Journal of Siberian Federal University. Mathematics & Physics 2014, 7(3), 331–338

УДК 515.17 + 517.545

Univalent Differentials of Integer Order on Variable Torus

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Received 06.05.2013, received in revised form 06.01.2014, accepted 20.06.2014

In this paper we give a full description for divisors of elementary differentials of all kinds. An analog of Appell's expansion formula for univalent functions on a variable torus is obtained. All basic type of vector bundles of meromorphic differentials of integer order over a Teichmüller space for torus are studied.

Keywords: univalent meromorphic differentials of integer order, divisors, vector bundles over Teichmüller space for torus.

Introduction

Univalent differentials (of order $q = 1$ and $q = 2$ in particular) even on a fixed surface have found a lot of applications in mathematical physics (algebraic-geometric integration of nonlinear equations in the works of S.P. Novikov, I.M. Krichever), in theoretical physics (R. Dick), and also in analytic number theory in the works by H.M. Farkas and I. Kra [1].

The main difference of the results of this paper from the classical ones found in the books by J. Springer [2], H.M. Farkas and I. Kra [1] and in other books on the geometric function theory on a compact Riemann surface is that we consider all objects on a variable compact Riemann surface F_μ of genus $g = 1$ (torus) [3, 4]. For the general theory of univalent differentials a big role is played by so called elementary differentials of integer order q that have the minimal number of poles: either one pole of order $\geqslant 2$, or two simple poles, and depend holomorphically on the modules of the torus F_μ. For the first time we give a full description for divisors of elementary abelian q-differentials of all kinds. An analog of Appel's expansion formula for univalent functions on a variable torus is obtained. We study also all basic types of vector bundles of meromorphic differentials of integer order $q \neq 1$ over a Teichmüller space for torus.

Preliminaries

Let F_0 be a fixed compact Riemann surface of genus $g = 1$, $F_0 = \mathbb{C}/\Gamma$, where Γ is a group with two generators $A_1(z) = z + \omega$, $B_1(z) = z + \omega'$, $\mathrm{Im}\dfrac{\omega'}{\omega} > 0$. Let $\mu_0 = \dfrac{\omega'}{\omega}$. The fundamental group of the surface F_0 has an algebraic representation

$$\Gamma \cong \pi_1(F_0) =< a_1, b_1 : a_1 b_1 = b_1 a_1 > .$$

The class $[F_0, \{a_1, b_1\}]$ of conformally equivalent marked compact Riemann surfaces of genus one is uniquely defined by a complex parameter (module) $\mu_0 = \dfrac{\omega'}{\omega}$, which lies in the upper half plane $H = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. Here $F_0 = \mathbb{C}/\Gamma_0$ where Γ_0 is the group generated by two generators

$$A_{01}(z) = z + 1,\ B_{01}(z) = z + \mu_0.$$

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Every other class $\left[F_\mu, \{a_1^\mu, b_1^\mu\}\right]$ of conformally equivalent marked compact Riemann surfaces of genus one is uniquely defined by a complex parameter (module) $\mu \in H$ and $F_\mu = \mathbb{C}/\Gamma_\mu$ where Γ_μ is generated by $A_{\mu 1}(z) = z+1$, $B_{\mu 1}(z) = z+\mu$. Moreover, there is a quasiconformal mapping $\widetilde{f}_\mu : F_0 \to F_\mu$, and its lifting $f_\mu : \mathbb{C} \to \mathbb{C}$ on the universal covering surface gives an isomorphism between the marked group Γ_0 and the marked group $\Gamma_\mu = f_\mu \Gamma_0 f_\mu^{-1}$ with $a_1^\mu = \widetilde{f}_\mu(a_1)$, $b_1^\mu = \widetilde{f}_\mu(b_1)$.

The Teichmüller space $\mathbb{T}_1 = \mathbb{T}_1(F_0)$, of the classes $\left[F_\mu, \{a_1^\mu, b_1^\mu\}\right]$ of conformally equivalent marked compact Riemann surfaces of genus one can be parametrized by points from H, and it is a 1-dimensional complex analytic manifold. This space with the Teichmüller metric is biholomorphically isometric to the space $\left(H, \dfrac{|dz|}{2y}\right)$, $z = x+iy$, with constant negative curvature [3].

Next, for every natural number $n \geqslant 1$ there is a fiber bundle over $\mathbb{T}_1$ such that its fibre over $\mu \in \mathbb{T}_1$ as the space of all integer divisors of degree n on F_μ. Locally holomorphic sections of this bundle define on every F_μ an integer divisor D^μ of degree n that holomorphically depends on μ [5, p. 261, 268].

Definition. *A q-differential ϕ with respect to the group Γ on $\mathbb{C}$ is a differential $\phi(z)dz^q$ such that*

$$\phi(Tz)(T'z)^q = \phi(z), \quad z \in \mathbb{C}, \quad T \in \Gamma.$$

In particular, for $q = 0$, this is a meromorphic function with respect to Γ.

Let D be a divisor on F. Introduce following the spaces: $L(D;F)$ of meromorphic functions f on F such that $(f) \geqslant D$, and $\Omega^q(D;F)$ of meromorphic q-differentials ω on F such that $(\omega) \geqslant D$. Denote by $r(D) = \dim_{\mathbb{C}} L(D;F)$ and $i_q(D) = \dim_{\mathbb{C}} \Omega^q(D;F)$ the dimensions of these complex vector spaces.

Theorem (Riemann-Roch) [1, p. 73]. *Let F be a compact Riemann surface of genus $g = 1$. Then for every divisor D on F*

$$r(D^{-1}) = \deg D + i(D).$$

Theorem (Riemann-Roch for q-differentials) [4, p.43]. *For every $q \in \mathbb{Z}$ on a compact Riemann surface F of genus one*

$$i_q(D) = -\deg D + r(1/D).$$

Theorem (Abel) [1, p.93; 4, p.67]. *Let $[F; \{a_1, b_1\}]$ be a marked compact Riemann surface of genus one and*

$$D = \frac{P_1^{\alpha_1} \ldots P_m^{\alpha_m}}{Q_1^{\beta_1} \ldots Q_s^{\beta_s}}$$

be a divisor of degree zero on F. Then there exists a function f on F with

$$(f) = D \Leftrightarrow \varphi(D) = \sum_{j=1}^{m} \alpha_j \varphi(P_j) - \sum_{k=1}^{s} \beta_k \varphi(Q_k) = 0$$

in $J(F) = \varphi(F)$, where φ is the Jacobi mapping from F to $J(F)$.

1. Univalent elementary q-differentials on a variable torus

In this section we establish the general form of elementary univalent q-differentials on the torus F_μ.

Let us find first the general form of q-differentials $\tau_{q;Q}^{(m)}$ with the only pole $Q = Q(\mu)$ exactly of order $m \geqslant 2$ on F_μ, $q \in \mathbb{Z}$.

By the Riemann-Roch theorem for q-differentials on F_μ [4, p.43] we find the dimension

$$i_q\left(\frac{1}{Q^m}\right) = \dim_{\mathbb{C}} \Omega^q\left(\frac{1}{Q^m}; F_\mu\right) = -\deg D + r(Q^m),$$

where $D = \dfrac{1}{Q^m}$. Hence $i_q\left(\dfrac{1}{Q^m}\right) = m \geqslant 2$. Here $r(Q^m) = 0$, so $\deg(Q^m) = m > 0$ under our conditions. This can also be proved by contradiction: if there existed a function g on F_μ such that $(g) \geqslant Q^m$, then $0 = \deg(g) \geqslant \deg(Q^m) \geqslant 2$.

Since $\deg Q^{m-1} = m - 1 \geqslant 1 > 0$,

$$i_q\left(\frac{1}{Q^{m-1}}\right) = -\deg\left(\frac{1}{Q^{m-1}}\right) + r(Q^{m-1}) = m - 1.$$

Therefore, $i_q\left(\dfrac{1}{Q^m}\right) = i_q\left(\dfrac{1}{Q^{m-1}}\right) + 1$. Hence there exists a q-differential $\tau_{q;Q}^{(m)}$ with the pole exactly of order m at the point Q on F_μ, i. e. the divisor $\left(\tau_{q;Q}^{(m)}\right) = \dfrac{R_1 \cdots R_m}{Q^m}$ on F_μ, $R_j \neq Q$, $j = 1, ..., m$.

Construct now such a differential explicitly: $\tau_{q;Q}^{(m)} = f dz^q$, $q \in \mathbb{Z}$, where dz is a holomorphic differential on F_μ that depends holomorphically on μ. The univalent function f has the divisor $(f) = \dfrac{R_1 \cdots R_m}{Q^m}$, since $(dz) = 1$. By the Abel theorem [4] we get the equation

$$\varphi_{P_0}(\mu)(R_1 \cdots R_m) - \varphi_{P_0}(\mu)(Q^m) = 0$$

in the Jacobi manifold $J(F_\mu)$, where P_0 is an initial point different from Q. We understand this equation as an equality in the variable Jacobian $J(F_\mu)$, i. e. in the fibre of the universal Jacobi bundle that lies over the marked surface F_μ. Therefore

$$\varphi(R_1) = \varphi(Q^m) - \varphi(R_2 \cdots R_m). \tag{1}$$

Thus, for zeros of the function f we have $m - 1 \geqslant 1$ free parameters that can be arbitrarily chosen on F_μ locally holomorphically depending on μ. By the theorem of C.Earle [5, p. 268] we can choose the divisor $R_2 \cdots R_m$ in such a way that it does not contain the point Q on F_μ and is a locally holomorphic section of the bundle of integer divisors of degree $m - 1$ over the Teichmüller space $\mathbb{T}_1$.

Solving the Jacobi problem in the universal bundle over $\mathbb{T}_1$, we find the divisor R_1 on F_μ, which is a unique solution to the equation (1) [1, p. 95, 97]. Here the point $R_1 \neq Q$ and R_1 depends holomorphically on our parameter, since the right hand side in (1) was chosen as holomorphically depending on μ. Indeed, if $R_1 = Q$ then consider the divisor $D = R_2 ... R_m$ with $m - 1$ free points. By the theorem on free points [1, p. 125] we have the inequality

$$m - 1 + 1 \leqslant \left(\frac{1}{D}\right) = m - 1 + i(D),$$

and hence $1 \leqslant i(D)$. Therefore we see that there exists a differential $\omega \neq 0$, $(\omega) \geqslant D$. Consequently, we have an impossible inequality

$$0 = \deg(\omega) \geqslant \deg D = m - 1 \geqslant 1.$$

Thus, the divisor $(\tau_{q;Q}^{(m)}) = \dfrac{R_1 R_2 \cdots R_m}{Q^m}$ is the most general for q-differentials $\tau_{q;Q}^{(m)}$ with the only pole exactly of order $m \geqslant 2$ on F_μ with the point $Q \in F_\mu$. Therefore, we have proved the following theorem.

Theorem 1.1. *On a variable torus F_μ for every natural number $m > 1$, $q \in \mathbb{Z}$ there exists an elementary q-differential $\tau^{(m)}_{q;Q}$ with the pole at the point $Q = Q(\mu) \in F_\mu$ exactly of order m locally holomorphically depending on μ, whose divisor is of the form*

$$\left(\tau^{(m)}_{q;Q}\right) = \frac{R_1 \cdots R_m}{Q^m},$$

where

$$\varphi(R_1) = \varphi(Q^m) - \varphi(R_2 \cdots R_m).$$

Here the divisors $R_2 ... R_m$ and $Q = Q(\mu)$ are chosen as locally holomorphic sections of the bundle of integer divisors over $\mathbb{T}_1$ of degrees $m-1$ and 1 respectively for μ from a sufficiently small neighborhood $U(\mu_0) \subset \mathbb{T}_1$.

Corollary 1.1. *Under the assumptions of theorem 1.1 there exists a q-differential*

$$\widetilde{\tau}^{(m)}_{q;Q} = \left(\frac{1}{z^m} + O(1)\right) dz^q$$

in a neighborhood of the point Q on F_μ.

Proof. For every $q \in \mathbb{Z}$, $m > 1$, there exists a q-differential

$$\tau^{(m)}_{q;Q} = \left(\frac{c_{-m}}{z^m} + ... + \frac{c_{-1}}{z^1} + O(1)\right) dz^q,\ c_{-m} \neq 0$$

in a neighborhood of the point Q on F_μ. The Abelian 1-differential $\dfrac{\tau^{(m)}_{q;Q}}{dz^{q-1}}$ has the residue $c_{-1} = 0$ at the point Q by the residue theorem on F_μ. For $m = 2$ we have a q-differential

$$\widetilde{\tau}^{(2)}_{q;Q} = \frac{1}{c_{-2}} \tau^{(2)}_{q;Q} = \left(\frac{1}{z^2} + O(1)\right) dz^q.$$

By induction, for every $m > 1$ we can get a q-differential

$$\widetilde{\tau}^{(m)}_{q;Q} = \left(\frac{1}{z^m} + O(1)\right) dz^q$$

in a neighborhood of the point Q on F_μ. Moreover, such q-differential can be obtained by differentiating with respect to the parameter $z(Q)$ from the formula

$$\widetilde{\tau}^{(m)}_{q;Q} = \frac{1}{(-m+1)...(-2)} \left[\widetilde{\tau}^{(2)}_{q;Q}\right]^{(m-2)}_{Q}.$$

Thus, we have proved the corollary. □

Remark 1.1. For every $q \in \mathbb{Z}$ by the Riemann-Roch theorem for q-differentials we have the equality

$$i_q(D) = -\deg D + r\left(\frac{1}{D}\right)$$

and $i_q(1) = 1$. Therefore $i_q\left(\dfrac{1}{Q}\right) = 1 + r(Q) = 1$. Also $i_q\left(\dfrac{1}{Q}\right) = r\left(\dfrac{1}{Q}\right) = 1$, where the first equality follows from the isomorphism given by division by the differential dz^q on the torus F_μ. Because of that we have $i_q\left(\dfrac{1}{Q}\right) = 1 = i_q(1)$. Therefore, there is no a q-differential $\tau_{q;Q}$ on the torus F_μ with the only pole at Q exactly of order one for every $q \in \mathbb{Z}$. This fact can be also proved by using the residue theorem for abelian differentials of order one on F_μ [7, 8].

Now we establish the general form for univalent q-differentials $\tau_{q;Q_1Q_2}$ of the third kind with exactly two simple poles at different points $Q_1 = Q_1(\mu)$ and $Q_2 = Q_2(\mu)$ on F_μ that depend holomorphically on the parameter μ.

Proposition 1.1. *On a variable torus F_μ for every integer q there exists an elementary q-differential $\tau_{q;Q_1Q_2}$ of the third kind with exactly two simple poles at different points $Q_1 = Q_1(\mu)$ and $Q_2 = Q_2(\mu)$ on F_μ locally holomorphically depending on μ with the divisor $(\tau_{q;Q_1Q_2}) = \frac{R_1R_2}{Q_1Q_2}$, where $\varphi(R_1) = \varphi(Q_1Q_2) - \varphi(R_2)$ in $J(F_\mu)$. Here the points R_2, $Q_1 = Q_1(\mu)$, $Q_2 = Q_2(\mu)$ can be chosen as locally holomorphic sections of the bundle of integer divisors of degree one over $\mathbb{T}_1$ for μ from a sufficiently small neighborhood $U(\mu_0) \subset \mathbb{T}_1$.*

Proof. For $q \in \mathbb{Z}$, set $\tau_{q;Q_1Q_2} = \tau_{Q_1Q_2}dz^{q-1}$, where $\tau_{Q_1Q_2}$ is the classical abelian differential of the third kind on F_μ that depends holomorphically on μ [1, p.51; 6].

Such a differential $\tau_{q;Q_1Q_2}$ can also be taken as $\tau = fdz^q$, where f is a univalent function with the divisor $(f) = \frac{R_1R_2}{Q_1Q_2}$. By Abel's theorem we have the equality

$$\varphi(R_1) = \varphi(Q_1Q_2) - \varphi(R_2) \tag{2}$$

in $J(F_\mu)$. The divisor R_1 is the only solution to the equation (2). Moreover, we can take the points such that $R_j \neq Q_1, Q_2$, $j = 1, 2$. Indeed, if $R_1 = Q_1$ for $R_2 \neq Q_1, Q_2$, then $\varphi(R_2) = \varphi(Q_2)$ and $R_2 = Q_2$. We arrive at a contradiction which proves the preposition. □

2. An analog of Appell's expansion formula for meromorphic functions on a variable torus

In this section we find an analog of Appell's formula where the terms (summands) have poles only at one point on F_μ and depend holomorphically on μ.

Let f be a function on a variable torus F_μ with s simple poles $Q_1, Q_2, ..., Q_s$ and residues $c_1, ..., c_s$ at them respectively. Consider the expression

$$f_1 = f - c_1T^{(1)}_{Q_1} - ... - c_sT^{(1)}_{Q_s},$$

where $T^{(1)}_{Q_k}(z) = -\int \tau^{(2)}_{Q_k}$ is a branch of the elementary abelian integral of the second kind [1, p.51] with only simple pole at Q_k and the residue $+1$ at Q_k depends holomorphically on μ, $k = 1, ..., s$. Then f_1 is an abelian integral of the first kind on the torus F_μ. Therefore $f_1 = C_1 \int dz + C = C_1z + C$ on F_μ.

Theorem 2.1. *Let f be a function on a variable torus F_μ with simple poles $Q_1, ..., Q_l$ and residues $c_1, ..., c_l$ at them, and poles at $Q_{l+1}, ..., Q_s$ with multiplicities $n_{l+1}, ..., n_s, n_k \geqslant 2, k = l+1, ..., s$, and given principal parts at them. Then*

$$f = C_1z + C + \sum_{j=1}^{l} c_jT^{(1)}_{Q_j} +$$

$$+ \sum_{k=l+1}^{s} \left[A_{k,1}T^{(1)}_{Q_k} + A_{k,2}\frac{\partial T^{(1)}_{Q_k}}{\partial Q_k} + \frac{A_{k,3}}{2!}\frac{\partial^2 T^{(1)}_{Q_k}}{\partial Q_k^2} + ... + \frac{A_{k,n_k}}{(n_k-1)!}\frac{\partial^{n_k-1}T^{(1)}_{Q_k}}{\partial Q_k^{n_k-1}} \right],$$

where C_1, C are complex numbers and

$$f = \frac{A_{k,n_k}}{(z - z(Q_k))^{n_k}} + ... + \frac{A_{k,2}}{(z - z(Q_k))^2} + \frac{A_{k,1}}{z - z(Q_k)} + O(1)$$

in a punctured neighborhood of Q_k, $k = l+1, ..., s$, on F_μ, and all terms depend holomorphically on μ.

Proof. If Q_1 is a pole of order $n_1, n_1 \geqslant 2$, then in the previous formula the term $c_1 T^{(1)}_{Q_1}$ is replaced by the sum

$$A_{11}T^{(1)}_{Q_1} + A_{12}\frac{\partial T^{(1)}_{Q_1}}{\partial Q_1} + \frac{A_{13}}{2}\frac{\partial^2 T^{(1)}_{Q_1}}{\partial Q_1^2} + ... + \frac{A_{1,n_1}}{(n_1-1)!}\frac{\partial^{n_1-1}T^{(1)}_{Q_1}}{\partial Q_1^{n_1-1}},$$

where A_{kj} are the coefficients of the principal part of the Laurent series for the function f in a punctured neighborhood of the point $Q_k, j = 1, ..., n_k(Q_k)$, $k = l+1, ..., s$. Indeed, in a neighborhood of the point Q_k we have the expansions $T^{(1)}_{Q_k} = \dfrac{1}{z - z(Q_k)} + O(1), z(Q_k) = a_k$; $(T^{(1)}_{Q_k})'_{a_k} = \dfrac{1}{(z-a_k)^2} + O(1); ...$; $(T^{(1)}_{Q_k})^{(m)}_{a_k} = \dfrac{m!}{(z-a_k)^{m+1}} + O(1)$, $1 \leqslant m \leqslant n_k(Q_k) - 1$, where $n_k(Q_k)$ is the order of the pole at the point Q_k for f, $k = l+1, ..., s$. The theorem is proved. □

3. The space of meromorphic q-differentials on a variable torus

Denote by $\Omega^q\left(\dfrac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}; F_\mu\right)$ the space of q-differentials on F_μ that are multiples of the divisor $\dfrac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}$, where $q \in \mathbb{Z}$, $\alpha_1, ..., \alpha_l \geqslant 2$, $n \geqslant 1$, $0 \leqslant l \leqslant n$, and the points $P_1, ..., P_n$ are pairwise distinct, and by $\Omega^q(1; F_\mu)$ the subspace of holomorphic q-differentials on F_μ. The divisor $P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n$ is chosen as a locally holomorphic section of the bundle of integer divisors of degree $\alpha_1 + \cdots + \alpha_l + n - l$ over $\mathbb{T}_1$.

By the Riemann-Roch theorem for q-differentials we find the dimensions of these spaces. For every q we have $\dim \Omega^q(1; F_\mu) = 1$, and

$$i_q\left(\frac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}\right) = -\deg\left(\frac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}\right) + r\left(P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n\right) =$$
$$= \alpha_1 + \cdots + \alpha_l + n - l \ \ (\geqslant 1).$$

Therefore

$$\dim \Omega^q\left(\frac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}; F_\mu\right)/\Omega^q(1; F_\mu) = \alpha_1 + \cdots + \alpha_l + n - l - 1 \ \ (\geqslant 1).$$

Consider the following collections of q-differentials:

$$\tau^{(2)}_{q;P_1}, ..., \tau^{(\alpha_1)}_{q;P_1}, ..., \tau^{(2)}_{q;P_l}, ..., \tau^{(\alpha_l)}_{q;P_l}, \tau_{q;P_1P_2}, ..., \tau_{q;P_1P_n}, \text{ for } l \geqslant 1; \tag{4}$$

$$\tau_{q;P_1P_2}, ..., \tau_{q;P_1P_n}, \text{ for } l = 0. \tag{5}$$

Let us show that the coset classes of q-differentials from (4) are linearly independent over $\mathbb{C}$. Assume that there exists a linear combination of differentials from (4)

$$C_1^{(2)}\tau^{(2)}_{q;P_1} + \cdots + C_1^{(\alpha_1)}\tau^{(\alpha_1)}_{q;P_1} + \cdots + C_l^{(2)}\tau^{(2)}_{q;P_l} + \cdots + C_l^{(\alpha_l)}\tau^{(\alpha_l)}_{q;P_l} + C_2\tau_{q;P_1P_2} + \cdots + C_n\tau_{q;P_1P_n} = \omega,$$

where ω is a holomorphic q-differential, such that not all its coefficients are zeroes.

The coefficients $C_1^{(2)} = \cdots = C_l^{(\alpha_l)} = 0$, since in the right hand side the points $P_1, ..., P_l$ are not poles of order $\geqslant 2$. We are left with the equality

$$C_2\tau_{q;P_1P_2} + \cdots + C_n\tau_{q;P_1P_n} = \omega.$$

Since the points $P_2, ..., P_n$ are not singular for the right hand side, $C_2 = \cdots = C_n = 0$. Thus, the coset classes for q-differentials from (4) is a base for the quotient space.

Let us now show that the collection (5) is linearly independent over $\mathbb{C}$. Suppose that there exists a linear combination $C_2\tau_{q;P_1P_2}+\cdots+C_n\tau_{q;P_1P_n} = \omega$, where ω is a holomorphic q-differential, such that not all its coefficients are zeroes. The coefficients $C_2 = \cdots = C_n = 0$, since $P_2, ..., P_n$ are not singular for the right hand side. Therefore the coset classes of q-differentials from (5) form a base for the quotient space. Thus, we have proved the following theorem.

Theorem 3.1. *The vector bundle* $E_1 = \bigcup\limits_{\mu} \Omega^q\left(\dfrac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}; F_\mu\right)/\Omega^q(1; F_\mu)$ *of rank* $d = \alpha_1 + \cdots + \alpha_l + n - l - 1$, *where* $\alpha_1, ..., \alpha_l \geqslant 2$, $n \geqslant 1$, $0 \leqslant l \leqslant n, q \in \mathbb{Z}$, *over* $\mathbb{T}_1$ *is complex analytic equivalent to the direct product* $\mathbb{T}_1 \times \mathbb{C}^d$, *and the coset classes of* q*-differentials from the collections* (4), (5) *give a base of locally holomorphic sections of this bundle over* $\mathbb{T}_1$.

Consider the collection of q-differentials

$$dz^q; \tau^{(2)}_{q;P_1}, ..., \tau^{(\alpha_1)}_{q;P_1}, ..., \tau^{(2)}_{q;P_l}, ..., \tau^{(\alpha_l)}_{q;P_l}, \tau_{q;P_1P_2}, ..., \tau_{q;P_1P_n}. \tag{6}$$

Let us show that q-differentials from(6) are linearly independent over $\mathbb{C}$. Assume again that there exists a linear combination

$$C_1dz^q + C_1^{(2)}\tau^{(2)}_{q;P_1} + \cdots + C_1^{(\alpha_1)}\tau^{(\alpha_1)}_{q;P_1} + C_l^{(2)}\tau^{(2)}_{q;P_l} + \cdots + C_l^{(\alpha_l)}\tau^{(\alpha_l)}_{q;P_l} +$$

$$+C_2\tau_{q;P_1P_2} + \cdots + C_n\tau_{q;P_1P_n} = 0,$$

such that not all its coefficients are zeroes. The coefficients $C_1^{(2)} = \cdots = C_l^{(\alpha_l)} = 0$ and $C_2 = \cdots = C_n = 0$, since in the right hand side there are no singular points. So we have $C_1dz^q = 0$, which implies $C_1 = 0$. Therefore the collection (6) of $q-$differentials is a base for the space $\Omega^q\left(\dfrac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}; F_\mu\right)$. This prove the following theorem.

Theorem 3.2. *The vector bundle* $E_2 = \bigcup\limits_{\mu} \Omega^q\left(\dfrac{1}{P_1^{\alpha_1}\cdots P_l^{\alpha_l}P_{l+1}\cdots P_n}; F_\mu\right)$ *of rank* $d_1 = \alpha_1 + \cdots + \alpha_l + n - l$ *over* $\mathbb{T}_1$ *is complex analytic equivalent to the direct product* $\mathbb{T}_1 \times \mathbb{C}^{d_1}$. *Moreover, q-differentials of (6) give a base of locally holomorphic sections of this bundle over* $\mathbb{T}_1$, *where* $\alpha_1, ..., \alpha_l \geqslant 2$, $n \geqslant 1, 0 \leqslant l \leqslant n$, *and* $q \in \mathbb{Z}$.

Remark 3.1. For $q = 0$, $l = 0$ the collection (6) is $1, \tau_{0;P_1P_2} = f_2, ..., \tau_{0;P_1P_n} = f_n$ and is a base for the space $L\left(\dfrac{1}{P_1...P_n}; F_\mu\right)$, where f_j are non-constant functions, and $(f_j) \geqslant \dfrac{1}{P_1P_j}$, $j = 2, ..., n$, on F_μ.

Remark 3.2. In particular, for $q = 1$ and for a fixed torus F Corollary 1.1 and Theorems 2.1, 3.1, 3.2 imply classical theorems on abelian 1-differentials found in [1, 2].

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Однозначные дифференциалы целого порядка на переменном торе

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В этой работе дано полное описание дивизоров элементарных дифференциалов всех родов. Получен аналог формулы Аппеля разложения однозначной функции на переменном торе. Исследованы основные типы векторных расслоений из мероморфных дифференциалов целого порядка над пространством Тейхмюллера для тора.

Ключевые слова: однозначные мероморфные дифференциалы целого порядка, дивизоры, векторные расслоения, пространство Тейхмюллера.