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# Conditions for Convergence of the Mellin-Barnes Integral for Solution to System of Algebraic Equations 

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In the article we consider the Mellin-Barnes integral that corresponds to a monomial function of a solution to a system of $n$ algebraic equations in $n$ unknowns. We obtain the necessary condition for the convergence domain of the integral to be non empty. For $n=2$ we prove that this condition is also sufficient.

Keywords: algebraic equations, Mellin-Barnes integral, convergence.

## Introduction

In 1921, H. Mellin [1] obtained a formula for a solution to the general reduced algebraic equation

$$
\begin{equation*}
y^{m}+x_{1} y^{m_{1}}+\ldots+x_{p} y^{m_{p}}-1=0 . \tag{1}
\end{equation*}
$$

He represented its solution $y(x)=y\left(x_{1}, \ldots, x_{p}\right)$ as a multiple integral (a Mellin-Barnes integral [2]), and as a power series of hypergeometric type as well.

Moving towards the case of a system instead of one equation (1), I. A. Antipova [3] follows Mellin's approach to obtain a solution to lower triangular system of algebraic equations, where the first equation depends only on the first unknown $y_{1}$, the second one on $y_{1}, y_{2}$, and so on, and the last one depends on all $n$ unknowns $y_{1}, \ldots, y_{n}$.

In the paper we consider a reduced system of $n$ equations

$$
\begin{equation*}
y_{j}^{m_{j}}+\sum_{\lambda \in \Lambda^{(j)}} x_{\lambda}^{(j)} y^{\lambda}-1=0, j=1, \ldots, n, \tag{2}
\end{equation*}
$$

in $n$ unknowns $y=\left(y_{1}, \ldots, y_{n}\right)$, where the set of exponents $\Lambda^{(j)} \subset \mathbb{Z}_{\geqslant 0}^{n}$ is fixed, and all the coefficients $x_{\lambda}^{(j)}$ are variables. We naturally assume that the set $\Lambda^{(j)}$ in the $j$ th equation does not contain points $\lambda=\left(0, \ldots, m_{j}, \ldots, 0\right)$ and $\lambda=0$, which are the exponents of the singled out monomials $y_{j}^{m_{j}}$ and $y^{0}$ with the fixed coefficients 1 and -1 . By algebraic manipulations any system of polynomial equations in $n$ unknowns can be reduced to the form (2) [4].

A power series for the solution to (2) (more generally, for the monomial function $y^{\mu}$ of the solution) was obtained in [5]. The case of the Mellin-Barnes integral for the solution is more difficult. The goal of this paper is to find a criterion for the convergence of the hypergeometric Mellin-Barnes integral for the solution to the system (2).

[^0]Denote by $\Lambda$ the disjunctive union $\bigsqcup \Lambda^{(j)}$, and let $N$ be the number of coefficients in the system (2) (i.e. the cardinality of the set $\Lambda$ ). The exponents $\lambda$ of the monomials $y^{\lambda}=y_{1}^{\lambda_{1}} \ldots y_{n}^{\lambda_{n}}$ in the system (2) can be represented as a $(n \times N)$ matrix

$$
\Phi=\left(\lambda^{1}, \ldots, \lambda^{N}\right)
$$

where $\lambda^{k}$ is a column vector from $\Lambda$. It is assumed that in each equation the order of the columns $\lambda$ is arbitrary but fixed. The elements $\lambda \in \Lambda$ are indices for the coordinates of vectors $x=\left(x_{\lambda}\right)$ of the system coefficients. Denote the space of all coefficients by $\mathbb{C}^{N}$.

Note that $\Phi$ naturally splits in blocks corresponding to $\Lambda^{(j)}$, therefore each its row $\varphi_{i}$ is the sequence of the vectors $\varphi_{i}^{(1)}, \ldots, \varphi_{i}^{(n)}$.

A formal integral formula (without studying its convergence) for the monomial $y^{\mu}, \mu>0$, of the solution to the system (2) appeared in [6]. After some tranformations the expression for the $y^{\mu}$ can be written as

$$
\begin{equation*}
\frac{1}{(2 \pi i)^{N}} \int_{\gamma+i \mathbb{R}^{N}} \frac{\prod_{j=1}^{n} \prod_{\lambda \in \Lambda^{(j)}} \Gamma\left(u_{\lambda}^{(j)}\right) \prod_{j=1}^{n} \Gamma\left(\frac{\mu_{j}}{m_{j}}-\frac{1}{m_{j}}\left\langle\varphi_{j}, u\right\rangle\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{\mu_{j}}{m_{j}}-\frac{1}{m_{j}}\left\langle\varphi_{j}, u\right\rangle+\sum_{\lambda \in \Lambda^{(j)}} u_{\lambda}^{(j)}+1\right)} Q(u) x^{-u} d u \tag{3}
\end{equation*}
$$

where $\gamma$ is from the polyhedron

$$
\left\{u \in \mathbb{R}_{>0}^{(N)},\left\langle\varphi_{j}, u\right\rangle<\mu_{j}\right\},
$$

and $Q(u)$ is the following polynomial

$$
Q(u)=\frac{1}{m_{1} \ldots m_{n}} \operatorname{det}\left\|\delta_{j}^{i}\left(\mu_{j}-\left\langle\varphi_{j}, u\right\rangle\right)+\left\langle\varphi_{j}^{(i)}, u^{(i)}\right\rangle\right\|_{i, j=1}^{n} .
$$

Theorem 1. If the convergence domain of the integral (3) is non empty then all the matrices

$$
\left(\begin{array}{ccc}
\lambda_{1}^{(1)} & \cdots & \lambda_{1}^{(n)}  \tag{4}\\
\vdots & \ddots & \vdots \\
\lambda_{n}^{(1)} & \cdots & \lambda_{n}^{(n)}
\end{array}\right)
$$

where each column vector $\lambda^{(j)}=\left(\lambda_{1}^{(j)} \ldots \lambda_{n}^{(j)}\right)^{T}$ runs over the corresponding set $\Lambda^{(j)}$, is positive definite.

The proof of the theorem is given below in Section 1.
In Section 2 we prove that for $n=2$ this condition is also sufficient.

## 1. The necessary condition for convergence of the integral for the solution to a system of algebraic equations

Here we give the proof of theorem 1. We use the description of the convergence domain for a Mellin-Barnes integral due to L. Nilsson, M. Passare, and A. Tsikh [7]. For the integral (3) we should consider the function

$$
g(v)=\sum_{j=1}^{n} \sum_{\lambda \in \Lambda^{(j)}}\left|v_{\lambda}^{(j)}\right|+\sum_{j=1}^{n}\left|\frac{1}{m_{j}}\left\langle\varphi_{j}, v\right\rangle\right|-\sum_{j=1}^{n}\left|\frac{1}{m_{j}}\left\langle\varphi_{j}, v\right\rangle-\sum_{\lambda \in \Lambda^{(j)}} v_{\lambda}^{(j)}\right|,
$$

where $v=\left(v_{\lambda}^{(j)}\right)$ is the imaginary part of $u=\left(u_{\lambda}^{(j)}\right)$ in (3). In this case the convergence domain of the integral (3) is defined by the following conditions on the arguments $\theta_{\lambda}^{(j)}=\arg x_{\lambda}^{(j)}$ of the variable coefficients $x_{\lambda}^{(j)}$ of the system (2):

$$
\begin{equation*}
\left|\left\langle v_{\nu}, \theta\right\rangle\right|<\frac{\pi}{2} g\left(v_{\nu}\right) \tag{5}
\end{equation*}
$$

where $\left\{v_{\nu}\right\}$ is the set of vectors that generate the fan $K$, corresponding to the decomposition of $\mathbb{R}^{N}$ by $N$ coordinate hypersurfaces

$$
v_{\lambda}^{(j)}=0, \quad \lambda \in \Lambda^{(j)}, \quad j=1, \ldots, n
$$

and $2 n$ hypersurfaces

$$
\left\langle\varphi_{j}, v\right\rangle=0, \quad \sum_{\lambda \in \Lambda^{(j)}} v_{\lambda}^{(j)}-\frac{1}{m_{j}}\left\langle\varphi_{j}, v\right\rangle=0, j=1, \ldots, n
$$

Thus, the necessary and sufficient condition for the convergence of the integral is the condition $g(v)>0$ for $v \neq 0$.

Our integral contains a polynomial factor $Q(u)$, therefore we begin with the following proposition.

Proposition 1. The convergence domain of the integral (3) does not depend on the polynomial factor $Q(u)$.

Proof. By using polylinearity of determinants, we write $Q(u)$ as

$$
\begin{aligned}
Q(u)= & \frac{1}{m_{1} \ldots, m_{n}} \operatorname{det}\left\|\delta_{j}^{i}\left(\mu_{j}-\left\langle\varphi_{j}, u\right\rangle\right)+\left\langle\varphi_{j}^{(i)}, u^{(i)}\right\rangle\right\|= \\
& =\sum_{q=1}^{n} \sum_{|I|=q} \prod_{j \notin I}\left(\mu_{j}-\left\langle\varphi_{j}, u\right\rangle\right) \operatorname{det}\left\|\left\langle\varphi_{r}^{(t)}, u^{(t)}\right\rangle\right\|_{r, t \in I} .
\end{aligned}
$$

Note that the determinants in this expression are linear with respect to each $u_{\lambda}^{(j)}$, so we can write $Q(u)$ as a sum of expressions of the form

$$
\text { const } \prod_{k \notin I}\left(\mu_{k}-\left\langle\varphi_{k}, u\right\rangle\right) u_{\lambda^{\left(i_{1}\right)}}^{\left(i_{1}\right)} \ldots u_{\lambda^{\left(i_{q}\right)}}^{\left(i_{q}\right)}
$$

Then the integral (3) is the sum of integrals of the form

$$
\begin{align*}
& \frac{\text { const }}{(2 \pi i)^{N}} \int_{\gamma+i \mathbb{R}^{N}} \frac{\prod_{j=1}^{n} \prod_{\lambda \in \Lambda^{(j)}} \Gamma\left(u_{\lambda}^{(j)}\right) \prod_{j=1}^{n} \Gamma\left(\frac{\mu_{j}}{m_{j}}-\frac{1}{m_{j}}\left\langle\varphi_{j}, u\right\rangle\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{\mu_{j}}{m_{j}}-\frac{1}{m_{j}}\left\langle\varphi_{j}, u\right\rangle+\sum_{\lambda \in \Lambda^{(j)}} u_{\lambda}^{(j)}+1\right)} \times  \tag{6}\\
& \quad \times \prod_{k \notin I}\left(\mu_{k}-\left\langle\varphi_{k}, u\right\rangle\right) u_{\lambda^{\left(i_{1}\right)}}^{\left(i_{1}\right)} \ldots u_{\lambda^{\left(i_{q}\right)}}^{\left(i_{q}\right)} x^{-u} d u .
\end{align*}
$$

Apply the identity $z \Gamma(z)=\Gamma(z+1)$ to these integrals, then the integrals (6) lose polynomial factors, but the linear homogeneous parts of arguments of the Gamma functions remain the
same. Thus we see that the polynomial $Q(u)$ does not affect the convergence domain of the integral (3).

Now we proceed to the proof of Theorem 1.
Note that the set of vectors $\left\{v_{1}, \ldots, v_{d}\right\}$, which generate the fan $K$, is the direction vectors for the lines that are the intersections of all possible independent subsets of $N-1$ hyperplanes from the list above. In practice, we can obtain such vectors as vector products of the normal vectors to these hyperplanes. Then the scalar product $\left\langle\varphi_{j}, v\right\rangle$ is nothing else but the determinant made up from $N-1$ normal vectors to $v$ and the vector $\varphi_{j}$.

To prove that all principal minors of the matrix (4) are positive we use induction on the order $k$ of these minors.

Note that the set $\left\{v_{1}, \ldots, v_{d}\right\}$ includes all the basis vectors $e_{\lambda}^{(i)}, \lambda \in \Lambda^{(i)}, i=1, \ldots, n$. By definition of $v$, the coordinate of the vector $\varphi_{j}$ on the position $\lambda^{(i)}$ is $\lambda_{j}^{(i)}$, therefore

$$
g\left(e_{\lambda}^{(i)}\right)=1+\sum_{j=1}^{n}\left|\frac{\lambda_{j}^{(i)}}{m_{j}}\right|-\sum_{j=1}^{n}\left|\frac{\lambda_{j}^{(i)}}{m_{j}}-\delta_{j}^{i}\right|=1+\left|\frac{\lambda_{i}^{(i)}}{m_{i}}\right|-\left|\frac{\lambda_{i}^{(i)}}{m_{i}}-1\right| .
$$

Since, by the theorem assumption, the polyhedron (5) is non empty, we have $g\left(e_{\lambda}^{(i)}\right)>0$, which implies that $\frac{\lambda_{i}^{(i)}}{m_{i}}$ and 1 have the same sign, i.e. $\lambda_{i}^{(i)}>0$ for all $\lambda \in \Lambda^{(i)}$. This proves that all principal minors of order 1 of the matrix (4) are positive. This constitutes the induction base.

Assume that all principal minors of order $k$ of the matrix (4) are positive. Consider the direction vector $v$ for the line obtained by the intersection of $N-(k+1)$ coordinate hyperplanes and hyperplanes $\left\langle\varphi_{j_{1}}, u\right\rangle=0, \ldots,\left\langle\varphi_{j_{k}}, u\right\rangle=0$. What is important here is that this set of coordinate hyperplanes does not include the hyperplanes orthogonal to $e_{\lambda^{\left(j_{1}\right)}}^{\left(j_{1}\right)}, \ldots, e_{\lambda^{\left(j_{k}\right)}}^{\left(j_{k}\right)}$ and a hyperplane orthogonal to some $e_{\lambda^{(s)}}^{(s)}, s \neq j_{1}, \ldots, s \neq j_{k}$.

Let us compute the function $g$ at the point $v$. The coordinates $v_{\lambda}^{(j)}=\left\langle v, e_{\lambda}^{(j)}\right\rangle$ of $v$ are equal to zero for all $e_{\lambda}^{(j)}$ except for $e_{\lambda^{j_{1}}}^{\left(j_{1}\right)}, \ldots, e_{\lambda^{j_{k}}}^{\left(j_{k}\right)}$ and $e_{\lambda^{s}}^{(s)}$, the scalar products $\left\langle\varphi_{j}, v\right\rangle$ are also equal to zero for all $j=j_{1}, \ldots, j_{k}$. Denote $J=\left\{j_{1}, \ldots, j_{k}\right\}$, and $\bar{J}=\{1, \ldots, n\} \backslash(J \cup\{s\})$, then

$$
\begin{aligned}
g(v)= & \sum_{p \in J}\left|v_{\lambda^{p}}^{(p)}\right|+\left|v_{\lambda^{(s)}}^{(s)}\right|+\sum_{q \in \bar{J}}\left|\frac{1}{m_{q}}\left\langle\varphi_{q}, v\right\rangle\right|+\left|\frac{1}{m_{s}}\left\langle\varphi_{s}, v\right\rangle\right|- \\
& -\sum_{p \in J}\left|v_{\lambda^{p}}^{(p)}\right|-\left|v_{\lambda^{(s)}}^{(s)}-\frac{1}{m_{s}}\left\langle\varphi_{s}, v\right\rangle\right|-\sum_{q \in \bar{J}}\left|\frac{1}{m_{q}}\left\langle\varphi_{q}, v\right\rangle\right| .
\end{aligned}
$$

This simplifies to

$$
g(v)=\left|v_{\lambda^{(s)}}^{(s)}\right|+\left|\frac{1}{m_{s}}\left\langle\varphi_{s}, v\right\rangle\right|-\left|v_{\lambda^{(s)}}^{(s)}-\frac{1}{m_{s}}\left\langle\varphi_{s}, v\right\rangle\right| .
$$

Given the fact that $v$ is the generalized vector product of $N-1$ vectors, its coordinates are minors of order $N-1$ of the matrix made up from the normal vectors to the hyperplanes. Hence the scalar product $\left\langle\varphi_{j}, v\right\rangle$ can be written as a determinant of order $N$. Thus we have the
following expression for $g(v)$

$$
\begin{aligned}
& \operatorname{det}\left(\begin{array}{ccc}
\lambda_{j_{1}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{1}}^{\left(j_{k}\right)} \\
\vdots & \ddots & \vdots \\
\lambda_{j_{k}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{k}}^{\left(j_{k}\right)}
\end{array}\right)\left|+\left|\operatorname{det}\left(\begin{array}{cccc}
\lambda_{j_{1}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{1}}^{\left(j_{k}\right)} & \lambda_{j_{1}}^{(s)} \\
\vdots & \ddots & \vdots & \vdots \\
\lambda_{j_{k}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{k}}^{\left(j_{k}\right)} & \lambda_{j_{k}}^{(s)} \\
\lambda_{s}^{\left(j_{1}\right)} & \ldots & \lambda_{s}^{\left(j_{k}\right)} & \lambda_{s}^{(s)}
\end{array}\right)\right|-\right. \\
& -\left|\operatorname{det}\left(\begin{array}{cccc}
\lambda_{j_{1}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{1}}^{\left(j_{k}\right)} \\
\vdots & \ddots & \vdots \\
\lambda_{j_{k}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{k}}^{\left(j_{k}\right)}
\end{array}\right)-\operatorname{det}\left(\begin{array}{cccc}
\lambda_{j_{1}}^{\left(j_{1}\right)} & \ldots & \lambda_{j_{1}}^{\left(j_{k}\right)} & \lambda_{j_{1}}^{(s)} \\
\vdots & \ddots & \vdots & \vdots \\
\lambda_{\left.j_{k}\right)}^{\left(j_{1}\right)} & \ldots & \lambda_{\left.j_{k}\right)}^{\left(j_{k}\right)} & \lambda_{j_{k}}^{(s)} \\
\lambda_{s}^{\left(j_{1}\right)} & \ldots & \lambda_{s}^{\left(j_{k}\right)} & \lambda_{s}^{(s)}
\end{array}\right)\right|,
\end{aligned}
$$

therefore the positivity of $g(v)$ yields the principal minor of order $(k+1)$ must have the same sign as the principal minor of order $k$, which is positive by the induction hypothesis.

## 2. The criterion for the convergence of the integral (3) for $\mathrm{n}=2$

In this section we prove that for a system of two algebraic equations the necessary condition for the convergence of the Mellin-Barnes integral is also sufficient.

Consider a system of two algebraic equations in two unknowns $y_{1}, y_{2}$

$$
\left\{\begin{array}{l}
y_{1}^{m_{1}}+\sum_{i=1}^{p_{1}} x_{i}^{(1)} y_{1}^{\alpha_{j}^{(1)}} y_{2}^{\beta_{i}^{(1)}}-1=0  \tag{7}\\
y_{2}^{m_{2}}+\sum_{j=1}^{p_{2}} x_{j}^{(2)} y_{1}^{\alpha_{j}^{(2)}} y_{2}^{\beta_{j}^{(2)}}-1=0
\end{array}\right.
$$

The Mellin-Barnes integral corresponding to the monomial function $y^{\mu}$ of its solution is the integral over the imaginary subspace $\gamma+i \mathbb{R}^{p_{1}+p_{2}}$ of

$$
\frac{\prod_{i=1}^{2} \prod_{j=1}^{p_{i}} \Gamma\left(u_{j}^{(i)}\right) \Gamma\left(\frac{\mu_{1}}{m_{1}}-\frac{1}{m_{1}}\langle\alpha, u\rangle\right) \Gamma\left(\frac{\mu_{2}}{m_{2}}-\frac{1}{m_{2}}\langle\beta, u\rangle\right) Q(u) x^{-u}}{\Gamma\left(\frac{\mu_{1}}{m_{1}}-\frac{1}{m_{1}}\langle\alpha, u\rangle+\left|u^{(1)}\right|+1\right) \Gamma\left(\frac{\mu_{2}}{m_{2}}-\frac{1}{m_{2}}\langle\beta, u\rangle+\left|u^{(2)}\right|+1\right)}
$$

Theorem 2. The domain of convergence of the integral (8) is non-empty if and only if all the exponents $\alpha_{j}^{(1)}, \beta_{i}^{(2)}$ and all determinants $\Delta_{i j}=\alpha_{i}^{(1)} \beta_{j}^{(2)}-\alpha_{j}^{(2)} \beta_{i}^{(1)}$ are positive.

Proof. Taking into account Theorem 1, we only need to prove that positivity of all these minors is sufficient for the convergence of the integral.

For our integral the function $g(v)$ has the form

$$
\begin{aligned}
g(v)= & \sum_{j=1}^{p_{1}}\left|v_{j}^{(1)}\right|+\frac{1}{m_{1}}|\langle\alpha, v\rangle|-\left|\sum_{j=1}^{p_{1}} v_{j}^{(1)}-\frac{1}{m_{1}}\langle\alpha, v\rangle\right|+ \\
& +\sum_{j=1}^{p_{2}}\left|v_{j}^{(2)}\right|+\frac{1}{m_{2}}|\langle\beta, v\rangle|-\left|\sum_{j=1}^{p_{2}} v_{j}^{(2)}-\frac{1}{m_{2}}\langle\beta, v\rangle\right| .
\end{aligned}
$$

It can be easily seen that $g(v)$ is a sum of two expressions and each of them is non-negative by the triangle inequality. Hence $g(v)$ is non-negative, moreover $g(v)=0$ if and only if both these expressions are zero. This happens if and only if in each group:

1) $\left\{v_{i}^{(1)}, i=1, \ldots, p_{1}\right\},-\frac{1}{m_{1}}\langle\alpha, v\rangle$;
2) $\left\{v_{j}^{(2)}, j=1, \ldots, p_{2}\right\},-\frac{1}{m_{2}}\langle\beta, v\rangle$,
all the elements have the same sign.
Thus, for $g$ there are four possibilities:
3) all elements in both groups are non-negative;
4) all elements in both groups are non-positive;
5) the elements of the first group are non-negative, and the elements of the second group are non-positive;
6) the elements of the first group are non-positive, and the elements of the second group are non-negative.

Now we shall consider cases 1 and 3 , since cases 2 and 4 are similar to them.

1. In this case we have a system of inequalities

$$
\begin{cases}v_{j}^{(1)} \geqslant 0, & j=1, \ldots, p_{1} \\ v_{j}^{(2)} \geqslant 0, & j=1, \ldots, p_{2} \\ -\langle\alpha, v\rangle \geqslant 0, \\ -\langle\beta, v\rangle \geqslant 0\end{cases}
$$

We multiply inequalities $v_{j}^{(1)} \geqslant 0$ by non-negative constants $\alpha_{j}^{(1)}, j=1, \ldots,[k], \ldots, p_{1}$, and inequalities $v_{j}^{(2)} \geqslant 0$ by non-negative constants $\alpha_{j}^{(2)}, j=1, \ldots, p_{2}$. After that we add the resulting inequalities to the inequality $-\langle\alpha, v\rangle \geqslant 0$. As a result, we obtain

$$
-\alpha_{k}^{(1)} v_{k}^{(1)} \geqslant 0,
$$

which in combination with the inequality $v_{k}^{(1)} \geqslant 0$ and positivity of $\alpha_{k}^{(1)}$ gives $v_{k}^{(1)}=0$ for all $k=1, \ldots, p_{1}$.

Similarly we obtain the inequality

$$
-\beta_{k}^{(2)} v_{k}^{(2)} \geqslant 0
$$

together with $v_{k}^{(2)} \geqslant 0$ and $\beta_{k}^{(2)}>0$ it gives $v_{k}^{(2)}=0$ for all $k=1, \ldots, p_{2}$.
Thus the system of inequalities has the only solution $v=0$.
2 . Let us prove that the only solution to the system

$$
\begin{cases}v_{j}^{(1)} \geqslant 0, & j=1, \ldots, p_{1} \\ -v_{j}^{(2)} \geqslant 0, & j=1, \ldots, p_{2} \\ -\langle\alpha, v\rangle \geqslant 0, & \\ \langle\beta, v\rangle \geqslant 0 & \end{cases}
$$

is $v=0$.
First $p_{1}+p_{2}$ inequalities of the system define an orthant $U$ in $\mathbb{R}^{|p|}$

$$
U:=\left\{v_{j}^{(1)} \geqslant 0, j=1, \ldots, p_{1},-v_{j}^{(2)} \geqslant 0, j=1, \ldots, p_{2}\right\} .
$$

Let $\tau^{(j)}=\left(\tau_{1}^{(j)}, \ldots \tau_{p_{j}}^{(j)}\right) \in \mathbb{R}_{\geqslant 0}^{p_{j}}$ such that $\left|\tau^{(j)}\right|=\sum_{i=1}^{p_{j}} \tau_{i}^{(j)}=1, j=1,2$. Then the orthant $U$ can be represented as a union of two dimensional sets

$$
U=\bigcup_{\tau} L_{\tau}
$$

where

$$
\begin{array}{r}
L_{\tau}=\left\{v \in \mathbb{R}^{|p|}: v_{j}^{(1)}=t_{1} \tau_{j}^{(1)}, j=1, \ldots, p_{1},\right. \\
\left.v_{j}^{(2)}=-t_{2} \tau_{j}^{(2)}, j=1, \ldots, p_{2}, t \in \mathbb{R}_{\geqslant 0}^{2}\right\} .
\end{array}
$$

Consider the restriction of the inequalities $-\langle\alpha, v\rangle \geqslant 0$ and $\langle\beta, v\rangle \geqslant 0$ to $L_{\tau}$ to get the system

$$
\left\{\begin{array}{l}
t_{1} \geqslant 0 \\
t_{2} \geqslant 0 \\
-t_{1}\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle+t_{2}\left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle \geqslant 0 \\
t_{1}\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle-t_{2}\left\langle\beta^{(2)}, \tau^{(2)}\right\rangle \geqslant 0
\end{array}\right.
$$

Introduce two non-negative parameters $s_{1}, s_{2}$ and rewrite the system of inequalities as

$$
\left\{\begin{array}{l}
t_{1} \geqslant 0, t_{2} \geqslant 0 \\
s_{1} \geqslant 0, s_{2} \geqslant 0 \\
-t_{1}\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle+t_{2}\left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle=s_{1} \\
t_{1}\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle-t_{2}\left\langle\beta^{(2)}, \tau^{(2)}\right\rangle=s_{2}
\end{array}\right.
$$

We solve this system using Cramer's rule and find

$$
\begin{aligned}
& t_{1}=\frac{-s_{1}\left\langle\beta^{(2)}, \tau^{(2)}\right\rangle-s_{2}\left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle}{\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle\left\langle\beta^{(2)}, \tau^{(2)}\right\rangle-\left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle}, \\
& t_{2}=\frac{-s_{2}\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle-s_{1}\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle}{\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle\left\langle\beta^{(2)}, \tau^{(2)}\right\rangle-\left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle} .
\end{aligned}
$$

Since

$$
\left|\begin{array}{cc}
\left\langle\alpha^{(1)}, \tau^{(1)}\right\rangle & \left\langle\alpha^{(2)}, \tau^{(2)}\right\rangle \\
\left\langle\beta^{(1)}, \tau^{(1)}\right\rangle & \left\langle\beta^{(2)}, \tau^{(2)}\right\rangle
\end{array}\right|=\sum_{j=1}^{p_{1}} \sum_{k=1}^{p_{2}} \tau_{j}^{(1)} \tau_{k}^{(2)}\left|\begin{array}{cc}
\alpha_{j}^{(1)} & \alpha_{k}^{(2)} \\
\beta_{j}^{(1)} & \beta_{k}^{(2)}
\end{array}\right|>0
$$

and $t_{1} \geqslant 0, t_{2} \geqslant 0, s_{1} \geqslant 0, s_{2} \geqslant 0$, the only solution to the obtained system of inequalities and equations is $t_{1}=t_{2}=0$, and $v=0$.

Thus in the inequality

$$
|\langle\theta, v\rangle| \leqslant \frac{\pi}{2} g(v)
$$

that defines the convergence domain for the integral the right hand side is always non-negative and equals zero only if $v=0$ It follows that convergence domain of the integral is non empty.

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## Условия сходимости интеграла Меллина-Барнса для решения системы алгебраических уравнений

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В статъе рассматривается интеграл Меллина-Барнса, соответствующий мономиальной функиии решения системы $n$ алгебраических уравнений от $n$ неизвестных. Найдено необходимое условие при котором интеграл имеет непустую область сходимости. Для $n=2$ показано, что приведенное условие является такэне и достаточным.

Ключевые слова: алгебрачческие уравнения, интеграл Меллина-Барнса, сходимость.


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