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On Initial Boundary Value Problem for Parabolic Differential Operator with Non-coercive Boundary Conditions

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Abstract. We consider initial boundary value problem for uniformly 2-parabolic differential operator of second order in cylinder domain in $\mathbb{R}^n$ with non-coercive boundary conditions. In this case there is a loss of smoothness of the solution in Sobolev type spaces compared with the coercive situation. Using by Faedo-Galerkin method we prove that problem has unique solution in special Bochner space.

Keywords: non-coercive problem, parabolic problem, Faedo-Galerkin method.

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Initial boundary value problems for parabolic (by Petrovsky) differential operators with coercive boundary conditions are well studied (see, for instance, [1–4]). However the problem with non-coercive boubdary conditions are also appeared in both theory and applications, see, for instance, pioneer work in this direction [5] and papers [6, 7] and [8] for such problems in the Elasticity Theory. Recent results in Fredholm operator equations, induced by boundary value problems for elliptic differential operators with non-coercive boundary conditions (see, for instance, [9–12]) allows us to apply these one for studying the parabolic problem. Consideration of such problems essentially extends variety of boundary operators, but there is a loss of regularity of the solution (see [13] for elliptic case). Namely, let Ω_T be a cylinder,

$$\Omega_T = \Omega \times (0, T),$$

where Ω is a bounded domain in $\mathbb{R}^n$.

Consider a second order differential operator

$$A(x, t, \partial) = -\sum_{i,j=1}^{n} \partial_i (a_{i,j}(x)\partial_j \cdot) + \sum_{j=1}^{n} a_j(x)\partial_l + a_0(x) + \frac{\partial}{\partial t}$$

of divergence form in the domain Ω_T. The coefficients $a_{i,j}$, a_j are assumed to be complex-valued functions of class $L^\infty(\Omega)$. We suppose that the matrix $\mathfrak{A}(x) = (a_{i,j}(x))_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ is Hermitian and satisfies

$$\sum_{i,j=1}^{n} a_{i,j}(x)\overline{w}_i w_j \geqslant 0 \text{ for all } (x, w) \in \overline{\Omega} \times \mathbb{C}^n, \tag{1}$$

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$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geqslant m\,|\xi|^2 \;\text{ for all }\; (x,\xi) \in \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}), \tag{2}$$

where m is a positive constant independent of x and ξ. Estimate (2) is nothing but the statement that the operator $A(x,t,\partial)$ is uniformly 2-parabolic.

We note that, since the coefficients of the operator and the functions under consideration are complex-valued, inequalities (1) and (2) are weaker than

$$\sum_{i,j=1}^{n} a_{i,j}(x)\,\overline{w}_i w_j \geqslant m\,|w|^2 \tag{3}$$

for all $(x,w) \in \overline{\Omega} \times (\mathbb{C}^n \setminus \{0\})$. Inequality (3) means that correspondent Hermitian form (see form (4)) is coercive.

Consider boundary operator of Robin type:

$$B(x,\partial) = b_1(x) \sum_{i,j=1}^{n} a_{i,j}(x)\,\nu_i\partial_j + b_0(x),$$

where b_0, b_1 are bounded functions on $\partial\Omega$ and $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the unit outward normal vector of $\partial\Omega$ at $x \in \partial\Omega$. Let S be an open connected subset of $\partial\Omega$ with piecewise smooth boundary ∂S. We allow the function $b_1(x)$ to vanish on S. In this case we assume that $b_0(x)$ does not vanish for $x \in S$.

Consider now the following mixed initial-boundary problem in a bounded domain Ω_T with Lipschitz boundary $\partial\Omega_T$.

Problem 1. *Find a distribution $u(x,t)$, satisfying the problem*

$$\left\{\begin{array}{lcccc} A(x,t,\partial)u & = & f & \textit{in} & \Omega_T, \\ B(x,\partial)u & = & 0 & \textit{on} & \partial\Omega \times (0,T), \\ u(x,0) & = & u_0 & \textit{on} & \Omega \end{array}\right.$$

with given data $f \in \Omega_T$.

For solving the problem we have to define appropriate functional spaces. Denote by $C^1(\overline{\Omega},S)$ the subspace of $C^1(\overline{\Omega})$ consisting of those functions whose restriction to the boundary vanishes on $\overline{S}$. Let $H^1(\Omega,S)$ be the closure of $C^1(\overline{\Omega},S)$ in $H^1(\Omega)$. Since on S the boundary operator reduces to $B = b_0(x)$ and $b_0(x) \neq 0$ for $x \in S$, then the functions $u \in H^1(\Omega)$ satisfying $Bu = 0$ on $\partial\Omega$ belong to $H^1(\Omega,S)$.

Split now both $a_0(x)$ and $b_0(x)$ into two parts

$$a_0 = a_{0,0} + \delta a_0,$$

$$b_0 = b_{0,0} + \delta b_0,$$

where $a_{0,0}$ is a non-negative bounded function in Ω and $b_{0,0}$ is a such function that $b_{0,0}/b_1$ is non-negative bounded function on S. Then, under reasonable assumptions, the Hermitian form

$$(u,v)_+ = \int_\Omega \sum_{i,j=1}^{n} a_{i,j}\partial_j u\overline{\partial_i v}\,dx + (a_{0,0}u,v)_{L^2(\Omega)} + (b_{0,0}/b_1\,u,v)_{L^2(\partial\Omega\setminus S)} \tag{4}$$

defines the scalar product on $H^1(\Omega,S)$. Denote by $H^+(\Omega)$ the completion of the space $H^1(\Omega,S)$ with respect to the corresponding norm $\|\cdot\|_+$. From now on we assume that the space $H^+(\Omega)$ is continuously embedded into the Lebesgue space $L^2(\Omega)$, i.e. there is a constant $c > 0$, independent of u, such that

$$\|u\|_{L^2(\Omega)} \leqslant c\|u\|_+ \text{ for all } u \in H^+(\Omega).$$

It is true, if there exist a positive constant c_1 such that

$$a_{0,0} \geqslant c_1 \text{ in } \Omega.$$

Actually we can get more subtle embedding for the space $H^+(\Omega)$.

Theorem 2. *Let the coefficients $a_{i,j}$ be C^∞ in a neighbourhood of the closure of Ω, inequalities (1), (2) hold and*

$$\frac{b_{0,0}}{b_1} \geqslant c_2 \text{ at } \partial\Omega \setminus S, \tag{5}$$

with some constant $c_2 > 0$. Then the space $H^+(\Omega)$ is continuously embedded into $H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ if there is a positive constant c_1, such that

$$a_{0,0} \geqslant c_1 \text{ in } \Omega \tag{6}$$

or the operator A is strongly elliptic in a neighborhood X of $\overline{\Omega}$ and

$$\int_X \sum_{i,j=1}^n a_{i,j}\partial_j u\overline{\partial_i u}\, dx \geqslant m\,\|u\|^2_{L^2(X)} \tag{7}$$

for all $u \in C^\infty_{\mathrm{comp}}(X)$, with $m > 0$ a constant independent of u.

Proof. See [12, Theorem 2.5]. □

Of course, under coercive estimate (3), the space $H^+(\Omega)$ is continuously embedded into $H^1(\Omega)$. However, in general, the embedding, described in Theorem 2 is rather sharp (see [12, Remark 5.1]).

The absence of coerciveness does not allows to consider arbitrary derivatives $\partial_j u$ for an element $u \in H^+(\Omega)$. To cope with this difficulty we note that the matrix $\mathfrak{A}(x) = (a_{i,j}(x))_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ admits a factorisation, i.e. there is an $(m \times n)$-matrix $\mathfrak{D}(x) = (\mathfrak{D}_{i,j}(x))_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ of bounded functions in Ω, such that

$$(\mathfrak{D}(x))^*\mathfrak{D}(x) = \mathfrak{A}(x) \tag{8}$$

for almost all $x \in D$ (see, for instance, [14]). For example, one could take the standard non-negative self-adjoint square root $\mathfrak{D}(x) = \sqrt{\mathfrak{A}(x)}$ of the matrix $\mathfrak{A}(x)$. Then

$$\sum_{i,j=1}^n a_{i,j}\partial_j u\overline{\partial_i v} = (\mathfrak{D}\nabla v)^*\mathfrak{D}\nabla u = \sum_{l=1}^m \overline{\mathfrak{D}_l v}\,\mathfrak{D}_l u,$$

for all smooth functions u and v in Ω, where ∇u is thought of as n-column with entries $\partial_1 u, \dots, \partial_n u$, and $\mathfrak{D}_l u := \sum_{s=1}^n \mathfrak{D}_{l,s}(x)\partial_s u$, $l = 1, \dots, m$. From now on we may confine ourselves with first order summand of the form

$$\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l, \quad \tilde{a}_l(x) \in L^\infty(\Omega),$$

instead of

$$\sum_{j=1}^{n} a_j(x)\partial_j.$$

Since the coefficients δa_0, $\tilde{a}_l$ belong to $L^\infty(\Omega)$ for all $l = 0, \dots, m$, it follows from Cauchy inequality that

$$\left| \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0 \Big) u, v \Big)_{L^2(\Omega)} \right| \leqslant c\, \|u\|_+ \, \|v\|_+. \tag{9}$$

Let now $H^-(\Omega)$ stand for the dual space for the space $H^+(\Omega)$ with respect to the pairing $< \cdot, \cdot >$ induced by the scalar product $(\cdot, \cdot)_{L^2(\Omega)}$, see [2, 15] and elsewhere. It is a Banach space with the norm

$$\|u\|_- = \sup_{\substack{v \in H^+(\Omega) \\ v \neq 0}} \frac{|(v, u)_{L^2(\Omega)}|}{\|v\|_+}.$$

The space $L^2(\Omega)$ is continuously embedded into $H^-(\Omega)$, if the space $H^+(\Omega)$ is continuously embedded into $L^2(\Omega)$ (see [9]). We denote by $i' : L^2(\Omega) \to H^-(\Omega)$ and $i : H^+(\Omega) \to L^2(\Omega)$ the operators of correspondent continuously embeddings. Thus we have a triple of the functional spaces

$$H^+(\Omega) \overset{i}{\hookrightarrow} L^2(\Omega) \overset{i'}{\hookrightarrow} H^-(\Omega),$$

where each embeddings is compact under the hypothesis of Theorem 2.

Denote by $L^2(0, T; H^+(\Omega))$ the Bochner space of L^2-functions

$$u(t) : [0, T] \to H^+(\Omega).$$

It is a Banach space with the norm

$$\|u\|^2_{L^2(0,T;H^+(\Omega))} = \int_0^T \|u(t)\|^2_+ dt.$$

Then an integration by parts in Ω leads to a weak formulation of Problem (1):

Problem 3. *Given $f \in L^2(0, T; H^-(\Omega))$ and $u_0 \in L^2(\Omega)$, find $u \in L^2(0, T; H^+(\Omega))$, such that*

$$(u, v)_+ + \Big(\big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0 \big) u, v \Big)_{L^2(\Omega)} + \frac{\partial}{\partial t} (u, v)_{L^2(\Omega)} =< f, v > \tag{10}$$

for all $v \in H^+(\Omega)$, and

$$u(0) = u_0. \tag{11}$$

In general case the condition (11) have no sense for functions $u \in L^2(0, T; H^+(\Omega))$. But we will see below that function $u(t) \in L^2(0, T; H^+(\Omega))$, satisfying (10), is continuous and (11) have a sense.

We want to apply the Faedo-Galerkin method for solving the Problem 3 (see, for instance, [2, 4]). For this purpose we need some complete system of vectors in the space $H^+(\Omega)$. As this system we take the set of eigenvectors of an operator, induced by the weak statement of elliptic selfadjoint problem, corresponding to the parabolic Problem 3. Namely, for given $f \in H^-(\Omega)$, find $u \in H^+(\Omega)$, such that

$$(u, v)_+ + \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0 \Big) u, v \Big)_{L^2(\Omega)} =< f, v > . \tag{12}$$

Equality (12) induces a bounded linear operator $L : H^+(\Omega) \to H^-(\Omega)$,

$$(u,v)_+ + \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u, v\Big)_{L^2(\Omega)} =< Lu, v > . \tag{13}$$

Denote by L_0 the operator L in the case, when $\delta a_0 = a_l = 0$ for all $l = 1, \dots, m$,

$$(u,v)_+ =< L_0 u, v > . \tag{14}$$

The operator $L_0 : H^+(\Omega) \to H^-(\Omega)$ is continuously invertible and $\|L_0\| = \|L_0^{-1}\| = 1$ (see [12, Lemma 2.6]). According to [12, Lemma 3.1], there is a system $\{h_j\}$ of eigenvectors of the compact positive selfadjoint operator $L_0^{-1} i' i : H^+(\Omega) \to H^+(\Omega)$, which is an orthonormal bases in $H^+(\Omega)$ and an orthogonal bases in $L^2(\Omega)$ and $H^-(\Omega)$.

Let now function $u \in L^2(0,T; H^+(\Omega))$ satisfies (10). We have from (13)

$$\Big(\frac{\partial u}{\partial t}, v\Big)_{L^2(\Omega)} =< \frac{\partial u}{\partial t}, v >=< f - Lu, v > .$$

Since $f \in L^2(0,T; H^-(\Omega))$ and operator $L : H^+(\Omega) \to H^-(\Omega)$ is bounded, then $\frac{\partial u}{\partial t} \in L^2(0,T; H^-(\Omega))$. It means, that

$$u \in C(0,T; L^2(\Omega)) \tag{15}$$

(see, for instance, [2] or [16]).

Using by the standard Faedo-Galerkin method (see, for instance, [1,2,4]) we get next Theorem.

Theorem 4. *Under the hypothesis of Theorem 2, the Problem 3 has at least one solution $u(t)$, and, moreover, $u(t) \in C(0,T; L^2(\Omega))$.*

Proof. For each k we are looking for approximate solution of Problem 3 on the next form

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t) h_j, \tag{16}$$

and function u_k satisfies

$$(u_k, h_i)_+ + \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u_k, h_i\Big)_{L^2(\Omega)} + \Big(\frac{\partial u_k}{\partial t}, h_i\Big)_{L^2(\Omega)} =< f, h_i >, \tag{17}$$

$$u_k(0) = \sum_{j=1}^{k} \frac{(u_0, h_j)_{L^2(\Omega)}}{\|h_j\|^2_{L^2(\Omega)}} h_j, \tag{18}$$

for each $j = 1, \dots, k$, where $\{h_j\}$ is the orthonormal bases in $H^+(\Omega)$. It means that (17) takes the form

$$g_{ik}(t) + \sum_{j=1}^{k} \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)h_j, h_i\Big)_{L^2(\Omega)} g_{jk}(t) + g'_{ik}(t)\|h_i\|^2_{L^2(\Omega)} =< f, h_i >, \tag{19}$$

where $i = 1, \dots, k$. It is a system of linear differential equations of first order with initial conditions

$$g_{ik}(0) = \frac{(u_0, h_i)_{L^2(\Omega)}}{\|h_i\|^2_{L^2(\Omega)}}, \quad i = 1, \dots, k. \tag{20}$$

Since $< f(t), h_i >$ is measurable function for all $i = 1, \dots, k$, then there is unique function $g_{ik}(t)$ for each $i = 1, \dots, k$, satisfying (19) and (20) for all $t \in [0, T]$ (see, for instance, [17]). Note, as the function $u(t)$ is complex-valued, then the functions $\{g_{ik}(t)\}$ may be complex-valued too and the system (19) consists $2k$ real-valued equations in general case.

Now we have to get some priori estimates for function $u_k(t)$ independent of k. Multiplying the equality (17) by the $\overline{g_{ik}(t)}$ and summing by $i = 1, \dots, k$ we get

$$\|u_k\|_+^2 + \left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)} =< f, u_k > - \Big(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u_k, u_k\Big)_{L^2(\Omega)}. \tag{21}$$

Hence, by the Cauchy inequality,

$$\begin{aligned} &2\left|\|u_k\|_+^2 + \left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right| = \\ &=2\Big| < f, u_k > - \Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l u_k, u_k\Big)_{L^2(\Omega)} - (\delta a_0 u_k, u_k)_{L^2(\Omega)}\Big| \leqslant \\ &\leqslant \|f\|_-^2 + \|u_k\|_+^2 + 2c_1\|u_k\|_+\|u_k\|_{L^2(\Omega)} + 2c_2\|u_k\|^2_{L^2(\Omega)} \leqslant \\ &\leqslant \|f\|_-^2 + \frac{3}{2}\|u_k\|_+^2 + (2c_2 + 2c_1^2)\|u_k\|^2_{L^2(\Omega)} \end{aligned} \tag{22}$$

for some positive constants c_1 and c_2. As the norm $\|u_k\|_+^2$ is a real-valued function, we have

$$\begin{aligned} &2\left|\|u_k\|_+^2 + \left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right| = \\ &= 2\left|\|u_k\|_+^2 + \mathfrak{Re}\left(\left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right) + i\mathfrak{Im}\left(\left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right)\right| \geqslant \\ &\geqslant 2\|u_k\|_+^2 + 2\mathfrak{Re}\left(\left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right), \end{aligned} \tag{23}$$

where $\mathfrak{Re}(g)$ and $\mathfrak{Im}(g)$ denote real and imaginary parts of function g respectively. On the other hand,

$$\begin{aligned} \frac{d}{dt}\|u_k\|^2_{L^2(\Omega)} &= \left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)} + \left(u_k, \frac{\partial u_k}{\partial t}\right)_{L^2(\Omega)} = \\ &= 2\mathfrak{Re}\left(\left(\frac{\partial u_k}{\partial t}, u_k\right)_{L^2(\Omega)}\right). \end{aligned} \tag{24}$$

It follows from (22), (23) and (24) that

$$\frac{1}{2}\|u_k(t)\|_+^2 + \frac{d}{dt}\|u_k(t)\|^2_{L^2(\Omega)} \leqslant \|f(t)\|_-^2 + (2c_2 + 2c_1^2)\|u_k\|^2_{L^2(\Omega)}. \tag{25}$$

Now, integrating (25) by t from 0 till some $s \in (0, T)$ we get

$$\begin{aligned} &\frac{1}{2}\int_0^s \|u_k(t)\|_+^2 dt + \|u_k(s)\|^2_{L^2(\Omega)} - \|u_k(0)\|^2_{L^2(\Omega)} \leqslant \\ &\leqslant \int_0^s \|f(t)\|_-^2 dt + (2c_2 + 2c_1^2)\int_0^s \|u_k\|^2_{L^2(\Omega)} dt. \end{aligned}$$

Since the sequence $\{u_k(0)\}$ seeks to u_0 with $k \to \infty$ strongly in $L^2(\Omega)$, it follows from Gronwall type lemma (see [18] or [19]), that

$$\|u_k(s)\|^2_{L^2(\Omega)} \leqslant \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_- dt \right) e^{(2c_2+2c_1^2)s}.$$

Hence

$$\sup_{s\in[0,T]} \|u_k(s)\|^2_{L^2(\Omega)} \leqslant \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_- dt \right) e^{(2c_2+2c_1^2)T}. \quad (26)$$

The right side of (26) independent of k, therefore the sequence $\{u_k(t)\}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Then there is a subsequence $\{u_{k'}(t)\}$ of the sequence $\{u_k(t)\}$ and an element $u(t) \in L^\infty(0,T;L^2(\Omega))$ such that $u_{k'}(t) \to u(t)$ in the weak-* topology of $L^\infty(0,T;L^2(\Omega))$, namely

$$\lim_{k'\to\infty} \int_0^T (u_{k'}(t) - u(t), v(t))_{L^2(\Omega)} dt = 0 \quad (27)$$

for all $v \in L^1(0,T;L^2(\Omega))$.

Integrating again (25) by t from 0 till T and applying Gronwall type lemma we have

$$\begin{aligned} &\frac{1}{2}\int_0^T \|u_k(t)\|^2_+ dt + \|u_k(T)\|^2_{L^2(\Omega)} \leqslant \\ &\leqslant \left(\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_- dt \right) e^{(2c_2+2c_1^2)T}. \end{aligned} \quad (28)$$

It means that the sequence $\{u_k(t)\}$ is bounded in $L^2(0,T;H^+(\Omega))$. In particular, the sequence $\{u_{k'}(t)\}$ is bounded in $L^2(0,T;H^+(\Omega))$ too. Hence there is a subsequence $\{u_{k''}(t)\}$ of the sequence $\{u_{k'}(t)\}$ and an element $\widetilde{u}(t) \in L^2(0,T;H^+(\Omega))$ such that $u_{k''}(t) \to u(t)$ in the weak topology of $L^2(0,T;H^+(\Omega))$,

$$\lim_{k''\to\infty} \int_0^T (u_{k''}(t), v)_+ \, dt = \int_0^T (u(t), v)_+ \, dt \quad (29)$$

for all $v \in L^2(0,T;H^+(\Omega))$ and

$$\lim_{k''\to\infty} \int_0^T < u_{k''}(t) - \widetilde{u}(t), v(t) > dt = 0 \quad (30)$$

for all $v \in L^2(0,T;H^-(\Omega))$. In particular

$$\lim_{k''\to\infty} \int_0^T (u_{k''}(t), v(t))_{L^2(\Omega)} dt = \int_0^T (\widetilde{u}(t), v(t))_{L^2(\Omega)} dt \quad (31)$$

for all $v \in L^2(0,T;L^2(\Omega))$.

From (27) and (31) we have

$$\int_0^T (u(t) - \widetilde{u}(t), v(t))_{L^2(\Omega)} dt = 0 \quad (32)$$

for all $v \in L^2(0,T;L^2(\Omega))$. Hence

$$u(t) = \widetilde{u}(t) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^+(\Omega)). \quad (33)$$

From now on we denote by $\{u_k(t)\}$ the subsequence $\{u_{k''}(t)\}$.

Let now $\psi(t)$ be a scalar differentiable function on $[0,T]$ such that $\psi(T)=0$. Multiplying (17) by $\psi(t)$ and integrating by t we get

$$\int_0^T (u_k(t),h_j)_+\psi(t)\,dt + \int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u_k(t),h_i\Big)_{L^2(\Omega)}\psi(t)\,dt+ \\ + \int_0^T \Big(\frac{\partial u_k(t)}{\partial t},h_i\Big)_{L^2(\Omega)}\psi(t)\,dt = \int_0^T < f(t),h_j > \psi(t)dt. \quad (34)$$

However

$$\int_0^T \Big(\frac{\partial u_k(t)}{\partial t},h_i\Big)_{L^2(\Omega)}\psi(t)\,dt \;=\; -\int_0^T (u_k(t),\psi'(t)h_j)_{L^2(\Omega)}\,dt \,-\, (u_k(0),h_j\psi(0))_{L^2(\Omega)}, \quad (35)$$

and it follows that

$$\int_0^T (u_k(t),h_j\psi(t))_+\,dt + \int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u_k(t),h_i\Big)_{L^2(\Omega)}\psi(t)\,dt- \\ - \int_0^T (u_k(t),\psi'(t)h_j)_{L^2(\Omega)}\,dt = (u_k(0),h_j\psi(0))_{L^2(\Omega)} + \int_0^T < f(t),h_j > \psi(t)dt. \quad (36)$$

Now we want to go to the limit in (36) with $k\to\infty$. It follows from 9, that

$$\int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u_k(t),h_i\Big)_{L^2(\Omega)}\psi(t)\,dt$$

is continuous linear functional on $L^2(0,T;H^+(\Omega))$. Since $u_k(t)\to u(t)$ with $k\to\infty$ in the weak topology of $L^2(0,T;H^+(\Omega))$, we have

$$\lim_{k\to\infty}\int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)(u_k(t)-u(t)),h_i\Big)_{L^2(\Omega)}\psi(t)\,dt = 0.$$

From (31), (29), (33) and the fact that $u_k(0)\to u_0$ strongly in $L^2(\Omega)$ with $k\to\infty$ we get

$$\int_0^T (u(t),h_j\psi(t))_+\,dt + \int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u(t),h_i\psi(t)\Big)_{L^2(\Omega)}dt- \\ - \int_0^T (u(t),\psi'(t)h_j)_{L^2(\Omega)}\,dt = (u_0,h_j\psi(0))_{L^2(\Omega)} + \int_0^T < f(t),h_j > \psi(t)dt. \quad (37)$$

As the system $\{h_j\}_{j=1,2,\ldots}$ is dense in $H^+(\Omega)$ and $L^2(\Omega)$, equality (37) holds by linearity and continuity for all $v\in H^+(\Omega)$,

$$\int_0^T (u(t),v)_+\psi(t)\,dt + \int_0^T \Big(\Big(\sum_{l=1}^m \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u(t),v\Big)_{L^2(\Omega)}\psi(t)\,dt- \\ - \int_0^T (u(t),v)_{L^2(\Omega)}\,\psi'(t)\,dt = (u_0,v)_{L^2(\Omega)}\psi(0) + \int_0^T < f(t),v > \psi(t)dt. \quad (38)$$

In particular, if we take by $\psi(t)$ differentiable functions with compact support in $(0, T)$, we get

$$(u(t), v)_+ + \left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u(t), v\right)_{L^2(\Omega)} + \frac{d}{dt}\,(u(t), v)_{L^2(\Omega)} =< f(t), v > \tag{39}$$

in the sense of distributions. Now we have to show that $u(0) = u_0$. Indeed, multiplying (39) by $\psi(t)$ and integrating by parts we get

$$\int_0^T (u(t), v)_+ \psi(t)\, dt + \int_0^T \left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)u(t), v\right)_{L^2(\Omega)} \psi(t)\, dt -$$
$$- \int_0^T (u(t), v)_{L^2(\Omega)}\, \psi'(t)\, dt = (u(0), v)_{L^2(\Omega)}\psi(0) + \int_0^T < f(t), v > \psi(t) dt.$$

Comparing it with (38) we get

$$(u(0) - u_0, v)_{L^2(\Omega)}\psi(0) = 0$$

for all $v \in H^+(\Omega)$. Taking $\psi(0) \neq 0$ we receive $u(0) = u_0$.

The continuity follows from (15). □

Corollary 5. *Under the hypothesis of Theorem 2, the Problem 3 has one and only one solution* $u(t) \in C(0, T; L^2(\Omega))$, *if*

$$\mathfrak{Re}\left(\left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)v, v\right)_{L^2(\Omega)}\right) \geqslant 0 \tag{40}$$

for all $v \in L^2(0, T; H^+(\Omega))$.

Proof. The existence of the solution follows from the Theorem 4. Let us now show, that the solution is unique, if the condition (40) and the hypothesis of Theorem 4 are fulfilled. Indeed, let $v \in L^2(0, T; H^+(\Omega))$ is another solution of Problem 3. Denote by $w = u - v$. Then w satisfies conditions of Problem 3 and

$$(w, v)_+ + \left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)w, v\right)_{L^2(\Omega)} + \frac{d}{dt}\,(w, v)_{L^2(\Omega)} = 0$$

for all $v \in H^+(\Omega)$, and $w(0) = 0$. It follows from (13), that

$$\frac{\partial w}{\partial t} + Lw = 0.$$

Multiplying scalar it by w we have

$$\|w\|_+^2 + \left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)w, w\right)_{L^2(\Omega)} + \left(\frac{\partial w}{\partial t}, w\right)_{L^2(\Omega)} = 0.$$

As the $\|w(t)\|_+^2$ is a real-valued function, therefore

$$\|w\|_+^2 + \mathfrak{Re}\left(\left(\frac{\partial w}{\partial t}, w\right)_{L^2(\Omega)}\right) + \mathfrak{Re}\left(\left(\Big(\sum_{l=1}^{m} \tilde{a}_l(x)\mathfrak{D}_l + \delta a_0\Big)w, w\right)_{L^2(\Omega)}\right) = 0.$$

On the other hand,

$$\mathfrak{Re}\left(\left(\frac{\partial w}{\partial t}, w\right)_{L^2(\Omega)}\right) = \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\Omega)}.$$

It follows from (40), that

$$2\mathfrak{Re}\left(\left(\frac{\partial w}{\partial t}, w\right)_{L^2(\Omega)}\right) = \frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leqslant 0$$

and

$$\|w(t)\|^2_{L^2(\Omega)} \leqslant \|w(0)\|^2_{L^2(\Omega)} = 0,$$

hence $w(t) = 0$ for almost all $t \in [0,T]$, that completes the proof. □

As we already mentioned, the embedding $H^+(\Omega)$ into $H^{1/2-\varepsilon}(\Omega)$ is rather sharp. Let us show, that the space $L^2(0,T;H^+(\Omega))$ can not be continuously embedded into $L^2(0,T;H^s(\Omega))$ for all $s > 1/2$.

Eexample 6. Let Ω be a unit sphere in $\mathbb{C}$, matrix $\mathfrak{A}(x)$ has a form

$$\mathfrak{A}(x) = (a_{ij}(x))_{\substack{i=1,2\\ j=1,2}} = \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix},$$

$S = \emptyset$, $a_l = 0$ for $l = 0,1,\dots,m$, and $b_1 = b_0 = 1$. Then the series

$$u_\varepsilon(z,t) = \sum_{k=0}^{\infty} \frac{z^k t^{k/2}}{T^{(k+1)/2}(k+1)^{\varepsilon/2}},$$

$\varepsilon > 0$, converges in $L^2(0,T;H^+(\Omega))$ and

$$\|u_\varepsilon\|^2_{L^2(0,T;H^+(\Omega))} = \|u_\varepsilon\|^2_{L^2(0,T;L^2(\mathbb{S}))} = 2\pi \sum_{k=0}^{\infty} \frac{1}{(k+1)^{1+\varepsilon}}.$$

According to [20, Lemma 1.4]

$$\left\|u_\varepsilon\right\|^2_{L^2(0,T;H^s(\mathbb{B}))} \geqslant \pi \sum_{k=0}^{\infty} \frac{k^{2s-1}}{(k+1)^{1+\varepsilon}},\ 0 < s \leqslant 1.$$

It means, that for each $s \in (1/2, 1)$ there exist $\varepsilon > 0$ such that $u_\varepsilon \notin L^2(0,T;H^s(\mathbb{B}))$. Hence, the space $L^2(0,T;H^+(\mathbb{B}))$ can not be continuously embedded into $L^2(0,T;H^s(\mathbb{B}))$ for all $s > 1/2$.

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О начально-краевой задаче для параболического дифференциального оператора с некоэрцитивными граничными условиями

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Аннотация. Мы рассматриваем начально-краевую задачу для равномерно 2-параболического дифференциального оператора второго порядка в цилиндрической области в $\mathbb{R}^n$ с некоэрцитивными граничными условиями. В отличие от коэрцитивного случая в данной ситуации происходит потеря гладкости решения в пространствах соболевского типа. Пользуясь методом Галеркина, мы доказываем, что проблема имеет единственное решение в специальных пространствах Бохнера.

Ключевые слова: некоэрцитивная задача, параболическая задача, метод Галеркина.