Mathematical Model of Fluids Motion in Poroelastic Snow-ice Cover

Margarita A. Tokareva[*]
Alexander A. Papin[†]
Altai State University
Barnaul, Russian Federation

Abstract. The dynamics of a snow-ice cover is considered within the theory of poroelasticity. The snow-ice cover is modeled by a three-phase medium consisting of water, air and ice. The governing equations are the equations of mass conservation for each phase with phase transitions, the equations of conservation of phase momentum in the form of Darcy's law, the equation of conservation of momentum of the whole system, the rheological equation for porosity and the equation of heat balance of snow. In the full formulation the liquid and air pressures are functions of the temperature and the corresponding densities, and the viscosity and compressibility coefficients of ice are functions of the temperature. The problem of two-dimensional nonstationary filtration of water in a thin poroelastic ice plate is considered in the model case. The solution is obtained in quadratures.

Keywords: poroelasticity, porosity, snow-ice cover, thin layer.

Citation: M.A. Tokareva, A.A. Papin, Mathematical Model of Fluids Motion in Poroelastic Snow-ice Cover, J. Sib. Fed. Univ. Math. Phys., 2021, 14(1), 47–56. DOI: 10.17516/1997-1397-2021-14-1-47-56.

A large number of studies are devoted to balance models and observations of the behavior of a snow cover [1–3]. These works primarily study snow stocks, flood risks and river flow formation. Models of a melting snow-ice cover based on the approaches of the mechanics of multiphase medium are intended to obtain a more detailed picture of the movement of water and air in melting snow, i.e. determination of the velocity field, concentration, phase pressures, etc. Only a few articles use the "full" models. The problems describing phase transitions and movement of the ice skeleton remain open.

The foundations of the theory of the movement of water and air in the melting snow within the mechanics of multiphase medium are formulated by S. C. Colbeck [4] and his followers [5, 6]. In these works, snow is considered as a multiphase medium, but the variable porosity of ice, its deformation and phase transitions are not taken into account. In this paper, a mathematical model of the movement of water and air in a deformable poroelastic medium is proposed. The model includes the laws of conservation of mass, momentum in the form of Darcy's law for fluid components, the rheological relation for the porosity and the equation for temperature [7]. The obtained system of the equations is a generalization of the well-known system of Musket-Leverett equations in the case of variable porosity [8]. We use the concept of effective stress introduced by Terzagi for the porous deformation model when modeling a poroelastic medium. In this approach, the effective stress is defined as the difference between the total stress and the

[*]tma25@mail.ru
[†]papin@math.asu.ru

pressure of the liquid phase [9]. This proposition means that the fluid carries part of the load. Fundamental in this approach is the relationship between the deformation of the skeleton of a solid matrix and the processes of fluid flow. Filtration in a thin poroelastic layer is considered in the case of one liquid. The model is based on the theory of multiphase flows in poroelastic medium. Structurally similar mathematical models are used to describe the movement of magma in the earth's crust, the flow of water and oil in a porous medium, the growth of tumors and other filtration processes in poroelastic medium [10–13].

1.   Governing equations

A system of differential equations describing the movement of air and water in the snow-ice cover taking into account the phase transitions is considered in the domain $\Omega = (x, z) = = [0, L] \times [0, H]$. The equations of continuity with the variable porosity ϕ are

$$\frac{\partial(1-\phi)\rho_i}{\partial t} + div((1-\phi)\rho_i\vec{v}_i) = I_{wi} + I_{ai}, \quad \frac{\partial(\rho_w s_w \phi)}{\partial t} + div(\rho_w \phi s_w \vec{v}_w) = I_{iw} + I_{aw},$$

$$\frac{\partial(\rho_a s_a \phi)}{\partial t} + div(\rho_w \phi s_a \vec{v}_w) = I_{iw} + I_{wa}, \quad s_a + s_w = 1.$$

Here ρ_a, ρ_w, ρ_i, $\vec{v}_a$, $\vec{v}_i$, $\vec{v}_w$ are the true density and phase velocity, respectively (a — air, w — water, i — ice), s_a, s_w are the saturations of air and water, respectively, ϕ is the porosity, I_{lm} is the intensity of mass transfer from l-phase to the m-phase per unit volume at time instant, $I_{lm} = -I_{ml}$.

Instead of the equation of dynamics in the theory of two-phase filtration, we use the general Darcy law for liquid phases, taking into account the movement of ice

$$\phi(\vec{v}_w - \vec{v}_i) = -\frac{k(\phi)}{\mu_w(\theta)} \left(\nabla p_w - \rho_w \vec{g} \right),$$

$$\phi(\vec{v}_a - \vec{v}_i) = -\frac{k(\phi)}{\mu_a(\theta)} \left(\nabla p_a - \rho_a \vec{g} \right),$$

where θ is the temperature of the medium ($\theta_i = \theta_w = \theta$), p_w, p_a are the water and air pressures, respectively, $k(\phi)$ is the permeability, $\mu_w(\theta)$, $\mu_a(\theta)$ are the dynamic viscosities of water and air, $\vec{g}$ is the gravity acceleration vector. The phase pressures p_a and p_w differ by the magnitude capillary jump p_c: $p_a - p_w = p_c(x, s_w)$.

The Maxwell-type relation between the porosity and the effective pressure p_e is [17]

$$div\vec{v}_i = -\phi \left(\alpha(\theta)p_e + \beta(\theta)\frac{dp_e}{dt} \right), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_i \cdot \nabla),$$

where $\alpha(\theta)$, $\beta(\theta)$ are the specified environment parameters. The effective pressure p_e and the pressures of the liquids p_w, p_a and solid p_i phases are related by the equations

$$p_{tot} = \phi p_f + (1-\phi)p_i, \quad p_e = (1-\phi)(p_i - p_f), \quad p_f = s_w p_w + s_a p_a.$$

The balance equation for the forces of the whole system has the form

$$\nabla p_{tot} = \rho_{tot}\vec{g} + div\left((1-\phi)\eta(\theta)\left(\frac{\partial \vec{v}_i}{\partial \vec{x}} + \left(\frac{\partial \vec{v}_i}{\partial \vec{x}} \right)^* \right) \right), \quad \rho_{tot} = \phi\rho_f + (1-\phi)\rho_i, \quad \rho_f = s_w\rho_w + s_a\rho_a,$$

where p_{tot} is the total pressure, ρ_{tot} is the total density, $\eta(\theta)$ is the viscosity of the porous skeleton, $*$ is the symbol of the transposition operation.

The energy conservation equation is taken as [1]

$$(\rho_a c_a s_a \phi + \rho_w c_w s_w \phi + \rho_i c_i (1 - \phi)) \frac{\partial \theta}{\partial t} + (\rho_a c_a s_a \phi \vec{v}_a + \rho_w c_w s_w \phi \vec{v}_w + \rho_i c_i (1 - \phi) \vec{v}_i) \nabla \theta =$$

$$= div \left(\lambda_{tot}(\phi) \nabla \theta \right) - L_i I_{ai} - \nu I_{wi}.$$

Here $c_i = const > 0$, $c_w = const > 0$, $c_a = const > 0$ are the heat capacity of ice, water and air at constant volume, respectively; $L_i = const > 0$ is the specific heat of ice sublimation, $\nu = const > 0$ is the specific heat of ice melting; λ_{tot} is the thermal conductivity of the medium as a whole ($\lambda_{tot} = a_{tot} + b_{tot} \rho_{tot}^2$, $a_{tot} = const > 0$, $b_{tot} = const > 0$). To close the equations, one can take into account the dependence of the densities of liquid phases on pressures and temperatures, or set the densities equal to constant.

Assuming that there is no air in the pores, we shell study the system of differential equations describing the filtration of liquid in the melting ice (the phase densities are constant):

$$\frac{\partial (1 - \phi) \rho_i}{\partial t} + div((1 - \phi) \rho_i \vec{v}_i) = I_{wi}, \quad \frac{\partial (\rho_w s_w \phi)}{\partial t} + div(\rho_w \phi s_w \vec{v}_w) = I_{iw}, \tag{1}$$

$$\phi(\vec{v}_w - \vec{v}_i) = -\frac{k(\phi)}{\mu_w(\theta)} \left(\nabla p_w - \rho_w \vec{g} \right), \tag{2}$$

$$div\vec{v}_i = -\phi \left(\alpha(\theta) p_e + \beta(\theta) \frac{dp_e}{dt} \right), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_i \cdot \nabla), \tag{3}$$

$$p_{tot} = \phi p_w + (1 - \phi) p_i, \quad p_e = (1 - \phi)(p_i - p_w), \tag{4}$$

$$\nabla p_{tot} = \rho_{tot} \vec{g} + div \left((1 - \phi) \eta(\theta) \left(\frac{\partial \vec{v}_i}{\partial \vec{x}} + \left(\frac{\partial \vec{v}_i}{\partial \vec{x}} \right)^* \right) \right), \quad \rho_{tot} = \phi \rho_w + (1 - \phi) \rho_i, \tag{5}$$

$$(\rho_w c_w \phi + \rho_i c_i (1 - \phi)) \frac{\partial \theta}{\partial t} + (\rho_w c_w \phi \vec{v}_w + \rho_i c_i (1 - \phi) \vec{v}_i) \nabla \theta = div \left(\lambda_{tot}(\phi) \nabla \theta \right) - \nu I_{wi}. \tag{6}$$

A structurally similar model is used to describe the nonisothermal motion of magma in a poroelastic medium. The system (1)–(6), in the one-dimensional case with constant temperature, the dependence of the density of the liquid phase on the pressure, and in the absence of the phase transitions is studies and the local solvability is proved in [18]. The global solvability is proved in [19] for the same problem when the densities are constant. The numerical analysis of the initial-boundary value problem for system (1)–(6) in the isothermal case is carried out in [20].

2.   Introduction of small parameter

We shell make equations (1)–(6) dimensionless. Let $\bar{x}$, $\bar{z}$, $\bar{t}$ be dimensionless variables defined by the equalities [21]

$$\bar{x} = \frac{x}{L}, \quad \bar{z} = \frac{z}{H}, \quad \bar{t} = \varepsilon^l \tau_0 t, \quad \varepsilon = \frac{H}{L} \ll 1,$$

where $[L] = [H] = [\text{ m}]$, $[\tau_0] = [1/\text{ s}]$; l is a fixed parameter.

The required functions in the dimensionless variables are

$$p_w(t, x, z) = p \bar{p}_w(\bar{t}, \bar{x}, \bar{z}) = p \bar{p}_w \left(\varepsilon^l \tau_0 t, \frac{x}{L}, \frac{z}{H} \right),$$

$$p_i(t, x, z) = p\bar{p}_i(\bar{t}, \bar{x}, \bar{z}) = p\bar{p}_i\left(\varepsilon^l \tau_0 t, \frac{x}{L}, \frac{z}{H}\right),$$

$$v_i^j(t, x, z) = v^j \bar{v}_i^j(\bar{t}, \bar{x}, \bar{z}) = v^j \bar{v}_i^j\left(\varepsilon^l \tau_0 t, \frac{x}{L}, \frac{z}{H}\right), \quad j = 1, 2,$$

$$v_w^j(t, x, z) = v^j \bar{v}_w^j(\bar{t}, \bar{x}, \bar{z}) = v^j \bar{v}_w^j\left(\varepsilon^l \tau_0 t, \frac{x}{L}, \frac{z}{H}\right), \quad j = 1, 2,$$

$$p_{tot}(t, x, z) = p\bar{p}_{tot}(\bar{t}, \bar{x}, \bar{z}), \quad p_e(t, x, z) = p\bar{p}_e(\bar{t}, \bar{x}, \bar{z}), \quad \theta = \delta\bar{\theta}, \quad I_{wi} = \gamma \bar{I}_{wi}, \quad I_{iw} = \gamma \bar{I}_{iw}.$$

Here $[v^i] = $ [m / s], $[p] = $ [Pa], $[\delta] = $ [K], $[\alpha] = $ [1 / c], the quantities p, v^j, δ, γ are the positive constants. Further, we use the assumption that $k(\phi) = k\phi^n$, $k = const$, $\eta(\theta) = \eta = const > 0$ [14, 17, 22].

Equations (1)–(6) in scalar form in the dimensionless variables are

$$\varepsilon^l \tau_0 \frac{\partial(1-\phi)}{\partial \bar{t}} + \frac{v^1}{L}\frac{\partial \bar{v}_i^1(1-\phi)}{\partial \bar{x}} + \frac{v^2}{H}\frac{\partial \bar{v}_i^2(1-\phi)}{\partial \bar{z}} = \gamma \bar{I}_{wi},$$

$$\varepsilon^l \tau_0 \frac{\partial \phi}{\partial \bar{t}} + \frac{v^1}{L}\frac{\partial \bar{v}_w^1 \phi}{\partial \bar{x}} + \frac{v^2}{H}\frac{\partial \bar{v}_w^2 \phi}{\partial \bar{z}} = \gamma \bar{I}_{iw}, \tag{7}$$

$$\phi(v^1 \bar{v}_w^1 - v^1 \bar{v}_i^1) = -\frac{k}{\mu}\phi^n \frac{p}{L}\frac{\partial \bar{p}_w}{\partial \bar{x}},$$

$$\phi(v^2 \bar{v}_w^2 - v^2 \bar{v}_i^2) = -\frac{k}{\mu}\phi^n \left(\frac{p}{H}\frac{\partial \bar{p}_w}{\partial \bar{z}} + \rho_w g\right), \tag{8}$$

$$\frac{v^1}{L}\frac{\partial \bar{v}_i^1}{\partial x} + \frac{v^2}{H}\frac{\partial \bar{v}_i^2}{\partial y} = -\phi\left(\alpha p\bar{p}_e - \beta\left(p\varepsilon^l \tau_0 \frac{\partial \bar{p}_e}{\partial \bar{t}} + \bar{v}_i^1 \frac{pv^1}{L}\frac{\partial \bar{p}_e}{\partial \bar{x}} + \bar{v}_i^2 \frac{pv^2}{H}\frac{\partial \bar{p}_e}{\partial \bar{z}}\right)\right), \tag{9}$$

$$2\frac{v^1}{L^2}\frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{x}}\right) + \frac{v^1}{H^2}\frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{z}}\right) +$$

$$+\frac{v^2}{HL}\frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{x}}\right) = \frac{p}{L\eta}\frac{\partial p_{tot}}{\partial \bar{x}}, \tag{10}$$

$$\frac{(\rho_w \phi + \rho_i(1-\phi))g}{\eta}\bar{\rho}\bar{g} + \frac{v^2}{L^2}\frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{x}}\right) + 2\frac{v^2}{H^2}\frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{z}}\right) +$$

$$+\frac{v^1}{HL}\frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{z}}\right) = \frac{p}{H\eta}\frac{\partial \bar{p}_{tot}}{\partial \bar{z}}, \tag{11}$$

$$\varepsilon^l \tau_0 \delta\left(\rho_w c_w \phi + \rho_i c_i(1-\phi)\right)\frac{\partial \bar{\theta}}{\partial \bar{t}} + \frac{v^1 \delta}{L}\left(\rho_w c_w \phi \bar{v}_w^1 + \rho_i c_i(1-\phi)\bar{v}_i^1\right)\frac{\partial \bar{\theta}}{\partial \bar{x}} +$$

$$+\frac{v^2 \delta}{H}\left(\rho_w c_w \phi \bar{v}_w^2 + \rho_i c_i(1-\phi)\bar{v}_i^2\right)\frac{\partial \bar{\theta}}{\partial \bar{z}} = \frac{\delta}{L^2}\frac{\partial}{\partial \bar{x}}\left((a + b(\phi\rho_w + (1-\phi)\rho_i))\frac{\partial \bar{\theta}}{\partial \bar{x}}\right) + \tag{12}$$

$$+\frac{\delta}{H^2}\frac{\partial}{\partial \bar{z}}\left((a + b(\phi\rho_w + (1-\phi)\rho_i)^2)\frac{\partial \bar{\theta}}{\partial \bar{z}}\right) - \nu\gamma \bar{I}_{wi}.$$

We now choose

$$v^1 = \varepsilon^l \tau_0 L, \quad v^2 = \varepsilon^l \tau_0 H.$$

Then equations (7)–(12) can be reduced to the form

$$\frac{\partial(1-\phi)}{\partial \bar{t}} + div((1-\phi)\bar{v}_i) = \frac{\gamma}{\varepsilon^l \tau_0}\bar{I}_{wi},$$

$$\frac{\partial \phi}{\partial \bar{t}} + div(\phi\bar{v}_w) = \frac{\gamma}{\varepsilon^l \tau_0}\bar{I}_{iw}, \tag{13}$$

$$\frac{\varepsilon^l \tau_0 \mu L^2}{kp} \phi(\bar{v}_w^1 - \bar{v}_i^1) = -\phi^n \frac{\partial \bar{p}_w}{\partial \bar{x}},$$

$$\frac{\varepsilon^{l+2} \tau_0 \mu L^2}{kp} \phi(\bar{v}_w^2 - \bar{v}_i^2) = -\phi^n \frac{\partial \bar{p}_w}{\partial \bar{z}} - \frac{\rho_w g L \varepsilon}{p}, \tag{14}$$

$$\frac{\varepsilon^l \tau_0}{\alpha p}\Big(\frac{\partial \bar{v}_i^1}{\partial \bar{x}} + \frac{\partial \bar{v}_i^2}{\partial \bar{z}}\Big) = -\phi \bar{p}_e - \frac{\varepsilon^l \tau_0 \beta}{\alpha}\left(\frac{\partial \bar{p}_e}{\partial \bar{t}} + \bar{v}_i^1 \frac{\partial \bar{p}_e}{\partial \bar{x}} + \bar{v}_i^2 \frac{\partial \bar{p}_e}{\partial \bar{z}}\right), \tag{15}$$

$$2\varepsilon^l \frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{x}}\right) + \varepsilon^{l-2} \frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{z}}\right) +$$

$$+\varepsilon^l \frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{x}}\right) = \frac{p}{\eta \tau_0}\frac{\partial p_{tot}}{\partial \bar{x}}, \tag{16}$$

$$\varepsilon^{l+2} \frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{x}}\right) + 2\varepsilon^l \frac{\partial}{\partial \bar{z}}\left((1-\phi)\frac{\partial \bar{v}_i^2}{\partial \bar{z}}\right) +$$

$$+\varepsilon^l \frac{\partial}{\partial \bar{x}}\left((1-\phi)\frac{\partial \bar{v}_i^1}{\partial \bar{z}}\right) = \frac{gL\varepsilon\rho_w}{\eta \tau_0}\phi + \frac{gL\varepsilon\rho_i}{\eta \tau_0}(1-\phi) + \frac{p}{\eta \tau_0}\frac{\partial \bar{p}_{tot}}{\partial \bar{z}}. \tag{17}$$

$$\varepsilon^l \left(\left(\frac{\rho_w c_w}{\rho_i c_i}\phi + (1-\phi)\right)\frac{\partial \bar{\theta}}{\partial \bar{t}} + \left(\frac{\rho_w c_w}{\rho_i c_i}\phi \bar{v}_w^1 + (1-\phi)\bar{v}_i^1\right)\frac{\partial \bar{\theta}}{\partial \bar{x}} + \left(\frac{\rho_w c_w}{\rho_i c_i}\phi \bar{v}_w^2 + (1-\phi)\bar{v}_i^2\right)\frac{\partial \bar{\theta}}{\partial \bar{z}}\right) =$$

$$= \frac{\partial}{\partial \bar{x}}\left(\left(\frac{a}{L^2 \tau_0 \rho_i c_i} + \frac{b\rho_i}{L^2 \tau_0 c_i}\left(\left(\frac{\rho_w}{\rho_i} - 1\right)\phi + 1\right)^2\right)\frac{\partial \bar{\theta}}{\partial \bar{x}}\right) +$$

$$+ \frac{1}{\varepsilon^2}\frac{\partial}{\partial \bar{z}}\left(\left(\frac{a}{L^2 \tau_0 \rho_i c_i} + \frac{b(\rho_w - \rho_i)}{L^2 \tau_0 \rho_i c_i}\phi + \frac{b}{L^2 \tau_0 c_i}\right)\frac{\partial \bar{\theta}}{\partial \bar{z}}\right) - \nu \gamma \bar{I}_{wi}. \tag{18}$$

3.   The passage to the limit and solution by quadratures

For small ε, $\varepsilon \to 0$, the form of equations (13)–(18) depends on the parameter l.

We consider the most physical case (for $l = -2$) describing a slow flow of the fluid in the poroelastic medium (below, the dashes are omitted):

$$\frac{\partial(1-\phi)}{\partial t} + div((1-\phi)v_i) = 0,$$

$$\frac{\partial \phi}{\partial t} + div(\phi v_w) = 0, \tag{19}$$

$$v_i^1 = v_w^1, \tag{20}$$

$$\frac{\tau_0 \mu L^2}{kp}\phi(v_w^2 - v_i^2) = -\phi^n \frac{\partial p_w}{\partial z}, \tag{21}$$

$$\frac{\partial v_i^1}{\partial x} + \frac{\partial v_i^2}{\partial z} = -\beta p \frac{d(p_{tot} - p_w)}{dt}, \tag{22}$$

$$\frac{\partial}{\partial z}\left((1-\phi)\frac{\partial v_i^1}{\partial z}\right) = 0, \tag{23}$$

$$2\frac{\partial}{\partial z}\left((1-\phi)\frac{\partial v_i^2}{\partial z}\right) + \frac{\partial}{\partial x}\left((1-\phi)\frac{\partial v_i^1}{\partial z}\right) = 0, \tag{24}$$

$$\left(\frac{\rho_w c_w}{\rho_i c_i}\phi + (1-\phi)\right)\frac{\partial \theta}{\partial t} + \left(\frac{\rho_w c_w}{\rho_i c_i}\phi v_w^1 + (1-\phi)v_i^1\right)\frac{\partial \theta}{\partial x} + \left(\frac{\rho_w c_w}{\rho_i c_i}\phi v_w^2 + (1-\phi)v_i^2\right)\frac{\partial \theta}{\partial z} =$$

$$= \frac{1}{L^2 \tau_0 c_i}\frac{\partial}{\partial z}\left(\left(\frac{a}{\rho_i} + b\left(\phi\left(\frac{\rho_w}{\rho_i} - 1\right) + 1\right)\right)^2\right)\frac{\partial \theta}{\partial z}\right). \tag{25}$$

We outline the scheme of the method of the solution for system (19)–(24). Some of the equations are integrated and as the result of some transformations we arrive at an equation for the function ϕ.

At the first stage, from equation (19) we find $div \vec{v}_i$ and substitute it into equation (22), then

$$\frac{\partial G(\phi)}{\partial t} + \vec{v}_i \cdot \nabla G(\phi) = \frac{\partial (p_{tot} - p_w)}{\partial t} + \vec{v}_i \cdot \nabla (p_{tot} - p_w),$$

where $\dfrac{dG(\phi)}{d\phi} = \dfrac{1}{\beta p \phi (1-\phi)}$.

From the last equation we get the equation for p_i (if ϕ and p_w are found):

$$\frac{\partial U}{\partial t} + v_i^1 \frac{\partial U}{\partial x} + v_i^2 \frac{\partial U}{\partial z} = 0,$$

where $U = G(\phi) + (1-\phi)(p_w - p_i)$.

This equation is solved by the method of characteristics, which are defined as the solution of the Cauchy problem

$$v_i^1 = \frac{dx}{dt}, \quad v_i^2 = \frac{dz}{dt}, \quad U\mid_{t=0} = G(\phi^0) + (1-\phi^0)(p_w^0 - p_i^0).$$

At the second stage, we find a representation for the velocity components. To do this, we twice integrate equation (23) over z and, taking into account (20), we obtain

$$v_i^1 = v_w^1 = A(x,t)\int_0^z \frac{1}{1-\phi}d\tau + B(x,t), \tag{26}$$

where the functions $A(x,t), B(x,t)$ are arbitrary functions of (x,t).

Equation (24) can be written, using (26), as

$$2\frac{\partial}{\partial z}\left((1-\phi)\frac{\partial v_i^2}{\partial z}\right) + \frac{\partial A(x,t)}{\partial x} = 0.$$

Integrating the last equation over z twice, we obtain a representation for v_i^2 in terms of ϕ and arbitrary functions $C(x,t), D(x,t)$:

$$v_i^2 = -\frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau + D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + C(x,t).$$

At the third stage, we find representations for the liquid phase. To do this, we summarize equations (19):

$$div(\phi(\vec{v}_w - \vec{v}_i) + \vec{v}_i) = 0.$$

Using equations (20) and (21), we have

$$div \vec{v}_i = \frac{kp}{\tau_0 \mu L^2}\frac{\partial}{\partial z}\left(\phi^n \frac{\partial p_w}{\partial z}\right).$$

Substituting into the last equation the representation for the velocity components of the solid medium, we get the equation for p_w:

$$\frac{\partial}{\partial x}\left(A(x,t)\int_0^z \frac{1}{1-\phi}d\tau\right) + \frac{\partial B(x,t)}{\partial x} - 1/2\frac{\partial A(x,t)}{\partial x}\frac{z}{1-\phi} +$$

$$+D(x,t)\frac{1}{1-\phi} = \frac{kp}{\tau_0\mu L^2}\frac{\partial}{\partial z}\left(\phi^n\frac{\partial p_w}{\partial z}\right).$$

Integrating this equation twice over z, we obtain a representation for p_w in terms of ϕ and arbitrary functions $E(x,t), F(x,t)$

$$p_w = \frac{\tau_0\mu L^2}{kp}\int_0^z \phi^{-n}\left[\int_0^\zeta \frac{\partial}{\partial x}\left(A(x,t)\int_0^\xi \frac{1}{1-\phi}d\tau\right)d\xi -\right.$$

$$\left. -\frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^\zeta \frac{\tau}{1-\phi}d\tau + \frac{\partial B(x,t)}{\partial x}\zeta + D(x,t)\int_0^\zeta \frac{1}{1-\phi}d\tau + E(x,t)\right]d\zeta + F(x,t).$$

Using a representation for p_w, from (21) we derive the equation for the second component of the velocity v_w^2 in terms of ϕ

$$v_w^2 = -\frac{1}{\phi}\left[\int_0^z \frac{\partial}{\partial x}\left(A(x,t)\int_0^\xi \frac{1}{1-\phi}d\tau\right)d\xi - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau +\right.$$

$$\left. +\frac{\partial B(x,t)}{\partial x}z + D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + E(x,t)\right] - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{1-\phi}d\tau +$$

$$+D(x,t)\int_0^z \frac{1}{1-\phi}d\tau + C(x,t).$$

Thus, the velocity components $\vec{v}_i, \vec{v}_w$ and the pressure p_w are obtained as the functions of ϕ.

At the last stage, the continuity equation for the solid phase is used to determine the porosity ϕ:

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}\left(sA(x,t)\int_0^z s^{-1}d\tau + sB(x,t)\right) +$$

$$+\frac{\partial}{\partial z}\left(-s\frac{1}{2}\frac{\partial A(x,t)}{\partial x}\int_0^z \frac{\tau}{s}d\tau + sD(x,t)\int_0^z s^{-1}d\tau + sC(x,t)\right) = 0, \quad s \equiv 1-\phi.$$

Note that the representation for p_i is found after obtaining the representation for the components of the velocity of the solid phase and the pressure of the liquid.

Now we consider the following initial-boundary value problem for equations (19)–(24). At $z = 0$, the rigid porous skeleton satisfies a plane stress state with the stress tensor $\sigma_{11} = \sigma_{22} = p_{tot}$, $\sigma_{12} = \sigma_{21} = 0$, and the fluid pressure is also set. The velocity of the rigid skeleton is specified at $z = H$, and the phase velocities are equal. The porosity and phase pressures are set at the initial moment of time. Thus, the following initial boundary conditions are considered:

$$\frac{\partial v_i^2}{\partial z}\Big|_{z=0} = 0, \qquad v_i^2\Big|_{z=H} = C = \text{const},$$

$$v_i^1\Big|_{z=H} = B = \text{const}, \qquad \frac{\partial v_i^1}{\partial z}\Big|_{z=0} = 0,$$

$$\phi\Big|_{t=0} = \phi^0(x,t), \qquad \frac{\partial p_w}{\partial z}\Big|_{z=H} = 0, \tag{27}$$

$$p_w\Big|_{t=0} = p_w^0(x,z), \qquad p_i\Big|_{t=0} = p_i^0(x,z), \qquad p_w\Big|_{z=0} = p_0(x,t),$$

where the constants B and C are given. This allow us to uniquely define the following functions:

$$A(x,t) = 0, \quad B(x,t) = B = \text{const}, \quad C(x,t) = C = \text{const},$$

$$D(x,t) = 0, \quad E(x,t) = 0, \quad F(x,t) = p_0(x,t),$$

$$v_i^1 = v_w^1 = B, \quad v_i^2 = v_w^2 = C, \quad p_w = p_0(x,t),$$

and the equation for s can be rewritten as

$$\frac{\partial s}{\partial t} + B\frac{\partial s}{\partial x} + C\frac{\partial s}{\partial z} = 0.$$

The solution of this equation is

$$\phi = \phi^0(x - Bt, z - Ct).$$

The solution for p_i is

$$p_i = p_0(x,t) - p_w^0(x - Bt, z - Ct) + p_i^0(x - Bt, z - Ct).$$

The equation for θ takes the form

$$\left(\frac{\rho_w c_w}{\rho_i c_i}\phi^0(x - Bt, z - Ct) + (1 - \phi^0(x - Bt, z - Ct))\right)\left(\frac{\partial \theta}{\partial t} + B\frac{\partial \theta}{\partial x} + C\frac{\partial \theta}{\partial z}\right) =$$

$$= \frac{1}{L^2\tau_0 c_i}\frac{\partial}{\partial z}\left(\left(\frac{a}{\rho_i} + b\rho_i\left(\phi^0(x - Bt, z - Ct)\frac{\rho_w}{\rho_i} + 1 - \phi^0(x - Bt, z - Ct)^2\right)\right)\frac{\partial \theta}{\partial z}\right)$$

and is supplemented by the following initial-boundary conditions

$$\theta\mid_{t=0} = \theta^0(x,z), \quad \frac{\partial \theta}{\partial z}\mid_{z=0,z=H} = 0, \quad \frac{\partial \theta}{\partial x}\mid_{x=0,x=L} = 0. \tag{28}$$

The equation for θ is linear uniformly parabolic. The maximum principle for the function θ is valid, and theorem of existences is also established for this problem (q.v. [23], pp. 29, 364).

Thus, the solution of problem (19)–(25), (27), (28) is found.

Conclusion

A new model of the dynamics of the snow-ice cover, taking into account the deformation of the porous iced skeleton, is considered in this work. The model is a generalization of the classical Musket-Leverett two-phase filtration model and it is applicable for a wide class of problems of multiphase flows in a poroelastic media. A feature of the model is the variable porosity. The problem of the justification of this model is open. There are also no results on the construction of simple solutions. On the other hand, the study of such a general problem is important for the analysis of complicated processes of salt-mass transfer in a melting snow. For example, the speed of pollution in spring during melting of the snow cannot be described without knowing the field of velocities and water saturation. Using a sophisticated model allows to get a complete picture of the dynamics of the process. The important point is the correct choice of initial data, which guarantees the existence of a solution. Only examples of single-phase numerical solutions are known without phase transitions and temperature effects [14, 22]. In this article, we consider a simplified model based on the selection of a small parameter, which is a common technique in

the study of complex problems. Solutions in quadratures are constructed. In the future, we plan to study the justification for the formulated problem and its applications in polar mechanics.

References

[1] L.S.Kuchment, P.Romanov, A.N.Gelfan, V.N.Demidov, Use of satellite-derived data for characterization of snow cover and simulation of snowmelt runoff through a distributed physically based model of runoff generation, *Hydrol. Earth Syst. Sci.*, **14**(2010), 339–350. DOI: 10.5194/hess-14-339-2010

[2] A.N.Gelfan, J.W.Pomeroy, L.S. Kuchment, Modeling Forest Cover Influences on Snow Accumulation, Sublimation, and Melt, *Journal of Hydrometeorology*, **5**(2004), 785–803.

[3] M.Sturm, J.Holmgren, G.Liston, A seasonal snow cover classification-system for local to global applications, *Journal of Climate*, **8**(1995), no. 5, 1261–1283. DOI: 10.1175/1520-0442(1995)008<1261:ASSCCS>2.0.CO;2

[4] S.C.Colbeck, A theory of water percolation in snow, *Journal Glaciol*, **11(63)**(1972), 369–385.

[5] J.M.N.T.Gray, Water movement in wet snow, *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, **354 (1707)**(1996), 465–500.

[6] S.Sellers, Theory of water transport in melting snow with a moving surface, *Cold Regions Science and Technology*, **31**(2000), no. 1, 47–57.   DOI: 10.1016/S0165-232X(00)00006-9

[7] R.I.Nigmatullin, Dynamics of multiphase media, Vol. 1, Moscow, Nauka, 1987 (in Russian).

[8] S.N.Antontsev, A.V.Kazhikhov, V.N.Monakhov, Boundary-Value Problems of the Mechanics of Inhomogeneous Fluids, Novosibirsk, Nauka, 1983 (in Russian).

[9] K.Terzaghi, Theoretical Soil Mechanics, New York, Jhon Wiley, 1943.

[10] V.V.Vedernikov, V.N.Nikolaevsii, Mechanics equations for porous medium saturated by a two-phase liquid, *Izvestiya Akademii Nauk SSSR, Mekhanika Zhidkosti i Gaza*, **5**(1978), 769–773.

[11] Y.Geng, L.Zhang, Bifurcations of traveling wave solutions for the magma equation, *Applied Mathematics and computation*, **217**(2010), 1741–1748.   DOI: 10.1016/j.amc.2009.11.035

[12] G.Simpson, M.Spiegelman, M.I.Weinstein, Degenerate dispersive equations arising in the study of magma dynamics, *Nonlinearity*, **20**(2007), 21–49.

[13] A.Friedman, Cancer as Multifaceted Disease, *Math. Model. Nat. Phenom.*, **7**(2012).

[14] J.A.D.Connolly, Y.Y.Podladchikov, Temperature-dependent viscoelastic compaction and compartmentalization in sedimentary basins, *Tectonophysics*, **324**(2000), 131–135.

[15] J.Bear, Dynamics of Fluids in Porous Media, Elsevier, New York, 1972.

[16] A.A.Papin, Y.Y.Podladchikov, Isothermal motion of two immiscible fluids in poroelastic medium, *Izvestiya Altai State University*, **1/2(85)**(2015), 131–135 (in Russian). DOI: 10.14258/izvasu(2015)1.2-24

[17] A.Fowler, Mathematical Geoscience, Springer-Verlag, London Limited, 2011.

[18] A.A.Papin, M.A.Tokareva, On Local Solvability of the System of the Equations of One Dimensional Motion of Magma, *Journal of Siberian Federal University. Mathematics and Physics*, **10**(2017), no. 3, 362–371.   DOI: 10.17516/1997-1397-2017-10-3-362-371

[19] M.A.Tokareva, A.A.Papin, Global solvability of a system of equations of one-dimensional motion of a viscous fluid in a deformable viscous porous medium, *Journal of Applied and Industrial Mathematics*, **13**(2019), no. 2, 350–362.   DOI: 10.33048/sibjim.2019.22.208

[20] M.N.Koleva, L.G.Vulkov, Numerical analysis of one dimensional motion of magma without mass forces, *Journal of Computational and Applied Mathematics*, **366**(2020), 112338. DOI: 10.1016/j.cam.2019.07.003

[21] J.Escher, M.Hillairet, P.Laurencot, C.Walker, Thin film equations with soluble surfactant and gravity: modeling and stability of steady states, *Mathematische Nachrichten*, **285**(2012), 210–222.

[22] C.Morency et al., A numerical model for coupled fluid flow and matrix deformation with applications to disequilibrium compaction and delta stability, *Journal of Geophysical Research: Solid Earth*, **112**(2007), no. B10.

[23] O.A.Ladyzhenskaya, V.A.Solonnikov, N.N.Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Moscow, Nauka, 1967 (in Russian).

Математическая модель движения жидкостей в пороупругом снежно-ледовом покрове

Маргарита А. Токарева
Александр А. Папин
Алтайский государственный университет
Барнаул, Российская Федерация

Аннотация. Задача динамики снежно-ледового покрова рассматривается в рамках теории пороупругости. Снежно-ледовый покров рассматривается как трехфазная среда, состоящая из воды, воздуха и льда. В основу математической модели положены уравнения сохранения массы для каждой из фаз с учетом фазовых переходов, уравнения сохранения импульсов фаз в форме законов Дарси, уравнение сохранения импульса системы в целом, реологического уравнения для пористости и уравнение теплового баланса снега. В полной постановке учтена зависимость давлений жидкости и воздуха от температуры и соответствующих плотностей, а также зависимость коэффициентов вязкости и сжимаемости льда от температуры. В модельном случае рассматривается фильтрация воды в тонкой пороупругой ледовой пластине. Получены решения в квадратурах.

Ключевые слова: пороупругость, пористость, снежно-ледовый покров, тонкий слой.