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Colorings of the Graph $K_2^m + K_n$

Le Xuan Hung*
Hanoi University of Natural Resources and Environment
Hanoi, Vietnam

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Abstract. In this paper, we characterize chromatically unique, determine list-chromatic number and characterize uniquely list colorability of the graph $G = K_2^m + K_n$. We shall prove that G is χ-unique, $\mathrm{ch}(G) = m + n$, G is uniquely 3-list colorable graph if and only if $2m + n \geqslant 7$ and $m \geqslant 2$.

Keywords: chromatic number, list-chromatic number, chromatically unique graph, uniquely list colorable graph, complete r-partite graph.

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1. Introduction and preliminaries

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by $\deg(v)$ (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. The independent sets and complete graphs of order n are denoted by O_n and K_n, respectively. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G = (V, E)$ is called *r-partite graph* if V admits a partition into r classes $V = V_1 \cup V_2 \cup \ldots \cup V_r$ such that the subgraphs of G induced by V_i, $i = 1, \ldots, r$, is independent set. An r-partite graph in which every two vertices from different partition classes are adjacent is called complete r-partite graph and is denoted by $K_{|V_1|,|V_2|,\ldots,|V_r|}$. The complete r-partite graph $K_{|V_1|,|V_2|,\ldots,|V_r|}$ with $|V_1| = |V_2| = \ldots = |V_r| = s$ is denoted by K_s^r

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Their *union* $G = G_1 \cup G_2$ has, as expected, $V(G) = V_1 \cup V_2$ and $E(G) = E_1 \cup E_2$. Their *join* defined is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. We call G_1 and G_2 *isomorphic*, and write $G_1 \cong G_2$, if there exists a bijection $f : V_1 \to V_2$ with $uv \in E_1$ if and only if $f(u)f(v) \in E_2$ for all $u, v \in V_1$.

Let $G = (V, E)$ be a graph and λ is a positive integer.

A λ-*coloring* of G is a bijection $f : V(G) \to \{1, 2, \ldots, \lambda\}$ such that $f(u) \neq f(v)$ for any adjacent vertices $u, v \in V(G)$. The smallest positive integer λ such that G has a λ-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. We say that a graph G is *n-chromatic* if $n = \chi(G)$.

*lxhung@hunre.edu.vn

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, two λ-colorings f and g are considered different if and only if $f(v_k) \neq g(v_k)$ for some $k = 1, 2, \ldots, n$. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of distinct λ-colorings of G. It is well-known that for any graph G, $P(G, \lambda)$ is a polynomial in λ, called the *chromatic polynomial* of G. The notion of chromatic polynomials was first introduced by Birkhoff [3] in 1912 as a quantitative approach to tackle the four-color problem. Two graphs G and H are called *chromatically equivalent* or in short *χ-equivalent*, and we write in notation $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is called *chromatically unique* or in short *χ-unique* if $G' \cong G$ (i.e., G' is isomorphic to G) for any graph G' such that $G' \sim G$. For examples, all cycles are χ-unique [8]. The notion of χ-unique graphs was first introduced and studied by Chao and Whitehead [4] in 1978. The readers can see the surveys [8,9] and [12] for more informations about χ-unique graphs. Recently, Ngo Dac Tan and Le Xuan Hung characterized chromatically unique split graphs [12] (A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs of G induced by I and K are independent sets and complete graphs, respectively).

Let $(L_v)_{v \in V}$ be a family of sets. We call a coloring f of G with $f(v) \in L_v$ for all $v \in V$ is a *list coloring from the lists L_v*. We will refer to such a coloring as an L-coloring. The graph G is called *λ-list-colorable*, or *λ-choosable*, if for every family $(L_v)_{v \in V}$ with $|L_v| = \lambda$ for all v, there is a coloring of G from the lists L_v. The smallest positive integer λ such that G has a λ-choosable is called the *list-chromatic number*, or *choice number* of G and is denoted by $\mathrm{ch}(G)$. In [7], we characterized list-chromatic number for split graphs, we have proved that if G is a split graphs then $\mathrm{ch}(G) = \chi(G)$.

Let G be a graph with n vertices and suppose that for each vertex v in G, there exists a list of k colors L_v, such that there exists a unique L-coloring for G, then G is called a *uniquely k-list colorable graph* or a UkLC graph for short. The idea of uniquely colorable graph was introduced independently by Dinitz and Martin [6] and by Mahmoodian and Mahdian [10] (Mahmoodian and Mahdian have obtained some results on the uniquely k-list colorable complete multipartite graphs, for example, they proved that graph $G = O_m + K_n$ is U3LC when $(m, n) \in \{(4, 6), (5, 5), (6, 4)\}$).

Finding a general result for the problems raised above is a difficult task, requiring a lot of time and effort for mathematicians. There have been many interesting and insightful research results on these issues for different graph classes. However, these are still issues that have not been resolved thoroughly, so much more attention is needed. In this paper, we shall characterize chromatically unique, determine list-chromatic number and characterize uniquely list colorability of the graph $G = K_2^m + K_n$. Namely, we shall prove that G is χ-unique (Section 2), $\mathrm{ch}(G) = m+n$ (Section 3), G is U3LC if and only if $2m+n \geqslant 7$ and $m \geqslant 2$ (Section 4). These results contribute to solving the coloring problem for a complete multipartite graph.

2. Chromatic uniqueness

We need the following Lemmas 1–4 to prove our results.

Lemma 1 ([2]). *If K_n is a complete graph on n vertices then $\chi(K_n) = n$.*

Lemma 2. *If $G = K_{n_1, n_2, \ldots, n_r}$ is a complete r-partite graph then $\chi(G) = r$.*

Lemma 3 ([11]). *Let G and H be two χ-equivalent graphs. Then*

(i) $|V(G)| = |V(H)|$;
(ii) $|E(G)| = |E(H)|$;
(iii) $\chi(G) = \chi(H)$;
(iv) G is connected if and only if H is connected;
(v) G is 2-connected if and only if H is 2-connected.

Lemma 4. *Let $G = (V_1 \cup V_2 \cup \ldots \cup V_{m+n}, E)$ be a $(m+n)$-partite graph with $m \geqslant 1$, $n \geqslant 1$, $|V_1| \geqslant |V_2| \geqslant \ldots \geqslant |V_{m+n}|$ and $|V_1| + |V_2| + \ldots + |V_{m+n}| = 2m+n$. Then*

$$|E| \leqslant \frac{(2m+n)^2 - 4m - n}{2}.$$

In particular,

$$|E| = \frac{(2m+n)^2 - 4m - n}{2}$$

if and only if G is a complete $(m+n)$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_{m+n}|}$ with

$$|V_1| = |V_2| = \ldots = |V_m| = 2,\ |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1.$$

Proof. We prove the lemma by induction on $t = m+n$. For $t = 2$ the assertion holds, so let $t > 2$ and assume the assertion for smaller values of t. If $|V_{m+n}| \geqslant 2$ then $|V_1| + |V_2| + \ldots + |V_{m+n}| \geqslant 2m + 2n > 2m + n$, a contradiction. So, $|V_{m+n}| = 1$. If $|V_m| \geqslant 3$ then $|V_1| + |V_2| + \ldots + |V_{m+n}| \geqslant 3m + n > 2m + n$, a contradiction. Therefore, $|V_m| \leqslant 2$. Now we consider separately two cases.

Case 1: There exists $i \in \{1, 2, \ldots, m\}$ such that $|V_i| = 2$.

Set $G' = G - V_i$. It is clear that G' is a $(m+n-1)$-partite graph

$$(V_1 \cup V_2 \cup \ldots \cup V_{i-1} \cup V_{i+1} \cup \ldots \cup V_{m+n}, E').$$

By the induction hypothesis,

$$|E'| \leqslant \frac{(2(m-1)+n)^2 - 4(m-1) - n}{2}.$$

We have

$$\begin{aligned} |E| &\leqslant |E'| + |V_i|(|V_1| + \ldots + |V_{i-1}| + |V_{i+1}| + \ldots + |V_{m+n}|) \leqslant \\ &\leqslant \frac{(2(m-1)+n)^2 - 4(m-1) - n}{2} + 2(2m+n-2) = \\ &= \frac{(2m+n)^2 - 4m - n}{2}. \end{aligned}$$

It is not difficult to see that

$$|E| = \frac{(2m+n)^2 - 4m - n}{2}$$

if and only if G is a complete $(m+n)$-partite graph $K_{|V_1|,|V_2|,\ldots,|V_{m+n}|}$ with

$$|V_1| = |V_2| = \ldots = |V_m| = 2, \quad |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1.$$

Case 2: $|V_i| \neq 2$ for every $i = 1, 2, \ldots, m$.

In this case, $|V_1| \geqslant 3$. Let $h \in \{1, 2, \ldots, m\}$ such that $|V_h| = 1$ and $|V_{h-1}| \geqslant 3$. Let $G_1 = K_{p_1,p_2,\ldots,p_{m+n}}$ be a complete $(m+n)$-partite graph such that $p_h = |V_h| + 1 = 2$, $p_{h-1} = |V_{h-1}| - 1$ and $p_i = |V_i|$ for every $i \in \{1, 2, \ldots, m+n\} \setminus \{h-1, h\}$. By Case 1,

$$|E(G_1)| \leqslant \frac{(2m+n)^2 - 4m - n}{2}.$$

We have

$$|E(G_1)| = \sum_{1 \leqslant i < j \leqslant m+n} p_i p_j =$$

$$\begin{aligned}
&= \sum_{i,j\in\{1,\dots,m+n\}\setminus\{h-1,h\}} p_ip_j + \sum_{i\in\{1,\dots,m+n\}\setminus\{h-1,h\}} p_ip_{h-1} + \\
&+ \sum_{i\in\{1,\dots,m+n\}\setminus\{h-1,h\}} p_ip_h + p_{h-1}p_h = \\
&= \sum_{i,j\in\{1,\dots,m+n\}\setminus\{h-1,h\}} |V_i||V_j| + \sum_{i\in\{1,\dots,m+n\}\setminus\{h-1,h\}} |V_i|(|V_{h-1}|-1) + \\
&+ \sum_{i\in\{1,\dots,m+n\}\setminus\{h-1,h\}} |V_i|(|V_h|+1) + (|V_{h-1}|-1)(|V_h|+1) = \\
&= \sum_{1\leqslant i<j\leqslant m+n} |V_i||V_j| + |V_{h-1}| - |V_h| - 1 \geqslant \\
&\geqslant |E|+1.
\end{aligned}$$

It follows that

$$|E| < \frac{(2m+n)^2-4m-n}{2}.$$

□

Now we characterize chromatically unique for the graph $G = K_2^m + K_n$.

Theorem 5. *The graph $G = K_2^m + K_n$ is χ-unique.*

Proof. It is clear that G is a complete $(m+n)$-partite graph $K_{p_1,p_2,\dots,p_{m+n}}$ with

$$p_1 = p_2 = \ldots = p_m = 2, \quad p_{m+1} = p_{m+2} = \ldots = p_{m+n} = 1.$$

Let $G' = (V', E')$ is a graph such that $G' \sim G$. Since Lemma 2 and (iii) of Lemma 3 we have

$$\chi(G') = \chi(G) = m+n.$$

Let G' has a coloring f using $m+n$ colors $1, 2, \dots, m+n$. Set

$$V_i' = \{u \in V' \mid f(u) = i\}.$$

for every $i = 1, 2, \dots, m+n$. It follows that G' is a $(m+n)$-partite graph $(V_1' \cup V_2' \cup \ldots \cup V_{m+n}', E')$. By (i) and (ii) of Lemma 3 we have

$$|V(G')| = |V(G)| = 2m+n, \quad |E(G')| = |E(G)| = \frac{(2m+n)^2-4m-n}{2}.$$

Without loss of generality we may

$$|V_1'| \geqslant |V_2'| \geqslant \ldots \geqslant |V_{m+n}'|.$$

By Lemma 4, we have

$$|V_1'| = |V_2'| = \ldots = |V_m'| = 2, \quad |V_{m+1}'| = |V_{m+2}'| = \ldots = |V_{m+n}|' = 1.$$

It follows that $G' \cong G$. Thus G is χ-unique. □

3. List-chromatic number

We need the following Lemmas 6–8 to prove our results.

Lemma 6 ([5])**.** *If G is a graph then* $\mathrm{ch}(G) \geqslant \chi(G)$.

Lemma 7 ([5]). *If G_1 is a subgraph of G_2 then* $\mathrm{ch}(G_1) \leqslant \mathrm{ch}(G_2)$.

We determine list-chromatic number for complete graphs.

Lemma 8. *If K_n is a complete graph on n vertices then* $\mathrm{ch}(K_n) = n$.

Now we determine list-chromatic number for the graph $G = K_2^r$.

Theorem 9. *List-chromatic number of $G = K_2^r$ is*

$$\mathrm{ch}(G) = r.$$

Proof. By Lemma 2 and Lemma 6, we have $\mathrm{ch}(G) \geqslant r$. Now we prove $\mathrm{ch}(G) \leqslant r$ by induction on r. For $r = 1$ the assertion holds, so let $r > 1$ and assume the assertion for smaller values of r.

Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ is a partition of $V(G)$ such that for every $i = 1, \ldots, r$, $|V_i| = 2$ and the subgraphs of G induced by V_i, is independent set. Set

$$V_i = \{v_{i1}, v_{i2}\}$$

for every $i = 1, \ldots, r$. Let $L_{v_{ij}}$ be the lists of colors of v_{ij} such that $|L_{v_{ij}}| = r$ for every $i = 1, 2, \ldots, r; j = 1, 2$. Now we consider separately two cases.

Case 1: There exists $i \in \{1, 2, \ldots, r\}$ such that $L_{v_{i1}} \cap L_{v_{i2}} \neq \emptyset$.

Without loss of generality we may assume that $L_{v_{11}} \cap L_{v_{12}} \neq \emptyset$ and $a \in L_{v_{11}} \cap L_{v_{12}}$. set $G' = G - V_1$. It is clear that G' is a graph K_2^{r-1}. Again set

$$L'_{v_{ij}} \subseteq L_{v_{ij}} \setminus \{a\}$$

such that $|L'_{v_{ij}}| = r - 1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

By the induction hypothesis, there exists $(r-1)$-choosable g of G' with the lists of colors $L'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

Let f be the coloring of G such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2,$

$f(v_{1j}) = a$ for every $j = 1, 2$.

Then f is a r-choosable for G, ie., $\mathrm{ch}(G) \leqslant r$.

Case 2: $L_{v_{i1}} \cap L_{v_{i2}} = \emptyset$ for every $i = 1, 2, \ldots, r$.

Let $b \in L_{v_{11}}$. Set $G' = G - V_1 = K_2^{r-1}$ and

$$L'_{v_{ij}} \subseteq L_{v_{ij}} \setminus \{b\}$$

such that $|L'_{v_{ij}}| = r - 1$ for every $i = 2, 3, \ldots, r; j = 1, 2$.

By the induction hypothesis, there exists $(r-1)$-choosable g of G' with the lists of colors $L'_{v_{ij}}$ for every $i = 2, 3, \ldots, r; j = 1, 2$. Since $|L_{v_{11}} \cup L_{v_{12}}| = 2r$ and $|V(G'| = 2(r-1)$, it follows that

$$|(L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))| \geqslant 2.$$

We again divide this case into two subcases.

Subcase 2.1: $((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}} \neq \emptyset$.

Let $c \in ((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}}$. Let f be the coloring of G such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 2, 3, \ldots, r; j = 1, 2,$

$f(v_{11}) = b$, $f(v_{12}) = c$.

Then f is a r-choosable for G, ie., $\mathrm{ch}(G) \leqslant r$.

Subcase 2.2: $((L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))) \cap L_{v_{12}} = \emptyset$.

By $|(L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))| \geqslant 2$, there exists $d \in (L_{v_{11}} \cup L_{v_{12}}) \setminus g(V(G'))$, $d \neq b$. It is clear that $b, d \in L_{v_{11}}$. Since $|L_{v_{12}}| = r$ and $|g(V(G'))| \leqslant 2(r-1)$, there exists $i \in \{2, 3, \ldots, r\}$ such that $g(v_{i1}), g(v_{i2}) \in L_{v_{12}}$. Without loss of generality we may assume that $g(v_{21}), g(v_{22}) \in L_{v_{12}}$. Let $e \in (L_{v_{21}} \cup L_{v_{22}}) \setminus g(V(G'))$. First assume that $e \in L_{v_{21}}$. If $e \neq b$ then coloring f of G such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = b$, $f(v_{12}) = g(v_{21})$.

is a r-choosable for G. If $e = b$ then coloring f of G such that

$f(v_{ij}) = g(v_{ij})$ for every $i = 3, 4, \ldots, r; j = 1, 2$,
$f(v_{22}) = g(v_{22})$, $f(v_{21}) = e$,
$f(v_{11}) = d$, $f(v_{12}) = g(v_{21})$.

is a r-choosable for G. By symmetry, we can show that $\mathrm{ch}(G) \leqslant r$ if $e \in L_{v_{22}}$. □

Theorem 10. *List-chromatic number of $G = K_2^m + K_n$ is*

$$\mathrm{ch}(G) = m + n.$$

Proof. It is clear that $G = K_2^m + K_n$ is a complete $(m+n)$-partite graph. By Lemma 2 and Lemma 6, we have $\mathrm{ch}(G) \geqslant m+n$. Now we prove $\mathrm{ch}(G) \leqslant m+n$. It is not difficult to see that G is a subgraph of K_2^{m+n}. By Lemma 7 and Theorem 9, $\mathrm{ch}(G) \leqslant m+n$. Thus, $\mathrm{ch}(G) = m+n$. □

4. Uniquely list colorability

If a graph G is not uniquely k-list colorable, we also say that G has property $M(k)$. So G has the property $M(k)$ if and only if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist at least two list colorings. The least integer k such that G has the property $M(k)$ is called the *m-number* of G, denoted by $m(G)$. This conception was originally introduced by Mahmoodian and Mahdian in [10].

We need the following Lemmas 11–16 to prove our results.

Lemma 11 ([10]). *A connected graph G has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

Lemma 12 ([10]). *For every graph G we have $m(G) \leqslant |E(\overline{G})| + 2$.*

Lemma 13 ([10]). *Every UkLC graph has at least $3k-2$ vertices.*

Lemma 14. *If $2m+n = 7$ and $m \geqslant 2$ then $G = K_2^m + K_n$ is U3LC.*

Proof. It is clear that $G = K_2^m + K_n$ is a complete $(m+n)$-partite graph. Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{m+n}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = |V_m| = 2, |V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1$ and for every $i = 1, \ldots, m$ the subgraphs of G induced by V_i, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, m$ and $V_{m+i} = \{v_i\}$ for every $i = 1, \ldots, n$. Now we consider separately two cases.

Case 1: $m = 2$ and $n = 3$.

We assign the following lists for the vertices of this graph:

$$L_{u_{11}} = \{1,2,3\}, \quad L_{u_{12}} = \{1,4,5\}, \quad L_{u_{21}} = \{1,2,3\}, \quad L_{u_{22}} = \{2,4,5\},$$

$$L_{v_1} = \{1,2,5\}, \quad L_{v_2} = \{1,2,4\}, \quad L_{v_3} = \{1,2,3\}.$$

A unique coloring f exists from the assigned lists:

$$f(u_{11}) = 1, \; f(u_{12}) = 1, \; f(u_{21}) = 2, \; f(u_{22}) = 2,$$

$$f(v_1) = 5,\ f(v_2) = 4,\ f(v_3) = 3.$$

Case 2: $m = 3$ and $n = 1$.

We assign the following lists for the vertices of this graph:

$$L_{u_{11}} = \{1,4,5\},\quad L_{u_{12}} = \{2,4,5\},\quad L_{u_{21}} = \{1,2,3\},\quad L_{u_{22}} = \{3,4,5\},$$

$$L_{u_{31}} = \{1,2,4\},\quad L_{u_{32}} = \{3,4,5\},\quad L_{v_1} = \{3,4,5\}.$$

A unique coloring f exists from the assigned lists:

$$f(u_{11}) = 1,\ f(u_{12}) = 2,\ f(u_{21}) = 3,\ f(u_{22}) = 3,$$

$$f(u_{31}) = 4,\ f(u_{32}) = 4,\ f(v_1) = 5.$$

□

Lemma 15. *If $m = 2$ and $n \geqslant 3$ then $G = K_2^m + K_n$ is U3LC.*

Proof. We prove G is U3LC by induction on n. If $n = 3$ then by Lemma 14, G is U3LC. So let $n > 3$ and assume the assertion for smaller values of n.

Let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{n+2}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = 2$, $|V_3| = |V_4| = \ldots = |V_{n+2}| = 1$ and for every $i = 1, 2$ the subgraphs of G induced by V_i, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2$ and $V_{i+2} = \{v_i\}$ for every $i = 1, \ldots, n$. Set $G' = G - v_n$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'.

We assign the following lists for the vertices of G:

$$L_{u_{11}} = L'_{u_{11}},\quad L_{u_{12}} = L'_{u_{12}},\quad L_{u_{21}} = L'_{u_{21}},\quad L_{u_{22}} = L'_{u_{22}},$$

$$L_{v_1} = L'_{v_1},\quad \ldots,\quad L_{v_{n-1}} = L'_{v_{n-1}},\quad L_{v_n} = \{f'(v_1), f'(v_2), t\},$$

with $t \notin L'_{u_{11}} \cup L'_{u_{12}} \cup L'_{u_{21}} \cup L'_{u_{22}} \cup L'_{v_1} \cup \ldots \cup L'_{v_{n-1}}$.

A unique coloring f of G exists from the assigned lists: $f(v) = f'(v)$ if $v \in V(G')$, $f(v_n) = t$.

□

Lemma 16. *If $m = 3$ and $n \geqslant 1$ then $G = K_2^m + K_n$ is U3LC.*

Proof. We prove G is U3LC by induction on n. If $n = 1$ then by Lemma 14, G is U3LC. So let $n > 1$ and assume the assertion for smaller values of n.

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{n+3}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = |V_3| = 2$, $|V_4| = |V_5| = \ldots = |V_{n+3}| = 1$ and for every $i = 1, 2, 3$ the subgraphs of G induced by V_i, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, 2, 3$ and $V_{i+3} = \{v_i\}$ for every $i = 1, \ldots, n$. Set $G' = G - v_n$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'.

We assign the following lists for the vertices of G:

$$L_{u_{11}} = L'_{u_{11}},\quad L_{u_{12}} = L'_{u_{12}},\quad L_{u_{21}} = L'_{u_{21}},\quad L_{u_{22}} = L'_{u_{22}},\quad L_{u_{31}} = L'_{u_{31}},\quad L_{u_{32}} = L'_{u_{32}}$$

$$L_{v_1} = L'_{v_1},\ \ldots,\ L_{v_{n-1}} = L'_{v_{n-1}},\quad L_{v_n} = \{f'(v_1), f'(v_2), t\},$$

with $t \notin L'_{u_{11}} \cup L'_{u_{12}} \cup L'_{u_{21}} \cup L'_{u_{22}} \cup L'_{u_{31}} \cup L'_{u_{32}} \cup L'_{v_1} \cup \ldots \cup L'_{v_{n-1}}$.

A unique coloring f of G exists from the assigned lists: $f(v) = f'(v)$ if $v \in V(G')$, $f(v_n) = t$.

□

Theorem 17. *$G = K_2^m + K_n$ is U3LC if and only if $2m + n \geqslant 7$ and $m \geqslant 2$.*

Proof. Firrst we prove the necessity. Suppose that $G = K_2^m + K_n$ is U3LC. By Lemma 13, $|V(G)| = 2m + n \geqslant 7$. If $m = 1$ then $|E(\overline{G})| = 1$, by Lemma 12, $m(G) \leqslant |E(\overline{G})| + 2 = 3$, a contradiction. Therefore, $m \geqslant 2$.

Now we prove the sufficiency. We prove G is U3LC by induction on m. If $m = 2$ then by Lemma 15, G is U3LC. If $m = 3$ then by Lemma 16, G is U3LC. So let $m > 3$ and assume the assertion for smaller values of m.

Let $V(G) = V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_{m+n}$ is a partition of $V(G)$ such that $|V_1| = |V_2| = \ldots = = |V_m| = 2$, $|V_{m+1}| = |V_{m+2}| = \ldots = |V_{m+n}| = 1$ and for every $i = 1, 2, \ldots, m$ the subgraphs of G induced by V_i, is independent set. Set $V_i = \{u_{i1}, u_{i2}\}$ for every $i = 1, \ldots, m$ and $G' = G - V_m$. By the induction hypothesis, for each vertex v in G', there exists a list of 3 colors L'_v, such that there exists a unique f' for G'.

We assign the following lists for the vertices of G: $L_{u_{m1}} = L_{u_{m2}} = \{f'(u_{11}), f'(u_{21}), t\}$ with $t \notin f'(G')$, $L_v = L'_v$ if $v \in V(G')$.

A unique coloring f of G exists from the assigned lists: $f(u_{m1}) = f(u_{m2}) = t,\ f(v) = f'(v)$ if $v \in V(G')$. □

5. Conclusion

The coloring problem, including the list coloring problem, has always been much researched in graph theory because it has many applications in computer science. The list coloring model can be used in the channel assignment. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [13]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [13]. The research of list coloring consists of two parts: the choosability and the unique list colorability.

The main results of the paper have identified the list-chromatic number (Theorem 10), characterized chromatically unique (Theorem 5) and characterized uniquely list colorability (Theorem 17) of the graph $G = K_2^m + K_n$. The desire in the future will achieve deeper results on the issues raised in this article.

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Раскраски графа $K_2^m + K_n$

Ле Хуан Хунг
Ханойский университет природных ресурсов и окружающей среды
Ханой, Вьетнам

Аннотация. В этой статье мы характеризуем хроматически уникальное хроматическое число в списке и однозначно характеризуем окрашиваемость графа списка $K_2^m + K_n$. Мы докажем, что G χ единственно, $\mathrm{ch}(G) = m + n$, G является однозначным трехцветным графом раскраски тогда и только тогда, когда $2m + n \geqslant 7$ и $m \geqslant 2$.

Ключевые слова: хроматическое число, хроматический номер списка, хроматически уникальный граф, однозначный список раскрашиваемого графа, полный r-раздельный граф.