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On the Differentiation in the Privalov Classes

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Abstract. The invariance of the Privalov classes with respect to the differentiation operator is studied.

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Introduction

Let $\mathbb{C}$ be the complex plane, D be the unit disk on $\mathbb{C}$, $H(D)$ be the set of all functions, holomorphic in D. For all $0 < q < +\infty$ we define the Privalov class of function Π_q as follows (see [11]):

$$\Pi_q = \left\{ f \in H(D) : \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta < +\infty \right\}.$$

$\ln^+ |a| = \max(\ln |a|, 0)$, $\forall a \in \mathbb{C}$.

The classes Π_q were first considered by I. I. Privalov in [11]. If $q = 1$ the Privalov class coincides with the Nevanlinna class N of analytic functions in D with bounded characteristic $T(r, f) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta$, $0 \leqslant r < 1$. This is well-known in scientific literature (see [9]). Using Hölder's inequality, it is easy to prove the inclusion chain:

$$\Pi_q\,(q > 1) \subset N \subset \Pi_q\,(0 < q < 1).$$

Since for all $0 < q < q'$

$$(\ln^+ |f|)^q < (\ln^+ |f| + 1)^q < (\ln^+ |f| + 1)^{q'} < 2^{q'} \cdot \left((\ln^+ |f|)^{q'} + 1\right),$$

we have

$$\Pi_{q'} \subset \Pi_q.$$

In the case of $1 \leqslant q < +\infty$ the Privalov spaces were studied by M. Stoll, V. I. Gavrilov, A. V. Subbotin, D. A. Efimov, R. Mestrovic, Z. Pavicevic, etc. The monograph [6] contains a brief overview of their results. Certain results were extended to the case $0 < q < 1$ by the first author of this paper (see [13]). Notice that the case $0 < q < 1$ was little studied. The questions

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of interpolation in the Privalov classes, as well as properties of root sets of analytic functions from these classes were investigated in recent works by the authors (see [14–16, 20]).

In this paper we study a question of the invariance of the classes Π_q with respect to the differentiation operator. In other words, we verify the validity of the Bloch-Nevanlinna conjecture in the Privalov spaces.

The assumption, known as the Bloch-Nevanlinna conjecture, was clearly formulated by Nevanlinna in 1929 (see [9]) as follows: a derivative of any analytic function in the unit disk with bounded characteristic is a function of bounded characteristic.

The famous result refuting this hypothesis belongs to O. Frostman (see [5]). He proved that there is a Blaschke product whose derivative is not a function with a bounded characteristic.

Subsequently, many counterexamples that refute the Bloch-Nevanlinna conjecture were constructed in the works of others such as H. Fried (1946), W. Rudin (1955), W. Hayman (1964), P. Duren (1969), J. Anderson (1971), L.-Sh. Khan (1972), et. al. D. Campbell and G. Weeks [1] provide a brief overview of these results, as well as a general approach to the construction of such examples.

The invariance with respect to the integro-differential operators of other classes of analytic functions have been studied by many mathematicians. A brief overview of their results is contained in the work of S. V. Shvedenko [22]. In particular, a closure of the classes of analytic functions in a disk with the restrictions on Nevanlinna's characteristic function regarding the operations of differentiation and integration was studied by F. A. Shamoyan, I. S. Kursina, V. A. Bednazh (see [19]).

We state the Bloch-Nevanlinna conjecture in the Privalov spaces: for whatever $q > 0$, the derivative of a function from the class Π_q belongs to the class Π_q.

The paper is organized as follows. In the first part of the article we refute the Bloch-Nevanlinna conjecture in the Privalov spaces for all $0 < q < +\infty$. In the second part of the article we indicate the class to which the derivative of any function from the Privalov space belongs.

1. The Bloch-Nevanlinna conjecture for the Privalov spaces

The following statement is true.

Theorem 1.1. *The Bloch-Nevanlinna conjecture fails in the spaces* Π_q, $0 < q < +\infty$.

In other words, the Privalov spaces Π_q are not invariant under the differentiation operator for all $0 < q < +\infty$, not only for $q = 1$.

In the sequel, unless otherwise noted, we denote by $c, c_1, \dots, c_n(\alpha, \beta, \dots)$ some arbitrary positive constants depending on α, $\beta, \dots$, whose specific values are immaterial.

Proof of this statement reproduces the arguments from [21], the method goes back to the work of Hayman [8].

Let λ be a sufficiently large positive integer, $0 < \alpha < 1$, H^∞ be the class of bounded analytic functions in D. We define a function f_λ as follows:

$$f_\lambda = \sum_{k=0}^{+\infty} \lambda^{-k(1-\alpha)} z^{\lambda^k}.$$

It is obvious that $f_\lambda \in H(D)$, and $|f_\lambda| \leqslant \sum\limits_{k=0}^{+\infty} \lambda^{-k(1-\alpha)} = \dfrac{\lambda^{1-\alpha}}{\lambda^{1-\alpha} - 1}$, that is $f_\lambda \in H^\infty$. Since $H^\infty \subset \Pi_q$, we have $f_\lambda \in \Pi_q$ for all $0 < q < +\infty$.

In the same time we have

$$f'_\lambda = \sum_{k=0}^{+\infty} \lambda^{\alpha k} z^{\lambda^k - 1}. \tag{1}$$

Show that $f'_\lambda \notin \Pi_q$. We fix $n \in \mathbb{N}$ and denote $r_n = \exp(-\alpha/\lambda^n)$, $r_n \to 1-0$, $n \to +\infty$. Let $u_n(z)$ be the n-th term of the series (1):

$$u_n(z) = \lambda^{\alpha n} z^{\lambda^n - 1}.$$

By $S_n(z)$ we denote the n-th partial sum of the series (1):

$$S_n(z) = \sum_{k=0}^{n-1} \lambda^{\alpha k} z^{\lambda^k - 1},$$

and by $R_n(z)$ we denote the n-th remainder of the series (1):

$$R_n(z) = \sum_{k=n+1}^{+\infty} \lambda^{\alpha k} z^{\lambda^k - 1}.$$

We estimate these sums on the circle $|z| = r_n$.

$$|S_n(z)| \leqslant \sum_{k=0}^{n-1} \lambda^{\alpha k} r_n^{\lambda^k - 1} = \sum_{k=0}^{n-1} \lambda^{\alpha k} \exp\left(-\frac{\alpha}{\lambda^n} \cdot (\lambda^k - 1)\right) = \exp\left(\frac{\alpha}{\lambda^n}\right) \sum_{k=0}^{n-1} \lambda^{\alpha k} \exp\left(-\alpha \cdot \lambda^{-(n-k)}\right) \leqslant$$

$$\leqslant \exp\left(\frac{\alpha}{\lambda^n}\right) \sum_{k=0}^{n-1} \lambda^{\alpha k} = \exp\left(\frac{\alpha}{\lambda^n}\right) \cdot \frac{\lambda^{n\alpha} - 1}{\lambda^\alpha - 1} = \lambda^{n\alpha} \exp(-\alpha - 1) \cdot A(\lambda, \alpha),$$

where $A(\lambda, \alpha) = \exp\left[\alpha(1 + \frac{1}{\lambda^n}) \cdot \frac{(1 - \lambda^{-n\alpha} \cdot e)}{\lambda^\alpha - 1}\right] < \frac{1}{4}$ for $\lambda > \lambda_0$.

Therefore we have $|S_n(z)| \leqslant \frac{1}{4}|u_n(z)|$.

Now we estimate $R_n(z)$ on the circle $|z| = r_n$.

$$|R_n(z)| \leqslant \sum_{k=n+1}^{+\infty} \exp\left(\frac{\alpha}{\lambda^n}\right) \lambda^{\alpha k} \sum_{m=1}^{+\infty} \frac{\lambda^{\alpha m}}{\exp(\alpha \lambda^m)}.$$

Since $\exp(\alpha \lambda^m) \geqslant \exp(m\alpha\lambda)$ for $m \geqslant 1$ and sufficient large λ,

$$\sum_{m=1}^{+\infty} \frac{\lambda^{\alpha m}}{\exp(\alpha \lambda^m)} \leqslant \frac{\lambda^\alpha}{e^{\alpha\lambda} - \lambda^\alpha},$$

so we have

$$|R_n(z)| \leqslant \exp(2\alpha + 1)|u_n(z)| \frac{\lambda^{\alpha m}}{\exp(\alpha \lambda^m)} \leqslant \frac{\lambda^\alpha}{e^{\alpha\lambda} - \lambda^\alpha} \leqslant \frac{1}{4}|u_n(z)|$$

for $\lambda > \lambda_1$.

As a result, we obtain:

$$|f'_\lambda(z)| \geqslant \frac{1}{2}|u(z)|,\ |z| = r_n,$$

for $\lambda > \max(\lambda_0, \lambda_1)$.

But

$$\ln|u_n(z)| \geqslant c_\alpha \ln\frac{1}{1 - r_n},\ \ n = 1, 2, \ldots$$

Thus, we have

$$\int_{-\pi}^{\pi} \left(\ln^+ |f'_\lambda(r_n e^{i\theta})|\right)^q d\theta \geqslant c_\alpha^q \ln^q \frac{1}{1 - r_n},$$

this means that $f'_\lambda \notin \Pi_q$. Theorem 1.1 is proved. □

2. On the differentiation in the Privalov spaces

An important place in the theory of analytic functions belongs to the Nevanlinna N-class of analytic functions in D with bounded characteristic $T(r,f)$. It was introduced by A. Ostrovsky and brothers R. Nevanlinna and F. Nevanlinna (see [10]). As noted above, $N = \Pi_1$. Unlike the class N, the area Nevanlinna class is defined as follows (see ibid.):

$$\mathbf{N} = \left\{ f \in H(D) : \iint\limits_D \ln^+ |f(z)| dxdy < +\infty \right\}, \ z = x + iy,$$

or equivalent to this

$$\mathbf{N} = \left\{ f \in H(D) : \int\limits_0^1 \int\limits_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta dr < +\infty \right\}.$$

The area Nevanlinna classes are a natural generalization of the classes N. As it was established in the works [2, 17], these classes are close with respect to the properties of root sets and the factorization of functions. The class $\mathbf{N}$ is included in the scale of the Nevanlinna-Djrbashian classes N_α (see ibid.):

$$N_\alpha = \left\{ f \in H(D) : \int_0^1 (1-r)^\alpha T(r,f) dr < +\infty \right\}, \ \alpha > -1,$$

and in the scale of S_α^q-classes of F.A. Shamoyan (see [18]):

$$S_\alpha^q = \left\{ f \in H(D) : \int_0^1 (1-r)^\alpha T^q(r,f) dr < +\infty \right\}, \quad \alpha > -1, \ 0 < q < +\infty.$$

Similar to the definition of the area Nevanlinna class, for all $0 < q < +\infty$ we introduce the area Privalov class:

$$\tilde{\Pi}_q = \left\{ f \in H(D) : \int\limits_0^1 \int\limits_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta dr < +\infty \right\}.$$

It is clear that $\tilde{\Pi}_1 = \mathbf{N}$. Using Hölder's inequality, it is easy to prove that $\tilde{\Pi}_q \subset S_0^q$ for $q > 1$ and $\tilde{\Pi}_q \supset S_0^q$ for $0 < q < 1$.

The main result of the second part of this paper is the following theorem.

Theorem 2.1. *If $f \in \Pi_q$ $(0 < q < +\infty)$ and function f has no zeros, then $f' \in \tilde{\Pi}_q$.*

To prove this statement, we need auxiliary statements.

Theorem 2.2 (see [13])**.** *If $f \in \Pi_q, (0 < q < 1)$, then*

$$\ln^+ M(r,f) = o((1-r)^{-1/q}), \ r \to 1-0, \tag{2}$$

where $M(r,f) = \max\limits_{|z|=r} |f(z)|$, and the estimate is exact.

Lemma 2.3 (The Minkowski inequality, see [7], p. 178)**.** *Let $\{f_k\}_{k=1}^{+\infty}$ be the sequence of non-negative functions. For all $0 < p < 1$ the following inequality is valid:*

$$\left[\int \left\{ \sum_k f_k(x) \right\}^p dx \right]^{1/p} \geqslant \sum_k \left\{ \int f_k^p(x) dx \right\}^{1/p}.$$

Lemma 2.4 (see [6], p. 144). *Let $P(r,\theta)$ denote the Poisson kernel in D, i.e.*

$$P(r,\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

For each real number q there exist finite positive constants c_q, d_q, such that

$$c_q\phi_q(r) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P^q(r,\theta)d\theta \leqslant d_q\phi_q(r),$$

where

$$\phi_q(r) = \begin{cases} (1-r)^q, \ q<\dfrac{1}{2}, \\ \sqrt{1-r}\ln\left(1+\dfrac{1}{1-r}\right), \ q=\dfrac{1}{2}, \\ (1-r)^{1-q}, \ q>\dfrac{1}{2}. \end{cases}$$

Proof of Theorem 2.1. Let $z = re^{i\theta}$, $t = Re^{i\varphi}$, $0<r<R<1$. Since $f \in H(D)$ and function f has no zeros, we have, by the Schwarz formula, that:

$$\ln f(z) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(t)| \cdot \frac{t+z}{t-z}d\varphi + iC, \tag{3}$$

where the main branch of the logarithm is chosen.

Differentiate (3) by z:

$$\frac{f'(z)}{f(z)} = \frac{1}{\pi}\int_0^{2\pi} \ln|f(t)| \cdot \frac{t}{(t-z)^2}d\varphi,$$

$$f'(z) = \frac{f(z)}{\pi}\int_0^{2\pi} \ln|f(Re^{i\varphi})| \cdot \frac{Re^{i\varphi}}{(Re^{i\varphi}-re^{i\theta})^2}d\varphi,$$

whence

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi}\int_0^{2\pi} \ln^+|f(Re^{i\varphi})| \cdot \frac{R}{R^2-2Rr\cos(\varphi-\theta)+r^2}d\varphi,$$

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R}\int_0^{2\pi} \ln^+|f(Re^{i\varphi})| \cdot \frac{1}{1-2\frac{r}{R}\cos(\varphi-\theta)+\frac{r^2}{R^2}}d\varphi. \tag{4}$$

Let us consider 3 cases.

Case 1. We assume that $0<q<1$.

Rewrite the last inequality in the form:

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R}\int_0^{2\pi} (\ln^+|f(Re^{i\varphi})|)^{q^2} \cdot (\ln^+|f(Re^{i\varphi})|)^{1-q^2} \cdot \frac{1}{1-2\frac{r}{R}\cos(\varphi-\theta)+\frac{r^2}{R^2}}d\varphi.$$

Applying Hölder's inequality with exponents $\dfrac{1}{q}$ and $\dfrac{1}{1-q}$, we have:

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R}\left[\int_0^{2\pi} (\ln^+|f(Re^{i\varphi})|)^q\right]^q \cdot \left[\int_0^{2\pi} \frac{(\ln^+|f(Re^{i\varphi})|)^{1+q}}{\left(1-2\frac{r}{R}\cos(\varphi-\theta)+\frac{r^2}{R^2}\right)^{1/(1-q)}}d\varphi\right]^{1-q}.$$

Since the function f belongs to the class Π_q, we have by Theorem 2.2:

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R} \cdot \frac{c_q \varepsilon_q}{(1-R)^{(1-q^2)/q}\left(1-\frac{r^2}{R^2}\right)} \left[\int_0^{2\pi} \left(P\left(\frac{r}{R}, \varphi - \theta\right)\right)^{1/(1-q)} d\varphi\right]^{1-q},$$

where $P\left(\frac{r}{R}, \varphi - \theta\right)$ is the Poisson kernel. We use the Poisson kernel estimate for $\dfrac{1}{1-q} > \dfrac{1}{2}$ from Lemma 2.4:

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R} \cdot \frac{c_q}{(1-R)^{(1-q^2)/q}} \cdot \frac{\varepsilon_q}{\left(1-\frac{r^2}{R^2}\right)} \cdot \frac{D_q}{\left(1-\frac{r}{R}\right)^q}.$$

Suppose $R = \dfrac{1+r}{2}$. After elementary transformations we obtain:

$$|f'(re^{i\theta})| \leqslant A_q \cdot |f(re^{i\theta})| \cdot \frac{1}{(1-r)^{(1+q)/q}}.$$

We proceed with the logarithm of the last inequality and take into account that $\ln^+|ab| \leqslant \ln^+|a| + \ln^+|b|$, $a > 0$, $b > 0$:

$$\ln^+|f'(re^{i\theta})| \leqslant \ln^+|f(re^{i\theta})| + \ln\left(\frac{A_q}{(1-r)^{(1+q)/q}}\right).$$

Next, raise both sides to the power q, and take into account $(a+b)^q \leqslant a^q + b^q$ for all $a > 0$, $b > 0$, $0 < q < 1$, after integration over $\theta \in [-\pi, \pi]$ we have:

$$\int_{-\pi}^{\pi} \left(\ln^+|f'(re^{i\theta})|\right)^q d\theta \leqslant \int_{-\pi}^{\pi} \left(\ln^+|f(re^{i\theta})|\right)^q d\theta + B_q + \left(\ln\frac{1}{(1-r)^{(1+q)/q}}\right)^q.$$

Since $f \in \Pi_q$ we have:

$$\int_{-\pi}^{\pi} \left(\ln^+|f'(re^{i\theta})|\right)^q d\theta \leqslant \tilde{B}_q + 2\pi\left(\ln\frac{1}{(1-r)^{(1+q)/q}}\right)^q.$$

Integrate over $r \in [0,1]$. In view of the convergence of the integrals on the right-hand side of the inequality, we conclude that $f' \in \tilde{\Pi}_q$.

Case 2. Now we suppose that $q > 1$.

Applying Hölder's inequality with exponents q and $1 + \dfrac{1}{q-1}$ in (4), we obtain

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R\left(1-\frac{r^2}{R^2}\right)} \left[\int_0^{2\pi} (\ln^+|f(Re^{i\varphi})|)^q d\varphi\right]^{1/q} \cdot \left[\int_0^{2\pi} P\left(\frac{r}{R}, \varphi - \theta\right)^{1+\frac{1}{q-1}} d\varphi\right]^{1-1/q}.$$

Since the function f belongs to the class Π_q, we have

$$|f'(z)| \leqslant \frac{|f(z)|}{\pi R\left(1-\frac{r^2}{R^2}\right)} c_q \cdot \left[\int_0^{2\pi} P\left(\frac{r}{R}, \varphi - \theta\right)^{1+\frac{1}{q-1}} d\varphi\right]^{1-1/q}.$$

We use the Poisson kernel estimate for $1 + \dfrac{1}{q-1} > \dfrac{1}{2}$ from Lemma 2.4:

$$|f'(z)| \leqslant \tilde{c}_q \frac{|f(z)|}{\pi R\left(1-\frac{r^2}{R^2}\right)} \cdot \frac{1}{\left(1-\frac{r}{R}\right)^{1/q}}.$$

Suppose that $R = \dfrac{1+r}{2}$, then we have:

$$|f'(z)| \leqslant C_q \frac{|f(z)|}{(1-r)^{\frac{q}{q-1}}}.$$

We proceed with the logarithm of the last inequality and take into account that $\ln^+ |ab| \leqslant \ln^+ |a| + \ln^+ |b|$, $a > 0$, $b > 0$:

$$\ln^+ |f'(z)| \leqslant \ln^+ |f(z)| + \ln \frac{C_q}{(1-r)^{\frac{q}{q-1}}}.$$

Further, raise both sides to the power q, and take into account $(a+b)^q \leqslant a^q + b^q$ for all $a > 0$, $b > 0$, $0 < q < 1$. After integration in $\theta \in [-\pi, \pi]$ we obtain:

$$\int_{-\pi}^{\pi} \left(\ln^+ |f'(re^{i\theta})|\right)^q d\theta \leqslant \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta + \ln \frac{\tilde{C}_q}{(1-r)^{\frac{q}{q-1}}}.$$

Since $f \in \Pi_q$, we see that:

$$\int_{-\pi}^{\pi} \left(\ln^+ |f'(re^{i\theta})|\right)^q d\theta \leqslant a_q + \ln \frac{\tilde{C}_q}{(1-r)^{\frac{q}{q-1}}}.$$

Integrate over $r \in [0, 1]$. In view of the convergence of the integrals on the right-hand side of the inequality, we conclude that $f' \in \tilde{\Pi}_q$.

Case 3. We assume $q = 1$. Using the estimate of S. N. Mergelyan for a function of the Nevanlinna class (see [12, c. 84]), we get from (4):

$$|f'(z)| \leqslant C \frac{|f(z)|}{\pi R(1-R)\left(1 - \frac{r^2}{R^2}\right)} \int_0^{2\pi} P\left(\frac{r}{R}, \varphi - \theta\right) d\varphi,$$

whence by the property of the Poisson integral

$$|f'(z)| \leqslant C \frac{|f(z)|}{\pi R(1-R)\left(1 - \frac{r^2}{R^2}\right)}.$$

Further, the proof repeats the argument for Case 2. Theorem 2.1 is completely proved. □

Remark 2.1. *Note that W. Hayman indicates the invariance of the class* Π_q, $(1 < q < +\infty)$ *with respect to the integration operator [8].*

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О дифференцировании в классах И. И. Привалова

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Аннотация. В статье исследуется инвариантность классов И. И. Привалова относительно оператора дифференцирования.

Ключевые слова: класс Привалова, гипотеза Блоха-Неванлинны, оператор дифференцирования.