(A_\varepsilon)$ can be defined on one probability space in such a way that

$$P\left(\Pi_n(A_\varepsilon) \neq B_n(A_\varepsilon)\right) \leqslant 2\lambda(A_\varepsilon). \tag{8}$$

Let us denote the convex hull generated by $\Pi_n(A_\varepsilon)$ by C'_ε. Then from Lemma 2 it follows that

$$\lim_{n\to\infty} \sup_{\varepsilon \geqslant \varepsilon_n} P\left(C'_n \neq C'_\varepsilon\right) = 0. \tag{9}$$

From (7)–(9) it follows that as $n \to \infty$

$$P\left(C'_n \neq C'_\varepsilon\right) \to 0. \tag{10}$$

Remark. Let $f_i,\ i = 1, 2, \ldots, k$ be a certain finite number of functionals defined on the set of convex polygons. If the joint distribution of random variables $f_i\left(C_n\right),\ i = 1, 2, \ldots, k$ converges to some distribution G, then it follows from (10) that $f_i\left(C'_n\right),\ i = 1, 2, \ldots, k$ also has this property. Thus, the problem of the limit distribution of the functionals ν_n and S_n, introduced in Section 1, is reduced to the study of ν'_n and S'_n are the corresponding characteristics of convex hulls generated by the h.p.p.p.

3. Convex hulls generated by the h.p.p.p.

3.1. Some properties of the h.p.p.p. Let K be a cone formed by two rays $l_i = (z : z = te_i,\, t > 0)$, $i = 1, 2$, where e_1 and e_2 are the unit vectors. Without loss of generality, we assume that e_1 and e_2 are the orthonormal vectors

$$e_0 = \frac{e_1 + e_2}{2}. \tag{11}$$

Let further $\Pi(\cdot)$ be a h.p.p.p. with intensity $\lambda(\cdot)$. We denote the narrowing on K by $\Pi(K)$. Consider the convex hull C' generated by K by $\Pi(K)$ and the set of its vertices Z.

Let us denote the vertex by $z_0 \in Z$ for which $(e_0, z - z_0) \geqslant 0$ for all $z \in Z$.

It is obvious that z_0 is determined unambiguously almost sure.

The straight line

$$(e_0, z - z_0) = 0 \tag{12}$$

is the supporting line for C'.

Consider a triangle formed by rays $l_i,\ i = 1, 2$ and a supporting line (12). We denote the set of interior points of this triangle by δ_0, and the area is denoted by ξ_0. It is easy to see that

$$\xi_0 = \frac{x_0^2}{2}, \tag{13}$$

where $x_0 = y_0 = u_0 + v_0$ and $z_0 = (u_0, v_0)$. Assume that

$$\eta_0 = \frac{v_0}{x_0}. \tag{14}$$

Then from (13) and (14) it is easy to obtain

$$u_0 = (1 - \eta_0)\sqrt{2\xi_0}, \quad v_0 = \eta_0\sqrt{2\xi_0}. \tag{15}$$

Let us label the vertices C', going around the boundary counterclockwise. Since z_0 is defined, each of the vertices gets its own number $j,\ -\infty < j < \infty$. Let us choose on the ray l_1 a sequence of points $x_j,\ j \geqslant 1$, lying on the intersection of l_1 and the lines passing through the vertices z_{j-1} and z_j, respectively. Likewise, on the ray l_2, points $y_j,\ j \leqslant -1$, are obtained as a result of intersections of l_2 and the lines passing through $z_j,\ z_{j+1}$, respectively.

Let $\delta_j,\ j \neq 0$; the set of interior points of a triangle with vertices z_{j-1}, $(x_{j-1}, 0)$, $(x_j, 0)$, if $j \geqslant 1$, and vertices z_{j+1}, $(0, y_{j+1})$, $(0, y_j)$, if $j \leqslant -1$. We denote the vertices of the triangle by $(x_0, 0)$, $(0, y_0)$, the set of interior points by δ_0. The third vertex of this triangle is the point $(0, 0)$. The figures are taken from [6] (see Fig. 1).

We assume that

$$\xi_j = \lambda(\delta_j).$$

Then it is easy to obtain

$$\xi_j = \begin{cases} v_{j-1}(x_j - x_{j-1})/2, \text{ if } j \geqslant 1 \\ u_{j+1}(y_j - y_{j+1})/2, \text{ if } j \leqslant -1 \end{cases}, \tag{16}$$

where $z_j = (u_j, v_j)$. If we assume that

$$\rho_j = \frac{u_j - u_{j-1}}{v_{j-1} - v_j}, \tag{17}$$

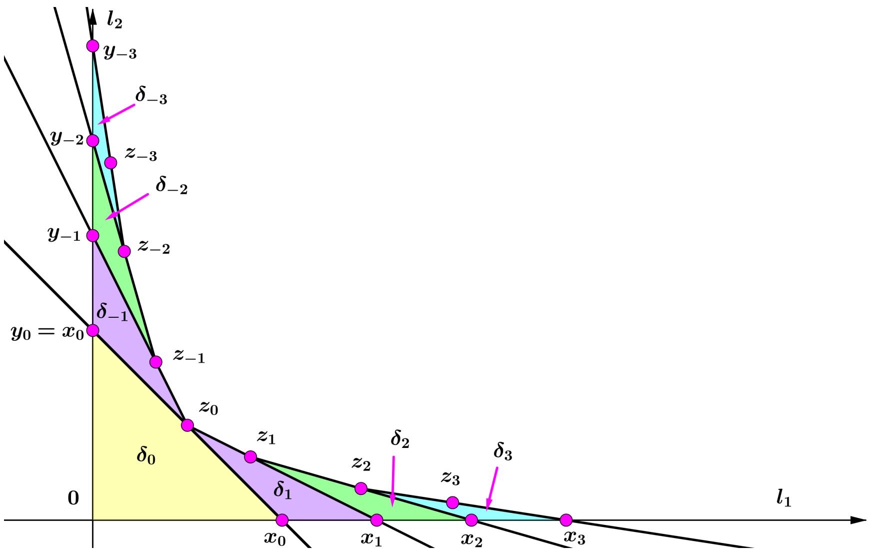

Fig. 1. Illustration of z_j and δ_j

then

$$\xi_j = \frac{v_{j-1}^2}{2}(\rho_j - \rho_{j-1}). \tag{18}$$

Now we define the boundary functionals

$$\theta_T = \inf\{j : x_j \geqslant T\} \quad \text{and} \quad \theta_T' = \inf\{-j : y_j \geqslant T\}, \tag{19}$$

where $T > 0$.

We assume that

$$\alpha(T) = \frac{2\log T}{3}, \quad \beta^2(t) = \frac{10\log t}{27}.$$

$$S_T = \begin{cases} \xi_1 + \xi_2 + \ldots + \xi_{\theta_T} & \text{if } \theta_T \geqslant 1 \\ 0 & \text{if } \theta_T = 0 \end{cases} \quad \text{and} \quad S_T' = \begin{cases} \xi_{-1} + \xi_{-2} + \ldots + \xi_{-\theta_T'} & \text{if } \theta_T' \geqslant 1 \\ 0 & \text{if } \theta_T' = 0 \end{cases}. \tag{20}$$

We present the following theorem with corollaries obtained in [6], which play the key role in this article (see Theorem 1, Corollaries 1, 2, 3 [6]).

Theorem 2 (Formanov and Khamdamov)**.** *Under our assumptions, as $T \to \infty$, we have*

$$(\beta(T))^{-1}(\theta_T - \alpha(T), S_T - \alpha(T)) \stackrel{d}{\Rightarrow} N(0, B) \quad \textit{with} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 14/5 \end{pmatrix}.$$

Here $N(0, B)$ is a normally distributed random vector with a zero vector of mean values and a covariance matrix B.

Corollary 1 (Formanov and Khamdamov)**.** *In our case $E\theta_T = \alpha(T) + o(\beta(T))$ and $Var\theta_T = \beta^2(T)(1 + o(1))$ as $T \to \infty$.*

Corollary 2 (Formanov and Khamdamov)**.** *Let $0 < T_1 \leqslant T_2$ such that $c_1T_1 < T_2 < c_2T_1$ for some $c_1 > 0,\ c_2 > 0$. Then $\theta_{T_2} - \theta_{T_1} = o_p(\beta(T_1))$ as $T_1 \to \infty$.*

Corollary 3 (Formanov and Khamdamov)**.** *Let $0 < T_1 \leqslant T_2$ such that $c_1T_1 < T_2 < c_2T_1$ for some $c_1 > 0,\ c_2 > 0$. Then $(S_{T_2} - S_{T_1})/\beta(T_1)$ converges in probability to zero as $T_1 \to \infty$.*

It is easy to see that at $\min\{T_1, T_2\} \to \infty$ the random vectors (θ_{T_1}, S_{T_1}) and $\left(\theta'_{T_2}, S'_{T_2}\right)$ are asymptotically independent. Moreover, the statements of Theorem 2 and its Corollaries 1–3 hold for $\left(\theta'_{T_2}, S'_{T_2}\right)$.

4. Proof of Theorem 1

The reasoning here is completely elementary. Generally, a verbal description of geometric objects is somewhat lengthy.

In accordance with the conclusions obtained at the end of Section 2 from Lemmas 1 and 2, it is sufficient to obtain the limit distribution for the number of vertices ν'_n and the area S'_n of the convex hull C'_n generated by the narrowing of the $\Pi_n(A)$ h.p.p.p. $\Pi_n(\cdot)$ on the set A. The scheme of further reasoning is as follows. First, we divide the boundary C'_n into $2r$ conditionally independent parts in such a way that each of the r angles of the polygon A corresponds to two elements of this partition. Thus, each of the functionals of interest to us ν'_n and S'_n is represented as a sum of $2r$ random variables. Then, using the properties of the h.p.p.p. stated in Section 3, the asymptotic independence and normality of these random variables are established.

Thus, the general principles for studying the problem are the same as in [7], although their implementation is completely different.

4.1. Dividing the boundary into conditionally independent parts. We denote the vertices of an r-gon of the support A of the initial uniform distribution by $a^{(i)},\ i = 1, 2, \ldots, r$. Let further, for some $\varepsilon > 0$

$$B_i = A \bigcap S\left(a^{(i)}, \varepsilon\right), \tag{21}$$

where $S(z, \varepsilon)$ is a disk of radius ε centered at z. Let us denote the narrowing $\Pi_n(\cdot)$ to a cone K_i with the vertex $a^{(i)}$ and generating rays l_{i1} and l_{i2} by $\Pi_{ni}(\cdot),\ i = 1, 2, \ldots, r$, passing through $a^{(i+1)}$ and $a^{(i-1)}$ respectively. It is clear that $a^{(-1)} = a^{(r)}$, $a^{(r+1)} = a^{(1)}$.

Let e_{0i} play the same role with respect to K_i as played by the vector with respect to K_i in Section 3. Note that e_0 is determined by the equality (11). More precisely,

$$e_{0i} = 2^{-1}\left(\frac{a^{(i+1)} - a^{(i)}}{\|a^{(i+1)} - a^{(i)}\|} + \frac{a^{(i-1)} - a^{(i)}}{\|a^{(i-1)} - a^{(i)}\|}\right).$$

We denote the convex hull as C_{ni} generated by $\Pi_{ni}(\cdot)$. Let us agree to denote the set of vertices C'_n by Z_{ni}. Recall that the set of vertices C'_n in Section 2 is denoted by W'_n. We select in Z_{ni} and W'_n the elements z_{0i} and w_{0i}, that possess the property that the straight lines $(e_{0i}, w - z_{0i}) = 0$ and $(e_{0i}, w - w_{0i}) = 0$ are the supporting lines for C_{ni} and C'_n, respectively.

Assume that

$$\Upsilon_1 = \{\pi :\ z_{0i} = w_{0i},\ i = 1, 2, \ldots, r\} \tag{22}$$

and

$$\Upsilon_2 = \{\pi :\ z_{0i} \in B_i,\ i = 1, 2, \ldots, r\}, \tag{23}$$

where π is the implementation of $\Pi_n(\cdot)$, and B_i is determined by equality (21).

It is easy to understand that as $n \to \infty$

$$P(\Upsilon_i) \to 1, \quad i = 1, 2. \tag{24}$$

As follows from (22)–(24), with probability close to 1, the boundary of each hull C_{ni} has a non-empty intersection with C'_n. Note that the points $w_{0i},\ i = 1, 2, \dots, r$ divide the boundary C'_n into r parts. We split each of them into two more parts. Let $w^{(i)}$ be the vertex $W'_n \subset C'_n$, for which the straight line $\left(p_i, w - w^{(i)}\right) = 0$, where $p_i \perp \left(a^{(i+1)} - a^{(i)}\right)$ is a supporting line to C'_n. It is easy to see that $w^{(i)}$ is the closest vertex to the ray l_{i1} from the vertices W'_n. Note that as the n vertex $w^{(i)}$ grows, it approaches this ray indefinitely, i.e., $\left(p_i, w^{(i)} - a^{(i)}\right) \to 0$. Since the conditional distribution on the section of the supporting line $\left(p_i, w - w^{(i)}\right) = 0$ lying in A, under the condition $\left(p_i, w^{(i)} - a^{(i)}\right) = t$ is uniform, we have

$$\lim_{\varepsilon \to 0} \lim_{n \to 0} \inf P\left(w^{(i)} \in \bigcap_{j=1}^{r} \overline{B}_j\right) = 1. \tag{25}$$

Hence it follows that

$$\lim_{\varepsilon \to 0} \lim_{n \to 0} \inf P\left(\overline{w}_i \in \bigcap_{j=1}^{r} \overline{B}_j\right) = 1, \tag{26}$$

where $\overline{w}_i$ is the base of the perpendicular drawn from w_i to l_{i1}.

Consider

$$\Upsilon_3 = \left\{\overline{w}_i \in \bigcap_{j=1}^{r} \overline{B}_j,\ i = 1, 2, \dots, r\right\}.$$

As follows from (25) and (26), for any $\varepsilon > 0$ one can find such $N > 0$ that, for all sufficiently large $n > N$, the following inequality holds

$$P\left(\Upsilon_3\right) \geqslant 1 - \varepsilon.$$

In what follows, without specifying, we consider only those implementations of $\Pi_n(\cdot)$ that are contained in $\bigcap_{j=1}^{3} \Upsilon_j$. For such implementations $w^{(i)},\ i = 1, 2, \dots, r$ lie between w_{0i} and $w_{0(i+1)}$. Thus, the boundary C'_n is divided into $2r$ parts. It is easy to see, that these parts are conditionally independent for the given $w_{0i},\ w^{(i)},\ i = 1, 2, \dots, r$.

4.2. Choice of approximating functionals. Let us consider the section of the boundary C'_n between the vertices w_{01} and $w^{(i)}$. The section between $w^{(r)}$ and w_{01} is studied in a similar way. Let us label the vertices C'_n, going around the boundary counterclockwise, starting from w_{01}. As a result, on the considered section of the boundary, we obtain $w_j,\ j = 0, 1, 2, \dots, \mu$, where $w_0 = w_{01},\ w_\mu = w^{(1)}$. We perform a similar operation with the vertices $z \in C'_{n1}$, obtaining $z_j,\ j = 0, 1, 2, \dots$, where, in view of (22) and (24) $z_0 = w_{01} = w_0$.

In order to use the h.p.p.p. properties described in Section 3, we need to proceed from $\Pi(\cdot)$ to $\Pi_n(\cdot)$. In such transition, the linear characteristics x_j, y_j, u_j, v_j, change to $x'_j = n^{-\frac{1}{2}} x_j$, $y'_j = n^{-\frac{1}{2}} y_j$, $u'_j = n^{-\frac{1}{2}} u_j$, $v'_j = n^{-\frac{1}{2}} v_j$ respectively, while the area ξ_j of the triangle δ_j becomes $\xi'_j = n^{-1} \xi_j$. Dimensionless quantities η_j, τ_j, ρ_j remain unchanged in such transition. We denote the images z_j of such a transformation by z'_j.

Let $T = \varepsilon\sqrt{n}$, $T_1 = h\sqrt{n}$ where h is the length of the side A connecting the vertices $a^{(1)}$ and $a^{(2)}$. In accordance with (19), we assume that

$$\theta = \theta_T \quad \text{and} \quad \chi = \theta_{T_1}.$$

It is clear that

$$\theta = \inf\left\{j : x'_j \geqslant \varepsilon\right\} \quad \text{and} \quad \chi = \inf\left\{j : x'_j \geqslant h\right\}.$$

Note that x_j and x'_j are constructed on the vertices z_{j-1}, z_j and z'_{j-1}, z'_j, respectively. Note that $w_j = z'_j$, at least for $0 \leqslant j \leqslant \chi - 1$.

Let further

$$p = \xi'_1 + \xi'_2 + \cdots + \xi'_\theta, \tag{27}$$

and

$$q = \xi'_1 + \xi'_2 + \cdots + \xi'_\chi. \tag{28}$$

Assume that

$$\theta^* = \frac{\theta - \alpha}{\beta_1}, \quad p^* = \frac{np - \alpha}{\beta_2}, \tag{29}$$

where

$$\alpha = \frac{1}{3}\log n, \quad \beta_1 = \sqrt{\frac{5\log n}{27}}, \quad \beta_2 = \sqrt{\frac{14\log n}{27}}. \tag{30}$$

From (20), (27), (28) and Theorem 2 it follows that

$$(\theta^*, p) \overset{d}{\Rightarrow} \omega, \tag{31}$$

where ω is determined from Theorem 1. Now we assume that

$$\chi^* = \frac{\chi - \alpha}{\beta_1}, \quad q^* = \frac{nq - \alpha}{\beta_2}. \tag{32}$$

According to Corollaries 1–3, in view of (28) and (30), we have

$$\frac{\theta - \chi}{\beta_1} \overset{p}{\to} 0, \quad \frac{n(p-q)}{\beta_1} \overset{p}{\to} 0. \tag{33}$$

From (29)–(33) follows that

$$(\chi^*, q^*) \overset{d}{\Rightarrow} \omega. \tag{34}$$

Similar characteristics θ', p' and χ', q' constructed along the section of the boundary C'_n between the vertices $w^{(r)}$ and $w_{01} = w$, also have properties (31) and (34). It is important that they are asymptotically independent of θ, χ, p and q. And no less important is the fact that θ, θ', p and p' are completely determined by the narrowing of $\Pi_n(\cdot)$ to B_1. It follows that similar characteristics θ_i, θ'_i, p_i, p'_i for the boundary sections corresponding to the angles with the vertices $a^{(i)}$, $i = 1, 2, \ldots, r$ are independent. By analogy with (29) and (32), we define

$$\Theta^* = \frac{\Theta - 2r\alpha}{\beta_1\sqrt{2r}} \quad \text{and} \quad P^* = \frac{nP - 2r\alpha}{\beta_2\sqrt{2r}}, \tag{35}$$

where

$$\Theta = \sum_{i=1}^{r} \left(\theta_i + \theta'_i\right), \quad P = \sum_{i=1}^{r} \left(p_i + p'_i\right).$$

Due to independence of $\left(\theta_i + \theta'_i, p_j + p'_j\right)$, $i, j - 1, 2, \ldots, r$, from (31) we obtain

$$(\Theta, P) \overset{d}{\Rightarrow} \omega.$$

Finally, by analogy with (35), we introduce

$$\mathbb{X}^* = \frac{\mathbb{X} - 2r\alpha}{\beta_1\sqrt{2r}}, \quad \mathbb{Q}^* = \frac{\mathbb{Q} - 2r\alpha}{\beta_2\sqrt{2r}}, \tag{36}$$

where (compare with (35))

$$\mathbb{X} = \sum_{i=1}^{r} (\chi_i + \chi_i'), \quad \mathbb{Q} = \sum_{i=1}^{r} (q_i + q_i').$$

Note that $(\chi_i + \chi_i', q_i + q_i')$, $i = 1, 2, \ldots, r$, generally speaking, are independent. However, in view of (33) and (34), we can assert that

$$(\mathbb{X}^*, \mathbb{Q}^*) \stackrel{d}{\Rightarrow} \omega. \tag{37}$$

It is the functionals $\mathbb{X}^*$ and $\mathbb{Q}^*$ that give us the required approximation for ν_n' and S_n'.

4.3. Estimation of the approximation accuracy. Let s be the area of the figure bounded by the section of the boundary C_n' between the vertices $w_0 = w_{01}$ and $w_\mu = w^{(1)}$, the segment of the ray l_{11} between the points $\overline{w}_1$ and $x_0' e_{11}$ and the supporting line $(e_{01}, w - w_{01}) = 0$. Here, the points $w_0, w_\mu, \overline{w}_1$ are defined in Sections 4.1 and 4.2,

$$e_{11} = \frac{a^{(2)} - a^{(1)}}{\|a^{(2)} - a^{(1)}\|}$$

and x_0' corresponds to x_0 when going from $\Pi(\cdot)$ to $\Pi_n(\cdot)$.

Let us construct similar left characteristics μ' and s' in the section of the boundary between the vertices $w^{(r)}$ and w_{01}.

In what follows, we denote μ_i, μ_i', s_i and s_i', the analogs of μ, μ', s and s', corresponding to the angle with the vertex $a^{(i)}$. It is easy to see that ν_n' is the total number of vertices C_n' and can be represented as

$$\nu_n' = \sum_{i=1}^{r} (\mu_i + \mu_i'). \tag{38}$$

And area $A - C_n'$ can be represented in the form

$$\lambda (A - C_n') = \sum_{i=1}^{r} (s_i + s_i') + \xi_{0i}', \tag{39}$$

where ξ_{0i}' is the area of the triangle cut by the supporting line $(e_{0i}, w - w_{0i}) = 0$.

Note that

$$n\xi_{01}' = \xi_0 = O(1), \tag{40}$$

where ξ_0 has an exponential distribution (see for example [6]). Similarly

$$n\xi_{0i}' = \xi_0 = O(1), \ i = 1, 2, \ldots, r. \tag{41}$$

As an approximation for μ_i, μ_i', s_i and s_i', we use χ_i, χ_i', q_i and q_i', introduced in Section 4.2. In this case, it is enough to evaluate the proximity of $(\mu_1, s_1) \stackrel{dis}{=} (\mu, s) + o_p(1)$ and $(\chi_1, q_1) \stackrel{dis}{=} (\chi, q)$. The remaining pairs of vectors are matched similarly.

To complete the proof of the theorem, it suffices to show the proximity of s and q, i.e.

$$\frac{n(s-q)}{\sqrt{\log n}} \xrightarrow{p} 0 \text{ at } n \to \infty \tag{42}$$

and proximity of μ and χ , i.e.

$$\frac{\mu - \chi}{\sqrt{\log n}} \xrightarrow{p} 0 \text{ at } n \to \infty. \tag{43}$$

We obtain the relation (42) from Corollary 3, and relation (43) from Corollary 2. The obtained relations (42) and (43) with the relations (36), (37)–(41) allow us to assert that a random vector with components $\dfrac{\nu_n' - 2r\alpha}{\beta_1\sqrt{2r}}$ and $\dfrac{n(1 - S_n' - 2r\alpha}{\beta_2\sqrt{2r}}$ converges in distribution to ω. Taking into account Remark given at the end of Section 3, we obtain the assertion of the theorem. The theorem is proved.

References

[1] C.Buchta, On the distribution of the number of vertices of a random polygon, *Anz. Osterreich. Akad. Wiss. Math. Natur. K1*, **139**(2003), 17–19.

[2] C.Buchta, Exact formulae for variances of functionals of convex hulls, *Advances in Applied Probability*, **45**(2013), no. 4, 917–924. DOI: 10.1239/aap/1386857850

[3] A.J.Cabo, P.Groeneboom, Limit theorems for functiohals of convex hulls, *Probab. Theory Relat. Fields*, **100**(1994), 31–55.

[4] H.Carnal, Die konvexe Hulle von n rotations symmetrisch verteilten Punkten, *Z. Wahrscheinlichkeits theorie verw. Geb.*, (1970), no. 15, 168–176.

[5] B.Efron, The convex hull of a random set of points. Biometrika, **52**(1965), 331–343.

[6] Sh.K.Formanov, I.M.Khamdamov, On joint probability distribution of the number of vertices and area of the convex hulls generated by a Poisson point process, *Statistics and Probability Letters*, **169**(2021), 108966, 1–7.

[7] P.Groeneboom, Limit theorems for convex hulls, *Probab. Theory Related Fields*, **79**(1988), 327–368.

[8] P.Groeneboom, Convex hulls of uniform samples from a convex polygon, *Adv. Appl. Prob. (SGSA)*, **44**(2012), 330–342. DOI: 10.1239/aap/1339878714

[9] T.Hsing, On the asymptotic distribution of the area outside a random convex hull in a disk, *The Annals of Applied Probability*, **4**(1994), no. 2, 478–493.

[10] I.M.Khamdamov, On Limit Theorem for the Number of Vertices of the Convex Hulls in a Unit Disk, *Journal of Siberian Federal University. Mathematics and Physics*, **13**(2020), no. 3, 275–284. DOI: 10.17516/1997-1397-2020-13-3-275-284

[11] I.M.Khamdamov, Properties of convex hull generated by inhomogeneous Poisson point process, *Ufa Mathematical Journal*, **12**(2020), no. 3, 81–96. DOI: 10.13108/2020-12-3-81

[12] A.V.Nagaev, Some properties of convex hulls generated by homogeneous Poisson point processes in an unbounded convex domain, *Ann. Inst. Statist. Math.*, **47**(1995), no. 1, 21–29.

[13] A.V.Nagaev, I.M.Khamdamov, Limit theorems for functionals of random convex hulls. Preprint of Institute of Mathematics, Academy of Sciences of Uzbekistan, Tashkent, 1991 (in Russian.)

[14] J.Pardon, Central limit theorems for random polygons in an arbitrary convex set, *The Annals of Probability*, **1**(2011), no. 3, 881–903.

[15] J.Pardon, Central Limit Theorems for Uniform Model Random Polygons, *J. Theor. Probab,* **25**(2012), 823–833.

[16] A.Reny, R.Sulanke, Uber diekovexe Hulle von zufalling gewahlten Punkte, *Z. Wahrscheinlichkeits theorie verw. Geb.*, **2**(1963), 75–84.

Совместное распределение числа вершин и площади выпуклых оболочек, порожденных равномерным распределением в выпуклом многоугольнике

Исакжан М. Хамдамов
Национальный университет Узбекистана им. Мирзо Улугбека
Ташкент, Узбекистан
Зоя С. Чай
Ташкентский университет информационных технологий
Ташкент, Узбекистан

Аннотация. Рассматривается выпуклая оболочка, порожденная выборкой, равномерно распределенной на плоскости для случая, когда носитель распределения представляет собой выпуклый многоугольник. Доказывается центральная предельная теорема для совместного распределения числа вершин и площади выпуклой оболочки с использованием пуассоновской аппроксимации биномиальных точечных процессов вблизи границы носителя распределения. Здесь применяются результаты [6] совместного распределения числа вершин и площади выпуклых оболочек, порожденных пуассоновским распределением. Из результатов, полученных в настоящей статье, в частности, следуют результаты [3, 7], когда носитель представляет собой выпуклый многоугольник, а выпуклая оболочка порождается однородным пуассоновским точечным процессом.

Ключевые слова: выпуклая оболочка, выпуклый многоугольник, пуассоновский точечный процесс, биномиальный точечный процесс, центральная предельная теорема.