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On a Transmission Problem Related to Models of Electrocardiology

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Abstract. We consider a generalization of a transmission problem for matrix elliptic operators related to mathematical models of cardiology. We find sufficient conditions when the approach developed for scalar elliptic operators is still valid in this much more general situation.

Keywords: transmission problems for elliptic systems, models of electrocardiology.

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Introduction

In this paper we consider a family of transmission problems for elliptic operators with constant coefficients related to models of electrocardiology. More precisely, for many years for satisfactory models of heart activity one uses Cauchy, Dirichlet, and Neumann problems for scalar strongly elliptic operators, see, for example, [1, 2]. A modification of such a model involving boundary problems for the Laplace operator has been recently studied in [3].

We consider similar problems for more general matrix linear elliptic operators and find sufficient conditions under which the scheme for solving the problems suggested in [3] allows to construct their solutions. Our approach is essentially based on the general theory of Fredholm problems for strongly elliptic (matrix) linear operators, see, e.g., [4], and the theory of regularization of an ill-posed Cauchy problem for operators with an injective principal symbol, see [3].

1. A model example

To begin with, we consider a basic example related to models of electrocardiology. As known from clinical practice, see, e.g., [1, 2], electrical activity of cardiac cells is crucial for pumping function of heart, which is the result of rhythmical cycles of contraction-relaxation of the cardiac tissue. Anomalies of electrical activity often cause heart diseases, which makes these investigations, in particular, development of adequate mathematical models, very relevant nowadays.

Let us illustrate this by one model of electrocardiology [1, 2, 5]. Denote by Ω_B and Ω_H three-dimensional domains with piecewise smooth boundaries with $\partial\Omega_B$ and $\partial\Omega_H$ corresponding to a body and a heart (see Fig. 1). Then the domain $\Omega = \Omega_B \setminus \Omega_H$ with the boundary $\partial\Omega = \partial\Omega_B \cup \partial\Omega_H$ corresponds to the body without heart.

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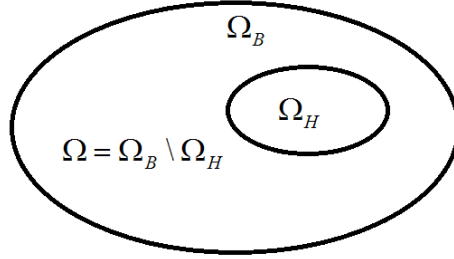


Fig. 1. Geometry of the model

Usually, in standard models one assumes that the cardiac tissue can be divided into two parts – intracellular and extracellular parts separated by a membrane – to which the electric potential u_i and u_e , respectively, is assigned. Regarding the cardiac tissue as a continuous medium we think of the potentials as defined in each point of Ω_H and satisfying the equation

$$\nabla^* M_i \nabla u_i + \nabla^* M_e \nabla u_e = 0, \quad (1)$$

where M_i and M_e are known tensor matrices that characterize intracellular and extracellular parts, and ∇ is the gradient operator in \mathbb{R}^3 .

One often considers the case when M_i and M_e are positively defined matrices with constant coefficients with entry values defined by conductivity of the cardiac tissue. For simplicity of the further analysis one assumes that these matrices are proportional

$$M_i = \lambda M_e, \quad \lambda > 0.$$

Based on equation (1) one considers two models of heart activity. In one model it is assumed that the heart is isolated and one considers the problem

$$\begin{aligned} \nabla^* M_i \nabla u_i + \nabla^* M_e \nabla u_e &= 0 \text{ in } \Omega_H, \\ (\nu_1, \nu_2, \nu_3) M_i \nabla u_i &= 0 \text{ on } \partial\Omega_H, \\ (\nu_1, \nu_2, \nu_3) M_e \nabla u_e &= -(\nu_1, \nu_2, \nu_3) M_b \nabla u_b \text{ on } \partial\Omega_H, \\ u_b &= u_e \text{ on } \partial\Omega_H, \end{aligned} \quad (2)$$

where M_b is the tensor matrix characterizing conductivity of the body, ν is the vector field of unit outward normal vectors to the boundary of the domain under the consideration and u_b is the electric potential of the body.

In the second model one takes the body into account, and from the electrodynamics of stationary currents it follows that the electric potential of the body u_b in the domain Ω is defined by the equations

$$\begin{aligned} \nabla^* M_b \nabla u_b &= 0 \text{ in } \Omega, \\ (\nu_1, \nu_2, \nu_3) M_b \nabla u_b &= 0 \text{ on } \partial\Omega_B. \end{aligned} \quad (3)$$

A feature of the model is the fact that one is more interested not in potentials u_i and u_e separately but in their difference $v = u_i - u_e$ in Ω_H or at least on its boundary.

Since matrices M_i and M_e are positively defined and not degenerate, the problems (2), (3) can be studied in the framework of the theory of boundary (maybe ill-posed) problems for elliptic

formally self-adjoint equations, see [1, 2, 5]. Moreover, notice that the problems above may be regarded as transmission problems for elliptic equations with discontinuous coefficients describing solutions in different domains of a continuum with the help of additional conditions on separating surfaces, see, for example, [6, 7].

Until now we have not used any functional spaces in the problems description, in the next section we give a precise formulation of a more general problem and specify functional classes for its solution.

2. Formulation of a problem

Let θ be a measurable set in \mathbb{R}^n , $n \geq 2$. Denote by $L^2(\theta)$ a Lebesgue space of complex-valued functions on θ with the scalar product

$$(u, v)_{L^2(\theta)} = \int_{\theta} \overline{v(x)} u(x) dx.$$

If D is a domain in \mathbb{R}^n with a piecewise smooth boundary ∂D , then for $s \in \mathbb{N}$ we denote by $H^s(D)$ the standard Sobolev space with the scalar product

$$(u, v)_{H^s(D)} = \int_D \sum_{|\alpha| \leq s} \overline{(\partial^\alpha v)} (\partial^\alpha u) dx.$$

It is well-known that this scale extends for all $s > 0$. Let now $H^s(D)$ for $s \in \mathbb{R}_+ \setminus \mathbb{Z}_+$ be the standard Sobolev-Slobodeckij spaces. Denote by $H_0^s(D)$ the closure of the subspace $C_{\text{comp}}^\infty(D)$ in $H^s(D)$, where $C_{\text{comp}}^\infty(D)$ is the linear space of functions with compact supports in D .

The space of k -vectors $u = (u_1, \dots, u_k)$ whose components lie in $H^s(D)$ equipped with the scalar product

$$(u, v)_{[H^s(D)]^k} = \sum_{j=1}^k \int_D \sum_{|\alpha| \leq s} \overline{(\partial^\alpha v_j)} (\partial^\alpha u_j) dx = \int_D \sum_{|\alpha| \leq s} (\partial^\alpha v)^* (\partial^\alpha u) dx$$

we shall denote by $[H^s(D)]^k$.

Further on, we shall consider linear matrix operators

$$A = \sum_{|\alpha| \leq p} A_\alpha \partial^\alpha, \quad x \in D,$$

where $p \in \mathbb{N}$ is the order of operator A , $\alpha \in \mathbb{Z}_+^n$, and A_α are $(l \times k)$ -matrices with constant coefficients. By a formal adjoint of A we call the differential operator

$$A^* = \sum_{|\alpha| \leq p} A_\alpha^* \partial^\alpha,$$

where A_α^* is the adjoint matrix for A_α or, equivalently,

$$(Au, v)_{[L^2(D)]^l} = (u, A^*v)_{[L^2(D)]^k} \quad \text{для всех } u \in [C_0^\infty(D)]^k, \quad v \in [C_0^\infty(D)]^l.$$

As usual, the principal symbol of an operator A is the matrix

$$\sigma(A)(x, \zeta) = \sum_{|\alpha|=p} A_\alpha \zeta^\alpha, \quad x \in D, \quad \zeta \in \mathbb{C}^n.$$

We say that the principal symbol of A is injective if $l \geq k$ and

$$\text{rang } \sigma(A)(x, \zeta) = k, \text{ для всех } \zeta \in \mathbb{R}^n \setminus \{0\} \text{ и всех } x \in \overline{D}.$$

If $l = k$ operators with injective principal symbols are called elliptic.

Let now A_e , A_i , and A_b be linear differential operators of the first order with constant coefficients on \overline{D}_m , i.e.

$$A_m = \sum_{j=1}^n a_j^{(m)} \frac{\partial}{\partial x_j} + a_0^{(m)},$$

where $m \in \{e, i, b\}$, $D_e \equiv D_i \equiv \Omega_H$, $D_b \equiv \Omega$.

Further on, we assume that principal symbols of operators A_m are injective in the corresponding domains.

Denote by A_m^* a formal adjoint of A_m and consider a generalized Laplacian $A_m^* A_m$.

Under assumptions made above, the operator $A_m^* A_m$ is a strongly elliptic $(k \times k)$ -matrix second order operator, i.e. it is elliptic and there exists a positive constant c such that

$$\Re\left(-w^* \sigma(A_m^* A_m)(x, \zeta) w\right) \geq c |w|^2 |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^n \setminus \{0\}, w \in \mathbb{C}^k \setminus \{0\}, x \in \overline{D}_m.$$

The operator $A_m^* A_m$ is also formally self-adjoint, i.e.

$$(A_m^* A_m u, v)_{[L^2(D_m)]^k} = (u, A_m^* A_m v)_{[L^2(D_m)]^k} = (A_m u, A_m v)_{[L^2(D_m)]^k} \text{ for all } u, v \in [C_0^\infty(D_m)]^k;$$

in particular, the operator $A_m^* A_m$ is (formally) positively defined

$$(A_m^* A_m u, u)_{[L^2(D_m)]^k} \geq 0 \text{ for all } u \in [C_0^\infty(D_m)]^k.$$

Let, as before, ν be the outward normal vector operator on the boundary of the domain of the operator A_m . Introduce the conormal derivatives

$$\nu_{A_m} = \sigma^*(A_m)(\nu) A_m,$$

associated with these operators via Green's formula:

$$\int_{\partial\Omega} \nu \nu_{A_m} u ds = \int_{\Omega} (v^* (A_m^* A_m u) - (A_m v)^* A_m u) dx \text{ for all } u, v \in [H^2(\overline{D}_m)]^k. \quad (4)$$

Assume that bounded domains Ω_H , Ω , and Ω_b have twice smooth boundaries and consider the following problem (5), (6): find vector-functions u_i , u_e from $[H^2(\Omega_H)]^k$ and a vector-function u_b from $[H^2(\Omega)]^k$ such that

$$\begin{cases} A_i^* A_i u_i + A_e^* A_e u_e = 0 \text{ in } \Omega_H, \\ \nu_{A_i} u_i = 0 \text{ on } \partial\Omega_H, \\ \nu_{A_e} u_e = -\nu_{A_b} u_b \text{ on } \partial\Omega_H, \\ u_e = u_b \text{ on } \partial\Omega_H, \end{cases} \quad (5)$$

$$\begin{cases} A_b^* A_b u_b = 0 \text{ in } \Omega, \\ \nu_{A_b} u_b = 0 \text{ on } \partial\Omega_B, \end{cases} \quad (6)$$

where the equality on the boundary is in the sense of traces, and the equality in the domains is in the sense of distributions. In this case we can assume that traces of functions and their conormal derivatives are well-defined.

It is obvious that the problem (5), (6) is a generalization of the problem (2), (3). Note also that it incorporates several classical boundary problems.

Example 2.1. Consider first the classical case $A_b = \nabla$ ($k = 1$, $l = n$), then $\nu_{A_b} = \frac{\partial}{\partial \nu}$ is a directional derivative along the outward normal vector to $\partial\Omega_B$. If we assume that u_e is known on $\partial\Omega_H$ and equal to a function $v_0 \in H^{3/2}(\partial\Omega_H)$, then (5), (6) gives the following problem: find a function $u_b \in H^2(\Omega)$ satisfying

$$\begin{cases} -\Delta u_b = 0 \text{ in } \Omega, \\ \frac{\partial u_b}{\partial \nu} = 0 \text{ on } \partial\Omega_B, \\ u_b = v_0 \text{ on } \partial\Omega_H. \end{cases} \quad (7)$$

This is a classical mixed problem that is often called a Zaremba problem, see, e.g. [4, 8]. This problem can be studied by standard methods in Sobolev and Hölder spaces. It is well-known that this problem has a unique solution in these classes that can be written with the help of the Green function $\mathcal{Z}_\Omega(x, y)$ having the standard properties

$$u_b(x) = \int_{\partial\Omega} \mathcal{Z}_\Omega(x, y) v_0(y) dS(y), \quad x \in \Omega_H,$$

where $dS(y)$ is the volume form on the surface $\partial\Omega$, see [4, 8].

Analogously, if we assume that $A_e = \nabla$ ($k = 1$, $l = n$), then $\nu_{A_e} = \frac{\partial}{\partial \nu}$ is a directional derivative along the outward normal vector to $\partial\Omega_H$. If the conormal derivative $\nu_{A_e} u_e$ is known on $\partial\Omega_H$ and equal to a function $v_1 \in H^{1/2}(\partial\Omega_H)$, then (5), (6) gives a special case of a classical Neumann problem for a Laplace operator: find a function $u_b \in H^2(\Omega)$ satisfying

$$\begin{cases} -\Delta u_b = 0 \text{ in } \Omega, \\ \frac{\partial u_b}{\partial \nu} = 0 \text{ on } \partial\Omega_B, \\ \frac{\partial u_b}{\partial \nu} = v_1 \text{ on } \partial\Omega_H, \end{cases} \quad (8)$$

see [4, 9]. It is known that this problem is Fredholm in Sobolev and Hölder spaces, its solution is defined up to an additive constant, and the necessary and sufficient condition for solvability is the following

$$\int_{\partial\Omega_H} v_1(y) dS(y) = 0. \quad (9)$$

If this condition is satisfied the problem has a unique solution u_b in these classes that satisfies, for example,

$$\int_{\partial\Omega_H} u_b(y) dS(y) = 0. \quad (10)$$

It can be written with the help of an appropriate parametrix $\mathcal{N}_\Omega(x, y)$ that has the standard properties

$$u_b(x) = \int_{\partial\Omega} \mathcal{N}_\Omega(x, y) v_0(y) dS(y), \quad x \in \Omega_H,$$

However, the general theory of boundary problems suggests that knowledge of u_e or $\nu_{A_e} u_e$ on $\partial\Omega_H$ does not allow to recover the potential u_i uniquely from the remaining data and equations

without additional conditions (see also Uniqueness Theorem 3.1 for the problem (5), (6) proved under additional assumptions below).

Besides that, cardiology models are special in the sense that additional conditions necessary for recovering of unknown potentials u_i , u_e , u_b in the problem (5), (6) should preferably be set on the boundary of ‘the body’ Ω , since all measurements must be less traumatic for a patient and not invasive.

3. Application of an ill-posed Cauchy problem

One of the simplest additional conditions mentioned above leads to using of an ill-posed Cauchy problem. More precisely, it implies measuring the potential u_b on the boundary of ‘the body’:

$$u_b = f \text{ on } \partial\Omega_B, \quad (11)$$

where f is a given vector-function from $[H^{3/2}(\Omega)]^k$.

Unfortunately, as known very well, the problem (6), (11) is nothing else but an ill-posed problem for an elliptic operator $A_b^*A_b$. Let us see what the addition of the property (11) gives in a more general problem than those in cardiology.

Denote by $N(\Omega)$ the set of solutions to the problem (5), (6), (11) under the condition $f = 0$. Let $S_{A_e}(\Omega_H)$ be the space of generalized solutions of the equation $A_e h = 0$ в Ω_H . Since the operator A_e has an injective symbol and its coefficients are real analytic, the Petrovsky theorem yields that the elements of the space $S_{A_e}(\Omega_H)$ are real analytic vector-functions in Ω_H .

Theorem 3.1. *Let bounded domains Ω_H , Ω , and Ω_b have twice smooth boundaries and let for some constant $\lambda > 0$,*

$$A_i = \lambda A_e. \quad (12)$$

Then the set $N(\Omega)$ consists of triples $(u_i, u_e, u_b) \subset [H^2(\Omega_H)]^k \times [H^2(\Omega_H)]^k \times [H^2(\Omega)]^k$ such that

$$u_i = \frac{h - w}{\lambda^2}, \quad u_e = w, \quad u_b = 0, \quad (13)$$

where h is an arbitrary function from the space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$, and w is an arbitrary function from $[H_0^2(\Omega_H)]^k$.

Proof. Let a vector h belong to $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ and a vector w belong to $[H_0^2(\Omega_H)]^k$. Then w satisfies the following conditions

$$w = 0 \text{ on } \partial\Omega_H, \quad \nu_{A_i}(w) = 0 \text{ on } \partial\Omega_H, \quad (14)$$

and $A_i^*A_i = \lambda^2 A_e^*A_e$. Therefore the vector functions from (13) give a solution to the problem (5), (6), (11) for $f = 0$.

Let $u_i, u_e \in [H^2(\Omega_H)]^k$, and $u_b \in [H^2(\Omega)]^k$ is a triple of functions from $N(\Omega)$. Then from (5), (6) it follows that u_b is a solution to the Cauchy problem for the operator $A_b^*A_b$:

$$A_b^*A_b u_b = 0 \text{ in } \Omega, \quad \nu_{A_b}(u_b) = 0 \text{ on } \partial\Omega_B, \quad u_b = 0 \text{ on } \partial\Omega_B.$$

Since the operators A_m have injective symbols, we have

$$\text{rang}(\nu_{A_m})(x, \nu(x)) = \sigma^*(A_m)(x, \nu(x))\sigma(A_m)(x, \nu(x)) = k$$

for any $m = e, i, b$ and all $x \in \partial\Omega_H$ or $\partial\Omega$, respectively. In particular, the systems of boundary operators $\{I, \nu_{A_e}\}$, $\{I, \nu_{A_i}\}$ are first order Dirichlet systems on $\partial\Omega_H$, while the system of boundary operators $\{I, \nu_{A_b}\}$ is a first order Dirichlet system on $\partial\Omega$ (see, for example, [3]). Then by the uniqueness theorem for a Cauchy problem for elliptic operators (see, for example, [3, Theorem 10.3.5]), $u_b \equiv 0$ in Ω . Now by the trace theorem for Sobolev spaces and by equations from (5) we see that $u_e \equiv 0$ on $\partial\Omega_H$ and $\nu_{A_e}(u_e) \equiv 0$ on $\partial\Omega_H$. However, since the system of boundary operators $\{I, \nu_{A_e}\}$ is a first order Dirichlet system on $\partial\Omega_H$, it follows from the theorem on spectral synthesis (see [10]) that $u_e \in [H_0^2(\Omega_H)]^k$.

To complete the proof of the theorem we need the following lemma.

Lemma 3.1. *Let Ω_H be a bounded domain in \mathbb{R}^n with a twice smooth boundary and (12). If the functions $u_e, u_i \in [H^2(\Omega_H)]^k$ satisfy the equations (5) then they are related in Ω_H by*

$$u_e + \lambda^2 u_i = h, \quad (15)$$

where h is a function from the space $S_{A_e^* A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

Moreover, if $u_b \equiv 0$ on $\partial\Omega_H$, then the functions u_e, u_i are related in Ω_H by 15, where h is a function from the space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

Proof. Since $A_i = \lambda A_e$, the first equation in (5) can be rewritten in the form

$$A_e^* A_e h = 0 \text{ in } \Omega_H, \quad (16)$$

with $h = u_e + \lambda^2 u_i$, and clearly $h \in S_{A_e^* A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

If we additionally know that $u_b \equiv 0$ on $\partial\Omega_H$ then, as noticed above, $u_b \equiv 0$ in Ω . Therefore $\nu_{A_e}(u_e) = 0$ on $\partial\Omega_H$, and $\nu_{A_i}(u_i) = 0$ on $\partial\Omega_H$, which implies that

$$\nu_{A_e}(h) = 0 \text{ on } \partial\Omega_H. \quad (17)$$

From this, by the Green formula (4) we obtain

$$\begin{aligned} 0 &= (A_e^* A_e h, h)_{[L^2(\Omega_H)]^k} = \int_{\Omega_H} h^* (A_e^* A_e h) dx = \\ &= \int_{\Omega_H} (A_e h)^* (A_e h) dx + \int_{\partial\Omega_H} h^* \nu_{A_e}(h) ds = \|A_e h\|_{[L^2(\Omega_H)]^k}^2. \end{aligned}$$

Therefore, the vector function h defined by the equality (15) belongs to $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$. \square

Thus, the functions $u_i, u_e \in [H^2(\Omega_H)]^k$ satisfy (5), and by Lemma 3.1 we get $u_i = \frac{h - v}{\lambda^2}$, where $v \in [H_0^2(\Omega_H)]^k$ and $h \in S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$. \square

In particular, it follows from Lemma 3.1 that the zero space of the problem (5) coincides with the space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

Denote by $\ker A_e$ the kernel of a continuous linear operator $A_e : [H^2(\Omega_H)]^k \rightarrow [H^1(\Omega_H)]^l$ and consider several examples. In fact, $\ker A_e = S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

Example 3.1. Let $A_e = \begin{pmatrix} \nabla \\ 1 \end{pmatrix}$, ($k = 1, l = n + 1$). Then $A_e^* = (-\operatorname{div}, 1)$, $\nu_{A_e} = \frac{\partial}{\partial \nu}$, $A_e^* A_e = -\Delta + 1$, and the problem (16)–(17) becomes a Neumann problem for the Helmholtz operator

$$\begin{cases} -\Delta h + h = 0 \text{ in } \Omega_H, \\ \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega_H, \end{cases} \quad (18)$$

and the equation $A_e h = 0$ takes the form

$$\begin{cases} \nabla h = 0 & \text{in } \Omega_H, \\ h = 0 & \text{in } \Omega_H. \end{cases}$$

Consequently, $\ker A_e = \{0\}$ and coincides with the space of solutions of the homogeneous problem (18).

Example 3.2. Let $A_e = \nabla$ then $A_e^* = -\text{div}$. In this case ($k = 1, l = n$), $A_e^* = -\text{div}$, $\nu_{A_e} = \frac{\partial}{\partial \nu}$, $A_e^* A_e = -\Delta$, and the problem (16), (17) becomes a Neumann problem for the Laplace operator

$$\begin{cases} \Delta h = 0 & \text{in } \Omega_H, \\ \frac{\partial h}{\partial \nu} = 0 & \text{on } \partial\Omega_H, \end{cases} \quad (19)$$

and the equation $A_e h = 0$ takes the form

$$\nabla h = 0 \text{ in } \Omega_H.$$

Therefore, $\ker A_e = \mathbb{R}$ and coincides with the space of solutions of the problem (19).

Example 3.3. Consider the case where $A_e = \bar{\partial} = \partial_x - i\partial_y$ is the Cauchy–Riemann operator in $\mathbb{R}^2 \cong \mathbb{C}$ where i stands for imaginary unit. Then $A_e^* = -\partial = -\partial_x - i\partial_y$, and the kernel of A_e is holomorphic functions. The problem (16)–(17) defines then the zero space of a non-coercive $\bar{\partial}$ -Neumann problem, see, for example, [11, 12].

It is clear that the operator A_e should be chosen in a way that its kernel is at least finite dimensional.

Under assumptions of Theorem 3.1 the rest of the scheme of solving the problem (5), (6), (11) differs little from the standard one, see [5]. Namely, first we introduce a function $h(x)$ such that $h(x) = \lambda^2 u_i + u_e$, where $x \in \Omega_H$. From the conditions on the boundaries in (5) and the fact that $\nu_{A_i} = \lambda \nu_{A_e}$ we get that

$$\nu_{A_e} h = -\nu_{A_b} u_b \text{ on } \partial\Omega_H.$$

Thus, we can rewrite the original problem (5), (6), (11) in new notation: knowing a vector $f \in [H^{3/2}(\partial\Omega_H)]^k$, find vectors $h \in [H^2(\partial\Omega_H)]^k$ and $u_b \in [H^2(\partial\Omega)]^k$ such that

$$\begin{cases} A_e^* A_e h = 0 & \text{in } \Omega_H, \\ \nu_{A_e} h = -\nu_{A_b} u_b & \text{on } \partial\Omega_H, \end{cases} \quad (20)$$

$$\begin{cases} A_b^* A_b u_b = 0 & \text{in } \Omega, \\ \nu_{A_b} u_b = 0 & \text{on } \partial\Omega_B, \\ u_b = f & \text{on } \partial\Omega_B. \end{cases} \quad (21)$$

The original problem splits into two – (20) and (21). The problem (21), as noticed above, is an ill-posed Cauchy problem for an elliptic operator $A_b^* A_b$. It is known that if a solution to this problem exists it is unique. The problem (20) is a Neumann problem for an elliptic operator $A_e^* A_e$. Unfortunately, in general the Neumann problem may also be ill-posed. For it to be Fredholm, the so called Shapiro-Lopatinsky conditions must be placed [13, Chapter 1, Sec. 3, condition II for $q = 0$], [14] on the pair $(A_e^* A_e, \nu_{A_e})$. In particular, they guarantee that the space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ is finite dimensional.

More precisely, let us consider the following Neumann problem: for a given vector $h_0 \in [H^{1/2}(\partial\Omega_H)]^k$ find a vector $h \in [H^2(\partial\Omega_H)]^k$ such that

$$\begin{cases} A_e^* A_e h = 0 \text{ in } \Omega_H, \\ \nu_{A_e} h = h_0 \text{ on } \partial\Omega_H, \end{cases} \quad (22)$$

and formulate conditions for solvability.

Theorem 3.2. *If for a pair of operators $(A_e^* A_e, \nu_{A_e})$ the Shapiro-Lopatinsky conditions are fulfilled then the problem (22) is Fredholm. To be precise,*

1) *the zero space of the problem coincides with the finite-dimensional space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$;*

2) *the problem is solvable if and only if*

$$(h_0, \varphi)_{[L^2(\partial\Omega_H)]^k} = 0 \text{ for all } \varphi \in S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k; \quad (23)$$

3) *under (23) there exists a unique solution h_1 of the problem (22) satisfying*

$$(h_1, \varphi)_{[L^2(\partial\Omega_H)]^k} = 0 \text{ for all } \varphi \in S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k. \quad (24)$$

Proof. See [4]. □

Thus, under hypothesis of Theorem 3.2 for solvability of the Neumann problem (20) it is necessary and sufficient that for the vector $h_0 = -\nu_{A_b} u_b$ the condition (23) is fulfilled. This can be achieved if we place additional conditions on relations between the operators A_e and A_b . Namely, as we have seen above, it is quite natural to assume that

$$A_e = \tilde{\lambda} A_b \text{ for some constant } \tilde{\lambda} > 0. \quad (25)$$

Denote by $S_{A_b}(\Omega)$ the zero space of solutions to the problem (21) in the domain Ω .

Corollary 3.1. *Let for the pair of operators $(A_e^* A_e, \nu_{A_e})$ the Shapiro-Lopatinsky conditions be fulfilled. Besides that assume that the identity (25) holds and the spaces $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ and $S_{A_b}(\Omega) \cap [H^2(\Omega)]^k$ coincide. Then for any vector $u_b \in [H^2(\Omega)]^k$ satisfying (21) there exists a unique vector $h_1 \in [H^2(\Omega_H)]^k$ that satisfies (20) and (24).*

Proof. By Theorem 3.2 for solvability of the problem (20) it is necessary and sufficient that

$$(\nu_{A_e} u_b, \varphi)_{[L^2(\partial\Omega_H)]^k} = 0 \text{ for all } \varphi \in S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k. \quad (26)$$

If the vector $u_b \in [H^2(\Omega)]^k$ satisfies (21), then by the Green formula (4) for the operator A_b

$$-\int_{\partial\Omega_H} \nu_{A_b} u_b \psi ds = \int_{\Omega} \nu_{A_b} u_b \psi ds = (\psi, A_b^* A_b u)_{[L^2(\Omega)]^k} - (A_b \psi, A_b u)_{[L^2(\Omega)]^k} = 0,$$

for any $\psi \in S_{A_b}(\Omega_H) \cap [H^2(\Omega_H)]^k$.

On the other hand, the relation (25) guarantees that $(\tilde{\lambda})^2 \nu_{A_b} = -\nu_{A_e}$, and therefore

$$(\nu_{A_e} u_b, \varphi)_{[L^2(\partial\Omega_H)]^k} = -(\tilde{\lambda})^2 (\nu_{A_b} u_b, \varphi)_{[L^2(\partial\Omega_H)]^k} = -(\tilde{\lambda})^2 \int_{\partial\Omega_H} \nu_{A_b} u_b \psi ds$$

for any $\varphi \in S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$. Due to the fact that the spaces $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ and $S_{A_b}(\Omega) \cap [H^2(\Omega)]^k$ coincide, (26) holds. Then by statement 3 of Theorem 3.2 for any vector u_b there exists a unique vector $h_1 \in [H^2(\Omega_H)]^k$ satisfying (20) and (24). □

The condition that the spaces $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ and $S_{A_b}(\Omega) \cap [H^2(\Omega)]^k$ coincide seems to be rather strong, especially since these are spaces of solutions to different differential equations in different domains. Nevertheless, provided (25) holds, such a coincidence is possible if the operator A_e is so much overdetermined that the space of its solutions in any domain is finite dimensional and coincides with the space of solutions in \mathbb{R}^n ; the typical examples are the so-called stationary holonomic systems. Let us illustrate this by the following examples.

Example 3.4. Let $A_e = \nabla$ and $A_b = \tilde{\lambda} \nabla$ ($k = 1, l = n$). The function $u = \text{const}$ is a solution to the equation $\nabla u = 0$ in Ω_H and extends to Ω , where it is a solution to $\tilde{\lambda} \nabla u = 0$. Thus we get that the spaces $S_{A_e}(\Omega_H)$ and $S_{A_b}(\Omega)$ coincide.

Example 3.5. Let $A_e = \begin{pmatrix} \nabla \\ 1 \end{pmatrix}$ and $A_b = \tilde{\lambda} \begin{pmatrix} \nabla \\ 1 \end{pmatrix}$, ($k = 1, l = n + 1$). A solution to $A_e u = 0$ in Ω_H is $u \equiv 0$ and it extends to Ω , where it is a solution $A_b u = 0$. Thus, the spaces $S_{A_e}(\Omega_H)$ and $S_{A_b}(\Omega)$ coincide.

Example 3.6. Consider the following operators A_i, A_e и A_b :

$$A_e = \begin{pmatrix} \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & \partial_y & 0 \\ -1 & 0 & \partial_x \\ 0 & -1 & \partial_y \end{pmatrix}, \quad A_i = \lambda \begin{pmatrix} \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & \partial_y & 0 \\ -1 & 0 & \partial_x \\ 0 & -1 & \partial_y \end{pmatrix}, \quad A_b = \tilde{\lambda} \begin{pmatrix} \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & \partial_y & 0 \\ -1 & 0 & \partial_x \\ 0 & -1 & \partial_y \end{pmatrix}.$$

These operators have injective principal symbols and are equivalent to second order operators

$$\tilde{A}_e = \begin{pmatrix} \partial_{xx} \\ \partial_{yy} \\ \partial_{xy} \end{pmatrix}, \quad \tilde{A}_i = \lambda \begin{pmatrix} \partial_{xx} \\ \partial_{yy} \\ \partial_{xy} \end{pmatrix}, \quad \tilde{A}_b = \tilde{\lambda} \begin{pmatrix} \partial_{xx} \\ \partial_{yy} \\ \partial_{xy} \end{pmatrix}.$$

Therefore the space of solutions of the system $A_e u = 0$ in Ω_H coincides with the set of all linear functions $u = c_1 x + c_2 y + c_3$, and any function of this form extends to Ω , where it is a solution to the equation $A_b u = 0$. Therefore, the spaces $S_{A_e}(\Omega_H)$ and $S_{A_b}(\Omega)$ coincide.

As noticed above, if a solution to the Neumann problem (20) exists, it is ‘unique’ up to an element of the space $S_{A_e}(\Omega_H) \cap [H^2(\Omega_H)]^k$ (additively).

Recall that the aim of solving the original problem (5), (6), (11) is to find the transmembrane potential v on the surface $\partial\Omega_H$. Let us write down the algorithm for solving the problem (20), (21):

1. Find a function u_b and its conormal derivative $\nu_{A_b}(u_b)$ on the surface $\partial\Omega_H$ by solving an ill-posed Cauchy problem (21) for an elliptic operator $A_b^* A_b$.
2. Compute values of $h(x)$ on the surface $\partial\Omega_H$ by solving a Neumann problem (20) for an elliptic operator $A_e^* A_e$ with the data $\nu_{A_b} u_b$ on $\partial\Omega_H$ obtained in Step 1. The possibility of this depends on whether the restrictions on operators A_e and A_b described above hold.
3. Find the transmembrane potential v on the surface $\partial\Omega_H$ by using the relation (15) together with u_b and h on $\partial\Omega_H$, found in Steps 1 and 2, respectively

$$v = u_i - u_e = \frac{h - u_b}{\lambda^2} - u_b \quad \text{on } \partial\Omega_H. \quad (27)$$

In conclusion we note that solvability conditions for an ill-posed Cauchy problem in Sobolev spaces for a rather wide class of operators with real analytic coefficients are well known, see, for example, [3]. Moreover, in [3, 15] one can find constructive procedures for its regularization, i.e. for construction exact and approximate solutions (the so called Carleman formulas). Regarding areas with models with the geometry corresponding to that in cardiology and operators that are first order matrix factorizations of the Laplace operator, or more generally, of a Lamé-type operator such Carleman formulas were obtained in [17].

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Об одной задаче трансмиссии, связанной с моделями электрокардиологии

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Аннотация. В настоящей работе рассмотрено одно обобщение задачи трансмиссии для матричных эллиптических операторов, связанной с математическими моделями кардиологии. Указаны достаточные условия, при которых подход, разработанный для скалярных операторов, все еще работает в новой, гораздо более общей ситуации.

Ключевые слова: задачи трансмиссии для эллиптических операторов, модели электрокардиологии.