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Problems on Structure for Quasifields of Orders 16 and 32

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Well-known method of the construction of finite projective translation planes (analogously, semifield planes) uses their correspondence with quasifields (resp., semifields). We distinguish certain questions on the structure of any finite quasifield (possible maximal subfields, the property of cyclicity of multiplicative loop of non-zero elements and possible orders of elements). In the present paper we discover some anomalous properties of finite quasifields of small even orders.

Keywords: projective translation plane, quasifield, semifield, multiplicative loop, orders of elements.

A ring $S = \langle S, +, \circ \rangle$ with the identity $e \neq 0$ is said to be *a semifield* (or *"quasitelo"*, according to [1, II. 6.1]), if $S^* = (S \setminus \{0\}, \circ)$ is a loop, i.e., for any $a \in S^*$ and $b \in S$ each equation $a \circ x = b$ and $y \circ a = b$ is uniquely solvable in S. For finite S the weakening of two-sided distributivity to one-sided one gives a *quasifield* [2, 3]. It is well-known that there exists its unique minimal subfield of a prime order p and hence the order $|S|$ is p-primary.

The construction of *proper* (or not being a field) quasifields is closely related to construction of non-Desargues projective translation planes and from the middle of last century it is based on computer calculations. Unlike finite fields, finite quasifields and semifields are poorly studied [4].

By the order $|v|$ of element v of the loop we shall call (generalizing the notion of the order of a group element) the smallest integer $m \geqslant 1$ such that at least one m-th degree of element v at all possible positioning of brackets shall be equal e; the order is infinite when such m does not exist. The set of orders of all elements of a loop is called *a spectrum.*

For a finite proper quasifield the first author wrote down the following questions.

(A) *Enumerate maximal subfields and their possible orders.*

(B) *What loop spectrums S^* of finite semifields and quasifields are possible ?*

(C) *Enumerate finite quasifields, in particular, semifields where the loop S^* is not singly-generated.*

It remains open still *the hypothesis of Wene on right primitive* of semifield of order > 32 ([5, 6]): Is it true that the right-ordered degrees of a fixed element give all elements in S^*? See also N.D. Podufalov [7] and his questions 9.43, 10.48, 11.76, 11.77 and 12.66 in [8].

Smallest even orders of non-Desargues projective semifield planes and translation planes are the same (unlike odd orders) and they are equal to 16, by virtue of [9] and [10]. Such planes are enumerated in [11–13] (see also [14, 15]), and of order 32 — in [12, 16]. Up to isotopisms, corresponding semifields and quasifields exhaust all ones of the same orders. For their P.K.Shtukkert have studied the issues **(A)**–**(C)** in the case of semifields, see [17] and Section 2 below. In the present paper we use the Kleinfeld's classification of semifields of order 16, up to isomorphisms.

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According to [11], the number of isotopic classes and the number of isomorphic classes of proper semifields of order 16 is equal to 2 and 23, respectively. Theorems 2.1, 2.2 and Tab. 4 in Section 2 solve questions **(A)**–**(C)** for all semifields of order 16. Note that up to isomorphisms and anti-isomorphisms the number of such semifields is equal to 16. Using [16] we show that there exists a quasifield Q of order 16, which are the set-theoretic union of its 7 subfields of order 4; in particular, the loop Q^* is not singly-generated and its spectrum is $\{1, 3\}$ (Theorem 3.3 in Section 3).

Anomalous properties of proper semifields of order 32 are considered in Section 4.

1. The tie of quasifields and translation planes

The projective plane is defined as a set of points with certain subsets, which are called lines [18, Section 20.1]; ibid see definition of the plane order.

For the construction of a translation plane π of rank n it must be chosen a n-dimensional linear space W over a field F (*coordinatized set*), the outer direct sum of two copies of W,

$$V = W \oplus W = \{(x, y) \mid x, y \in W\},$$

and the spread μ of additive group $(V, +)$ such that $V = M \oplus N$ for any $M \neq N$ from μ. By definition, 1-dimensional subspaces in V are points of projective translation plane $\pi = \pi(V, \mu)$, subgroups from μ and their cosets are lines of π and, also, different cosets on the same subgroup have a unique general point (∞) and all such *singular points* give *a singular line* $[\infty]$ of π, [2].

Recall that the spread of an additive group is a set of its subgroups (components of spread), which have trivial pairwise intersections and their set-theoretic union gives whole group. The components of our spread μ are n-dimensional subspace in V [3]. It is well-known the following lemma [19], where

$$V(\sigma) = \{(v, v^\sigma) \mid v \in W\} \quad (\sigma \in GL(W)), \quad V(0) = (W, 0), \quad V(\infty) = (0, W).$$

Lemma 1.1. *Let us assume that $V(0), V(\infty) \in \mu$. Then:*

a) if $M \in \mu$ and $M \neq V(0), V(\infty)$, then $M = V(\sigma)$ at unique $\sigma \in GL(W)$ and, in particular, $\mu = \{V(\sigma) \mid \sigma \in R^* \cup \{0\}\} \cup \{V(\infty)\}$ at $R^* = \{\sigma \in GL(W) \mid V(\sigma) \in \mu\}$;

b) if $u, v \in W \setminus \{0\}$, then $u^\sigma = v$ at unique $\sigma \in R^*$;

c) if $\tau, \rho \in R^*$ and $\tau \neq \rho$, then $\tau - \rho \in GL(W)$.

Conversely if a subset R^* in $GL(W)$ satisfies b) and c), then $\mu = \{V(0),\ V(\infty)\} \cup \{V(\sigma) \mid \sigma \in R^*\}$ is a spread of the group $(V, +)$ such that $V = M \oplus N$ for any $M \neq N$ from μ.

Taking into account b) there exists a bijective mapping $\theta : W \to R^* \cup \{0\}$ such that

$$\theta(v) = \sigma \quad (v \in W \setminus \{0\}, \quad u^\sigma = v), \quad \theta(0) = 0.$$

The totality R of a subset R^* in $GL(W)$ with the identity satisfying b), c) and the null map is said to be *a regular set* of plane π. Writing the vectors from W as coordinate rows and setting

$$x \circ y := x \cdot \theta(y) \quad (x, y \in W) \tag{1}$$

we obtain a quasifield $W = (W, +, \circ)$. If W is a semifield then π is called *a semifield plane.*

Definition 1.1. *Quasifields $\langle S_1, +, \circ\rangle$ and $\langle S_2, +, \cdot\rangle$ are called isotopic, if there exist isomorphisms F, G, H of additive groups $S_1 \to S_2$ such that*

$$x^F \cdot y^G = (x \circ y)^H \qquad (x, y \in S_1).$$

It is well-known the following lemmas.

Lemma 1.2. (Albert, 1960) *Two semifield planes are isomorphic if and only if the corresponding semifields are isotopic.*

Lemma 1.3. *The projective translation plane is Desargues if and only if the corresponding quasifield is a field.*

Lemma 1.4. *Coordinatized set is a field if and only if the regular set is subfield of a ring $M(n, F)$ of all $n \times n$-dimentions matrixes over F.*

We now consider the structure of semifield planes of order 16. Any such plane π can be coordinatizate by 4-dimension space W over Z_2. Choosing a regular set $R = \theta(W)$ with the Z_2-linear map $\theta : W \to M(4, Z_2)$, which is identical onto $(1, 0, 0, 0)$. When $(W, +, \circ)$ is a finite field with multiplication (1), by lemma 1.4, R is a subfield of order 16 in the ring $M(4, Z_2)$. Then R^* is a cyclic group of order 15 which is generated by a matrix $A \in GL(4, 2)$ with irreducible characteristic polynomial over Z_2; for construction of such matrices we may use the natural normal form of matrices [20, Section 15.5]. In general we have

$$\theta(x_1, x_2, x_3, x_4) = x_1 \cdot E + x_2 \cdot B + x_3 \cdot C + x_4 \cdot D \quad (B, C, D \in GL(4, Z_2)).$$

By computer calculations we obtain exactly 19936 different sets $\{B, C, D\}$ in which 336 cases give a field $R = \theta(W)$. In other cases any set $\{B, C, D\}$ uniquely defines a non-Desargues semifield plane π (together with R) and a semifield W with multiplication (1). All non-Desargues semifield planes of rang 2 over $GF(4)$ are pairwise isomorphic [21]. In fact by methods [21] we may show that the enumeration of nonisomorphic non-Desargues semifield planes gives two cases:

$$\theta(x, y, z, w) = \begin{pmatrix} x & y & z & w \\ w & x+w & z+w & y+z+w \\ z & z+w & x+y+w & y+w \\ y & z & y+w & x \end{pmatrix},$$

$$\theta(x, y, z, w) = \begin{pmatrix} x & y & z & w \\ w & x+z & z+w & y+w \\ z & w & x+y+z+w & y+z+w \\ y+z+w & z & y+z & x+w \end{pmatrix} \quad (x, y, z, w \in Z_2).$$

Taking into account (1) and lemma 1.2, we find, up to isotopism, exactly 2 proper semifields of order 16 with the multiplication, respectively,

$$\begin{gathered}(a, b, c, d) \circ (u, v, z, w) = (au + bv + cz + dw, av + bu + bw + cz + cw + dz, \\ az + bz + bw + cu + cv + cw + dv + dw, aw + bv + bz + bw + cv + cw + du); \end{gathered} \tag{2}$$

$$\begin{gathered}(a, b, c, d) \circ (u, v, z, w) = (au + bv + cz + dv + dz + dw, av + bu + bz + cw + dz, \\ az + bz + bw + cu + cv + cz + cw + dv + dz, aw + bv + bw + cv + cz + cw + du + dw). \end{gathered} \tag{3}$$

We now obtain the following theorem which was earlier proved also with computer calculations, see E.Kleinfeld [11] and D.Knuth [12].

Theorem 1.1. *There exist only 3 nonisomorphic semifield planes of order 16.*

In the investigation of any quasifield the impotent role play the left, middle and right kernels. For any semifield S it is, respectively,

$$N_l(S) = \{x \in S \mid x \circ (y \circ z) = (x \circ y) \circ z, \quad \forall y, z \in S\},$$

$$N_m(S) = \{y \in S \mid x \circ (y \circ z) = (x \circ y) \circ z, \quad \forall x, z \in S\},$$

$$N_r(S) = \{z \in S \mid x \circ (y \circ z) = (x \circ y) \circ z, \quad \forall x, y \in S\}.$$

2. The construction of semifields of order 16

In 1960 E.Kleinfeld [11] obtained the classification of proper semifields of order 16, up to isomorphisms. It had been shown that one isotopic class has 18 pairwise nonisomorphic semifields $V_1, V_2, \cdots, V_{18}$, and second has 5 pairwise nonisomorphic semifields $T_{24}, T_{25}, T_{35}, T_{45}, T_{50}$.

Note that the constructed semifields with the multiplication (2) and (3) in Section 2 are isomorphic to semifields V_7 and T_{25}, respectively. We now investigate their structure.

Theorem 2.1. *Let S be the semifield with multiplication (2), i.e., $S \simeq V_7$. Then:*

(i) the minimal subfield $Z_2 e$ of S is maximal;

(ii) the loop S^* is generated by each nonidentity element;

(iii) the spectrum of loop S^* is $\{1, 4, 5, 6\}$.

Proof. Firstly we find Cayley's table of loop multiplication S^*, see Tab. 1. (In the table multiplications on identity element $(1, 0, 0, 0)$ is omitted.)

Further we denote by g^k, the $k-$th degree $(k \geqslant 1)$ of element g of semifield with the right (or the right-normalized) positioning of brackets. The Tab. 1 shows that the right k-th degrees $(1 \leqslant k \leqslant 15)$ of each element

$$m_1 = (0, 1, 1, 1),\ m_2 = (1, 1, 0, 0),\ m_3 = (1, 1, 0, 1),\ m_4 = (1, 1, 1, 0)$$

give all elements of the loop S^*. In particular, each element m_i generates loop S^* and $(m_i)^{15} = e$.

Since for the element $h = (0, 0, 0, 1)$ all products of length < 5 differ from e and $h^2 \cdot h^3 = e$, so $|h| = 5$. On the other hand, for the element m_1 all possible products of length $\leqslant 5$ also do not equal e and $m_1^2 \cdot (m_1 \cdot (m_1)^3) = e$. Therefore $|m_1| = 6$. Analogously we show that any element of the loop S^* has the order $\leqslant 6$. The orders of all elements of the loop S^* are given in Tab. 2.

In particular, the analog of group-theoretic Lagrange's theorem is not satisfied even for the orders of elements of the loop S^*. Also we consider a table of left and right invertible elements.

The Tabs. 1 and 3 show that any nonidentity element of the loop S^* is a suitable degree of some element of m_i and, therefore, it generates the loop S^*. In particular, S^* is the right-cyclic or the right-primitive in terms of [6].

It is clear that for each element of any subfield left and right invertible elements are coincide. In according to Tabs. 2 and 3, such nonidentity elements are only elements (1,1,0,0) and (1,1,1,0) of order > 3. Consequently, the semifield S has no a subfield of order > 2. □

For the semifield with multiplication (3) it is true

Theorem 2.2. *Let S be the semifield with multiplication (3),i.e., $S \simeq T_{25}$. Then:*

(i) there exist exactly 2 maximal subfields H_1 and H_2, and $|H_1| = |H_2| = 4$;

(ii) the loop S^* is generated by each element from $S \setminus \{H_1 \cup H_2\}$;

(iii) the spectrum of loop S^* is $\{1, 3, 4, 5, 6\}$.

In fact, it was obtained the analogous structural description for all 23 proper Kleinfeld's semifields of order 16. It seems, up to isomorphisms and antiisomorphisms, there exist only 16 of proper semifields of order 16. More exactly it is proved

Theorem 2.3. *Any proper semifield of order 16 up to isomorphisms is either one of 7 semifields $V_1, V_3, V_4, V_8, V_{11}, V_{15}, T_{25}$ or one of opposite semifields to them $V_6, V_7, V_5, V_9, V_{14}, V_6, T_{50}$, respectively, or one of 9 semifields $V_2, V_{10}, V_{12}, V_{13}, V_{17}, V_{18}, T_{21}, T_{35}, T_{45}$.*

For any semifield W the opposite semifield is denoted by W^{op}. The following Tab. 4 gives the structure of proper semifields of order 16 up to isomorphisms and antiisomorphisms.

Table 1. Loop S^* with multiplication

	(0,0,0,1)	**(0,0,1,0)**	**(0,0,1,1)**	**(0,1,0,0)**	**(0,1,0,1)**	**(0,1,1,0)**	**(0,1,1,1)**
(0,0,0,1)	(0,0,1,0)	(0,1,0,0)	(0,1,1,0)	(1,0,1,0)	**(1,0,0,0)**	(1,1,1,0)	(1,1,0,0)
(0,0,1,0)	(0,1,1,1)	(1,1,0,0)	(1,0,1,1)	(0,0,1,1)	(0,1,0,0)	(1,1,1,1)	**(1,0,0,0)**
(0,0,1,1)	(0,1,0,1)	**(1,0,0,0)**	(1,1,0,1)	(1,0,0,1)	(1,1,0,0)	(0,0,0,1)	(0,1,0,0)
(0,1,0,0)	(1,1,1,1)	(0,0,1,1)	(1,1,0,0)	(0,0,0,1)	(1,1,1,0)	(0,0,1,0)	(1,1,0,1)
(0,1,0,1)	(1,1,0,1)	(0,1,1,1)	(1,0,1,0)	(1,0,1,1)	(0,1,1,0)	(1,1,0,0)	(0,0,0,1)
(0,1,1,0)	**(1,0,0,0)**	(1,1,1,1)	(0,1,1,1)	(0,0,1,0)	(1,0,1,0)	(1,1,0,1)	(0,1,0,1)
(0,1,1,1)	(1,0,1,0)	(1,0,1,1)	(0,0,0,1)	**(1,0,0,0)**	(0,0,1,1)	(0,0,1,1,)	(1,0,0,1)
(1,0,0,1)	(0,0,1,1)	(0,1,1,0)	(0,1,0,1)	(1,1,1,0)	(1,1,0,1)	**(1,0,0,0)**	(1,0,1,1)
(1,0,1,0)	(0,1,1,0)	(1,1,1,0)	**(1,0,0,0)**	(0,1,1,1)	(0,0,0,1)	(1,0,0,1)	(1,1,1,1)
(1,0,1,1)	(0,1,0,0)	(1,0,1,0)	(1,1,1,0)	(1,1,0,1)	(1,0,0,1)	(0,1,1,1)	(0,0,1,1)
(1,1,0,0)	(1,1,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,0,1)	(1,0,1,1)	(0,1,0,0)	(1,0,1,0)
(1,1,0,1)	(1,1,0,0)	(0,1,0,1)	(1,0,0,1)	(1,1,1,1)	(0,0,1,1)	(1,0,1,0)	(0,1,1,0)
(1,1,1,0)	(1,0,0,1)	(1,1,0,1)	(0,1,0,0)	(0,1,1,0)	(1,1,1,1)	(1,0,1,1)	(0,0,1,0)
(1,1,1,1)	(1,0,1,1)	(1,0,0,1)	(0,0,1,0)	(1,1,0,0)	(0,1,1,1)	(0,1,0,1)	(1,1,1,0)

	(1,0,0,1)	**(1,0,1,0)**	**(1,0,1,1)**	**(1,1,0,0)**	**(1,1,0,1)**	**(1,1,1,0)**	**(1,1,1,1)**
(0,0,0,1)	(0,0,1,1)	(0,1,0,1)	(0,1,1,1)	(1,0,1,1)	(1,0,0,1)	(1,1,1,1)	(1,1,0,1)
(0,0,1,0)	(0,1,0,1)	(1,1,1,0)	(1,0,0,1)	(0,0,0,1)	(0,1,1,0)	(1,1,0,1)	(1,0,1,0)
(0,0,1,1)	(0,1,1,0)	(1,0,1,1)	(1,1,1,0)	(1,0,1,0)	(1,1,1,1)	(0,0,1,0)	(0,1,1,1)
(0,1,0,0)	(1,0,1,1)	(0,1,1,1)	**(1,0,0,0)**	(0,1,0,1)	(1,0,1,0)	(0,1,1,0)	(1,0,0,1)
(0,1,0,1)	**(1,0,0,0)**	(0,0,1,0)	(1,1,1,1)	(1,1,1,0)	(0,0,1,1)	(1,0,0,1)	(0,1,0,0)
(0,1,1,0)	(1,1,1,0)	(1,0,0,1)	(0,0,0,1)	(0,1,0,0)	(1,1,0,0)	(1,0,1,1)	(0,0,1,1)
(0,1,1,1)	(1,1,0,1)	(1,1,0,0)	(0,1,1,0)	(1,1,1,1)	(0,1,0,1)	(0,1,0,0)	(1,1,1,0)
(1,0,0,1)	(1,0,1,0)	(1,1,1,1)	(1,1,0,0)	(0,1,1,1)	(0,1,0,0)	(0,0,0,1)	(0,0,1,0)
(1,0,1,0)	(1,1,0,0)	(0,1,0,0)	(0,0,1,0)	(1,1,0,1)	(1,0,1,1)	(0,0,1,1)	(0,1,0,1)
(1,0,1,1)	(1,1,1,1)	(0,0,0,1)	(0,1,0,1)	(0,1,1,0)	(0,0,1,0)	(1,1,0,0)	**(1,0,0,0)**
(1,1,0,0)	(0,0,1,0)	(1,1,0,1)	(0,0,1,1)	(1,0,0,1)	(0,1,1,1)	**(1,0,0,0)**	(0,1,1,0)
(1,1,0,1)	(0,0,0,1)	**(1,0,0,0)**	(0,1,0,0)	(0,0,1,0)	(1,1,1,0)	(0,1,1,1)	(1,0,1,1)
(1,1,1,0)	(0,1,1,1)	(0,0,1,1)	(1,0,1,0)	**(1,0,0,0)**	(0,0,0,1)	(0,1,0,1)	(1,1,0,0)
(1,1,1,1)	(0,1,0,0)	(0,1,1,0)	(1,1,0,1)	(0,0,1,1)	**(1,0,0,0)**	(1,0,1,0)	(0,0,0,1)

Table 2. The orders of elements of the loop S^*

y	(1,0,0,0)	(0,0,0,1)	(0,0,1,0)	(0,0,1,1)	(0,1,0,0)	(0,1,0,1)	(0,1,1,0)	(0,1,1,1)
|y|	1	5	5	5	5	5	5	6

y	(1,0,0,1)	(1,0,1,0)	(1,0,1,1)	(1,1,0,0)	(1,1,0,1)	(1,1,1,0)	(1,1,1,1)
|y|	6	5	4	6	6	5	4

3. Quasifields of order 16

Kleinfeld [11] classified quasifields of order 16 with kernel of order 4 using computer calculations. Also he noted: "The problem of determining all such Veblen-Wedderburn systems turns out to be more difficult in the previous case and we abandon it in favor of determining all such division rings".

U.Dempwolff and A.Reifart (see works [13, 16]) have completely classified translation planes of order 16. Up to isomorphisms, there are exactly 8 planes and the number of classes containing

Table 3. Right and left invertible elements in the loop S^*

The element	**The left invertible**	**The right invertible**
(1,0,0,0)	**(1,0,0,0)**	**(1,0,0,0)**
(1,1,0,0)	**(1,1,1,0)**	**(1,1,1,0)**
(1,1,1,0)	**(1,1,0,0)**	**(1,1,0,0)**
(0,0,0,1)	(0,1,1,0)	(0,1,0,1)
(0,0,1,0)	(0,0,1,1)	(0,1,1,1)
(0,1,0,0)	(0,1,1,1)	(1,0,1,1)
(0,0,1,1)	(1,0,1,0)	(0,0,1,0)
(0,1,0,1)	(0,0,0,1)	(1,0,0,1)
(0,1,1,0)	(1,0,0,1)	(0,0,0,1)
(1,0,0,1)	(0,1,0,1)	(0,1,1,0)
(1,0,1,0)	(1,1,0,1)	(0,0,1,1)
(0,1,1,1)	(0,0,1,0)	(0,1,0,0)
(1,1,0,1)	(1,1,1,1)	(1,0,1,0)
(1,0,1,1)	(0,1,0,0)	(1,1,1,1)
(1,1,1,1)	(1,0,1,1)	(1,1,0,1)

Table 4. The structure of nonisomorphic semifields of order 16

Semifield	$\|N_l\|$	The number of subfields of order 4	The spectrum of the loop	The number of elements with equal left and right invertible	The opposite semifield
V_1	**2**	–	$\{1,4,5\}$	1	$V_1^{op} \simeq V_6$
V_2	**2**	**1**	$\{1,3,4,5,6\}$	3	$V_2^{op} = V_2$
V_3	**2**	–	$\{1,4,5,6\}$	3	$V_3^{op} \simeq V_7$
V_4	**2**	**1**	$\{1,3,4,5,6\}$	3	$V_4^{op} \simeq V_5$
V_8	**2**	**2**	$\{1,3,4,5,6\}$	7	$V_8^{op} \simeq V_9$
V_{10}	**2**	**1**	$\{1,3,5,6\}$	3	$V_{10}^{op} = V_{10}$
V_{11}	**2**	**1**	$\{1,3,4,5,6\}$	7	$V_{11}^{op} \simeq V_{14}$
V_{12}	**2**	–	$\{1,4,5,6\}$	1	$V_{12}^{op} = V_{12}$
V_{13}	**2**	**4**	$\{1,3,5\}$	9	$V_{13}^{op} = V_{13}$
V_{15}	**2**	**2**	$\{1,3,4,5\}$	7	$V_{15}^{op} \simeq V_{16}$
V_{17}	**2**	**1**	$\{1,3,4,5,6\}$	3	$V_{17}^{op} = V_{17}$
V_{18}	**2**	**2**	$\{1,3,5,6\}$	5	$V_{18}^{op} = V_{18}$
T_{24}	**4**	**2**	$\{1,3,4,5,6\}$	5	$T_{24}^{op} = T_{24}$
T_{25}	**4**	**2**	$\{1,3,4,5,6\}$	5	$T_{25}^{op} \simeq T_{50}$
T_{35}	**4**	**1**	$\{1,3,4,5,6\}$	3	$T_{35}^{op} = T_{35}$
T_{45}	**4**	**3**	$\{1,3,5\}$	7	$T_{45}^{op} = T_{45}$

semifield planes is 3. Regular sets of representatives of 5 other isomorphic classes of translation planes are described in [16]. We write them in the following fixed order where O and E are zero and identity matrices, respectively:

$$\Bigg\{O, E, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\Bigg\};$$

$$\Bigg\{O, E, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\Bigg\};$$

$$\Bigg\{O, E, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\Bigg\};$$

$$\Bigg\{O, E, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\Bigg\};$$

$$\Bigg\{O, E, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Bigg\}.$$

Determining for each of them on the multiplication on formula (1) we get pairwise nonisotopic quasifields $Q_i,\ i = 1, 2, 3, 4, 5$, respectively. For each of them we found the Cayley's table of loop Q_i^* and investigate questions **(A)–(C)**.

The following two Theorems show structure of quasifields Q_2 and Q_5 which are most similar properties to finite fields.

Theorem 3.1. *The quasifield Q_2 has a single maximal subfield H and $|H| = 4$. Each element of $Q_2 \setminus H$ has order 5 and generates a loop Q_2^*.*

Unlike Q_2, the quasifield Q_5 has 3 subfields of order 4:

$$G_1 = \{0, e, (0,0,1,0), (1,0,1,0)\},\ G_2 = \{0, e, (0,1,0,1), (1,1,0,1)\},$$

$$G_3 = \{0, e, (0,1,1,1), (1,1,1,1)\}$$

Theorem 3.2. *Every maximal subfield of the quasifield Q_5 coincides with G_1, G_2 or G_3 and any element from $Q_5^* \setminus \{G_1 \cup G_2 \cup G_3\}$ has the order 5 and generates loop Q_5^*. In particular, spectrum of loop Q_5^* is $\{1, 3, 5\}$.*

Quasifields Q_1, Q_3, Q_4 has an essential anomalous properties; in particular, any element of each of them is element from some subfield of order 4. One shows

Theorem 3.3. *Any semifield Q_i, $i = 1, 3, 4$, has 7 maximal subfields of order 4 and their set-theoretic union coincides with Q_i. In particular, spectrum of loop Q_i^* is $\{1, 3\}$.*

Proof. Using the first regular set we define multiplication (1) in quasifield Q_1. Further we find Cayley's table of loop Q_1^* (multiplication on the identity element $(1, 0, 0, 0)$ is omitted) (Tab. 5).

Similarly, we find a Cayley's tables for loops Q_3^* and Q_4^*. Any element of each loop Q_i^*, $i = 1, 3, 4$, has identical left and right invertible elements as it show the Cayley's tables. Also each quasifield Q_i has the following 7 subfields:

$$F_1 = \{0, e, (0,0,0,1), (1,0,0,1)\},\ F_2 = \{0, e, (0,0,1,0), (1,0,1,0)\},$$

$$F_3 = \{0, e, (0,0,1,1), (1,0,1,1)\},\ F_4 = \{0, e, (0,1,0,0), (1,1,0,0)\},$$

$$F_5 = \{0, e, (0,1,0,1), (1,1,0,1)\},\ F_6 = \{0, e, (0,1,1,0), (1,1,1,0)\},$$

$$F_7 = \{0, e, (0,1,1,1), (1,1,1,1)\}.$$

Clear that the set-theoretic union of 7 different subfields coincides with Q_i. In particular, the spectrum of loop Q_i^* is $\{1, 3\}$. □

Remark 3.1. It was also established that there are quasifields of order 16 with kernel of order 4 having elements of order 3, which are not lie in the any subfield of order 4. For instance, these are Kleinfeld's quasifields S_3 and S_{10}.

Table 5. Cayley's table of loop Q_1^*

	(0,0,0,1)	**(0,0,1,0)**	**(0,0,1,1)**	**(0,1,0,0)**	**(0,1,0,1)**	**(0,1,1,0)**	**(0,1,1,1)**
(0,0,0,1)	(1,0,0,1)	(1,1,0,1)	(1,1,1,0)	(0,1,1,0)	(1,0,1,1)	(0,0,1,1)	(0,1,0,0)
(0,0,1,0)	(1,1,1,1)	(1,0,1,0)	(0,1,0,1)	(1,0,1,1)	(0,1,0,0)	(0,0,0,1)	(1,1,1,0)
(0,0,1,1)	(0,1,1,0)	(0,1,1,1)	(1,0,1,1)	(1,1,0,1)	(1,1,1,1)	(0,0,1,0)	(1,0,1,0)
(0,1,0,0)	(1,0,1,0)	(0,0,1,1)	(1,0,0,1)	(1,1,0,0)	(0,1,1,0)	(1,1,1,1)	(0,1,0,1)
(0,1,0,1)	(0,0,1,1)	(1,1,1,0)	(0,1,1,1)	(1,0,1,0)	(1,1,0,1)	(1,1,0,0)	(0,0,0,1)
(0,1,1,0)	(0,1,0,1)	(1,0,0,1)	(1,1,0,0)	(0,1,1,1)	(0,0,1,0)	(1,1,1,0)	(1,0,1,1)
(0,1,1,1)	(1,1,0,0)	(0,1,0,0)	(0,0,1,0)	(0,0,0,1)	(1,0,0,1)	(1,1,0,1)	(1,1,1,1)
(1,0,0,1)	**(1,0,0,0)**	(1,1,1,1)	(1,1,0,1)	(0,0,1,0)	(1,1,1,0)	(0,1,0,1)	(0,0,1,1)
(1,0,1,0)	(1,1,1,0)	**(1,0,0,0)**	(0,1,1,0)	(1,1,1,1)	(0,0,0,1)	(0,1,1,1)	(1,0,0,1)
(1,0,1,1)	(0,1,1,1)	(0,1,0,1)	**(1,0,0,0)**	(1,0,0,1)	(1,0,1,0)	(0,1,0,0)	(1,1,0,1)
(1,1,0,0)	(1,0,1,1)	(0,0,0,1)	(1,0,1,0)	**(1,0,0,0)**	(0,0,1,1)	(1,0,0,1)	(0,0,1,0)
(1,1,0,1)	(0,0,1,0)	(1,1,0,0)	(0,1,0,0)	(1,1,1,0)	**(1,0,0,0)**	(1,0,1,0)	(0,1,1,0)
(1,1,1,0)	(0,1,0,0)	(1,0,1,1)	(1,1,1,1)	(0,0,1,1)	(0,1,1,1)	**(1,0,0,0)**	(1,1,0,0)
(1,1,1,1)	(1,1,0,1)	(0,1,1,0)	(0,0,0,1)	(0,1,0,1)	(1,1,0,0)	(1,0,1,1)	**(1,0,0,0)**

	(1,0,0,1)	**(1,0,1,0)**	**(1,0,1,1)**	**(1,1,0,0)**	**(1,1,0,1)**	**(1,1,1,0)**	**(1,1,1,1)**
(0,0,0,1)	**(1,0,0,0)**	(1,1,0,0)	(1,1,1,1)	(0,1,1,1)	(1,0,1,0)	(0,0,1,0)	(0,1,0,1)
(0,0,1,0)	(1,1,0,1)	**(1,0,0,0)**	(0,1,1,1)	(1,0,0,1)	(0,1,1,0)	(0,0,1,1)	(1,1,0,0)
(0,0,1,1)	(0,1,0,1)	(0,1,0,0)	**(1,0,0,0)**	(1,1,1,0)	(1,1,0,0)	(0,0,0,1)	(1,0,0,1)
(0,1,0,0)	(1,1,1,0)	(0,1,1,1)	(1,1,0,1)	**(1,0,0,0)**	(0,0,1,0)	(1,0,1,1)	(0,0,0,1)
(0,1,0,1)	(0,1,1,0)	(1,0,1,1)	(0,0,1,0)	(1,1,1,1)	**(1,0,0,0)**	(1,0,0,1)	(0,1,0,0)
(0,1,1,0)	(0,0,1,1)	(1,1,1,1)	(1,0,1,0)	(0,0,0,1)	(0,1,0,0)	**(1,0,0,0)**	(1,1,0,1)
(0,1,1,1)	(1,0,1,1)	(0,0,1,1)	(0,1,0,1)	(0,1,1,0)	(1,1,1,0)	(1,0,1,0)	**(1,0,0,0)**
(1,0,0,1)	(0,0,0,1)	(0,1,1,0)	(0,1,0,0)	(1,0,1,1)	(0,1,1,1)	(1,1,0,0)	(1,0,1,0)
(1,0,1,0)	(0,1,0,0)	(0,0,1,0)	(1,1,0,0)	(0,1,0,1)	(1,0,1,1)	(1,1,0,1)	(0,0,1,1)
(1,0,1,1)	(1,1,0,1)	(1,1,1,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(1,1,1,1)	(0,1,1,0)
(1,1,0,0)	(0,1,1,1)	(1,1,0,1)	(0,1,1,0)	(0,1,0,0)	(1,1,1,1)	(0,1,0,1)	(1,1,1,0)
(1,1,0,1)	(1,1,1,1)	(0,0,0,1)	(1,0,0,1)	(0,0,1,1)	(0,1,0,1)	(0,1,1,1)	(1,0,1,1)
(1,1,1,0)	(1,0,1,0)	(0,1,0,1)	(0,0,0,1)	(1,1,0,1)	(1,0,0,1)	(0,1,1,0)	(0,0,1,0)
(1,1,1,1)	(0,0,1,0)	(1,0,0,1)	(1,1,1,0)	(1,0,1,0)	(0,0,1,1)	(0,1,0,0)	(0,1,1,1)

4. Semifields of order 32

In 2011 all translation planes of order 32 and their regular sets were described by U. Dempwolff and R. Rockenfeller [16, 22]. (See also R. J. Walker [23].) There are 9 of these planes up to isomorphisms including 5 semifield planes and a Desargues plane. A coordinatizing set here is a 5-dimentional space W over the field Z_2.

Regular sets of non-Desargues semifield planes of order 32 are described in [16]. We denote their by R_i $(1 \leqslant i \leqslant 5)$, according to [17]. Let P_i $(1 \leqslant i \leqslant 5)$ be a semifield, which corresponding regular set R_i. In particular, using the regular set

$$\begin{pmatrix} x & y & z & w & s \\ z & x+z+w & y+w & w+s & w \\ z+s & w & x & y+w & z+w \\ z+w+s & z+s & s & x+z+w & y+z+s \\ y+z+w & w+s & y & z & x+z \end{pmatrix}$$

and multiplication (1) we obtain the semifield P_5. The following theorems are proved in [17].

Theorem 4.1. *The minimal subfield Z_2e in every semifield P_i, $i = 1, 2, 3, 4,$ is maximal and* *each element of order* > 1 *generates the loop* P_i^*. *The spectrum of loop* P_i^* *is* $\{1,4,5,6,7\}$ *for* $i = 1,2$*; it coincides with* $\{1,4,5,6,7,8\}$ *for* $i = 3$ *and with* $\{1,5,6,7,8,9\}$ *for* $i = 4$.

Theorem 4.2. *The semifield* P_5 *has a single subfield* H *and* $|H| = 4$. *The spectrum of the loop* P_5^* *is* $\{1,3,4,5,6,7,8\}$. *Each element of* $P_5 \setminus H$ *has order* > 3 *and generates the loop* P_5^*.

Remark 4.1. Theorem 5.2 distinguishes a semifield of order 32 having anomalous property of subfields (compared with finite fields): the semifield P_5 of order 2^5 has a subfield of order 2^2. The similar subfield S is constructed by Rua [6, Corollary 1]. In connection with the Wene's hypothesis he shows that the loop S^* is not right primitive. However, it is possible to prove that this loop S^* is one generated.

The problem on the structure of semifields of order 32 is more difficult than the case of semifields of order 16 which studied in Section 2. Up to isomorphism, there exist 2502 semifields of order 32 and they form 6 isotopic classes corresponding to six of pairswise nonisomorphic planes $\pi(i)$ $(0 \leqslant i \leqslant 5)$ [6]. This shows the following table from [6] (Tab. 6).

Table 6. Isomorphic classes of semifields of order 32

Plane	$\pi(0)$	$\pi(1)$	$\pi(2)$	$\pi(3)$	$\pi(4)$	$\pi(5)$
Left and right primitive	1	961	961	180	186	186
Only left primitive	0	0	0	6	0	7
Only right primitive	0	0	0	6	7	0
Neither L. nor R. prim.	0	0	0	1	0	0

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Вопросы строения квазиполей порядка 16 и 32

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Известный метод построения конечных проективных плоскостей трансляций (аналогично, полуполевых плоскостей) основан на их соответствии с квазиполями (соответственно, с полуполями) того же порядка. В статье рассматриваются вопросы структурного строения конечного квазиполя (возможные максимальные подполя, свойства цикличности мультипликативной лупы ненулевых элементов и возможные порядки элементов). Для конечных квазиполей малых четных порядков найдены аномальные свойства.

Ключевые слова: проективная плоскость трансляций, квазиполе, полуполе, мультипликативная лупа, порядки элементов.