

УДК 517.518.87

On Minimal Cubature Formulas Exact for Haar Polynomials of Low Degrees in the Two-Dimensional Case

Kirill A. Kirillov*

Institute of Space and Information Technologies,
Siberian Federal University,
Kirenskogo, 26, Krasnoyarsk, 660074
Russia

Received 26.02.2014, received in revised form 01.04.2014, accepted 20.05.2014

A lower estimate for number of nodes of cubature formulas possessing the Haar d -property ($d \geq 2$) is obtained in the two-dimensional case. Examples of minimal cubature formulas with Haar d -property for $d = 2$ and $d = 3$ are given.

Keywords: minimal cubature formulas, Haar functions, Haar polynomials, Haar d -property.

Introduction

The problem of constructing cubature formulas that integrate a given collection of functions exactly has been earlier considered mainly in the cases when these functions are algebraic or trigonometric polynomials. The approximate integration formulas exact for finite Haar sums can be found in the monograph [1], the accuracy of approximate integration formulas for finite Haar sums was used there for deriving the error of these formulas.

A description of all minimal weighted quadrature formulas possessing the Haar d -property, i.e., formulas exact for Haar polynomials of degree at most d , was given in [2].

In the two-dimensional case, the problem of constructing cubature formulas exact for Haar polynomials was considered in [3, 4]. In those papers the following lower estimate for number N of nodes of cubature formulas possessing the Haar d -property was obtained:

$$N \geq 2^d - \lambda(d), \quad (1)$$

where

$$\lambda(d) = \begin{cases} 2^{\frac{d}{2}+1} - 2, & d = 2k, \\ 3 \times 2^{\frac{d-1}{2}} - 2, & d = 2k - 1, \end{cases} \quad (2)$$

$k = 1, 2, \dots$ If $d \geq 5$, then the estimate (1) is used for constructing minimal cubature formulas possessing the Haar d -property, because in the case of $d \geq 5$ the number of nodes of these formulas is equal to $2^d - \lambda(d)$, where $\lambda(d)$ is defined by (2). In [4] examples of minimal formulas for $d = 5, 6, 7$ were given, and for any integer $d \geq 8$ minimal cubature formulas possessing the Haar d -property based on those formulas were constructed.

In this paper in the two-dimensional case the following lower estimate for number N of nodes of cubature formulas possessing the Haar d -property is derived:

$$N \geq 2^{d-1} + 1, \quad (3)$$

where $d \geq 2$ (see the Theorem 3). If $d \geq 4$, then the estimate (1) is more accurate than (3). In the cases of $d = 2$ and $d = 3$ the estimate (3) refines (1). If $d = 2$ or $d = 3$, then the estimate

*kkirillow@yandex.ru

(3) is used for constructing minimal cubature formulas possessing the Haar d -property, because in the cases of $d = 2$ and $d = 3$ the number of nodes of these formulas is equal to $2^{d-1} + 1$. In this paper for $d = 2$ and $d = 3$ examples of minimal cubature formulas possessing the Haar d -property are given (see the Examples 1 and 2).

Moreover, in this paper in the two-dimensional case the following statement is proved: if the coordinates of nodes of cubature formula possessing the Haar d -property ($d \geq 2$) not are the points of discontinuity of any Haar function of the $(d - 1)$ -th group, then

$$N \geq 2^d,$$

where N is the number of nodes of the formula (see the Theorem 1).

1. Basic definitions and auxiliary statements

In the present paper we use the original definition of the functions $\chi_{m,j}(x)$ introduced by Haar [5], which differs from the definition of these functions at discontinuity points used in [1].

Binary intervals $l_{m,j}$ are intervals with their endpoints at $(j - 1)/2^{m-1}$ and $j/2^{m-1}$ ($m = 1, 2, \dots$, $j = 1, 2, \dots, 2^{m-1}$). If the left endpoint of a binary interval coincides with 0, this interval is considered closed on the left. If the right endpoint of a binary interval coincides with 1, this interval is considered closed on the right. The other binary intervals are considered open. The left and right halves of $l_{m,j}$ (without its midpoint) are denoted by $l_{m,j}^-$ and $l_{m,j}^+$, respectively.

The set $l_{m_1,j_1} \times l_{m_2,j_2}$ is called a binary rectangle, and its closures we call a closed binary rectangle, where $m_n = 1, 2, \dots$, $j_n = 1, 2, \dots, 2^{m_n-1}$, $n = 1, 2$.

The system of Haar functions is constructed in groups: the group indexed by m contains 2^{m-1} functions $\chi_{m,j}(x)$, where $m = 1, 2, \dots$ and $j = 1, 2, \dots, 2^{m-1}$. The Haar functions $\chi_{m,j}(x)$ are defined as

$$\chi_{m,j}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & \text{if } x \in l_{m,j}^-, \\ -2^{\frac{m-1}{2}}, & \text{if } x \in l_{m,j}^+, \\ 0, & \text{if } x \in [0, 1] \setminus \overline{l_{m,j}}, \\ \frac{1}{2}[\chi_{m,j}(x-0) + \chi_{m,j}(x+0)], & \text{if } x \text{ is an interior discontinuity point,} \end{cases}$$

where $\overline{l_{m,j}} = \left[\frac{j-1}{2^{m-1}}, \frac{j}{2^{m-1}} \right]$ for $m = 1, 2, \dots$ and $j = 1, 2, \dots, 2^{m-1}$. The function $\chi_{0,1}(x) \equiv 1$ is also included in the system of Haar functions (it forms a zero group).

Let d be a fixed nonnegative integer.

Below is the definition of Haar polynomials in the two-dimensional case. The Haar monomials of degree d are the functions $\chi_{m_1,j_1}(x_1)\chi_{m_2,j_2}(x_2)$, where $m_1 + m_2 = d$, $j_n = 1, 2, \dots, 2^{m_n-1}$ for $m_n \neq 0$, $j_n = 0$, for $m_n = 0$ and $n = 1, 2$. The Haar polynomials of degree d are linear combinations (with real coefficients) of Haar monomials of degrees not exceeding d such that at least one of the coefficients multiplying the monomials of degree d is nonzero.

Consider the cubature formulas

$$I[f] = \int_0^1 \int_0^1 f(x_1, x_2) dx_1 dx_2 \approx \sum_{j=1}^N C_j f(x_1^{(j)}, x_2^{(j)}) = Q_N[f], \quad (4)$$

where $(x_1^{(j)}, x_2^{(j)}) \in [0, 1]^2$ are the nodes of a cubature formula; C_j are the coefficients of the formula at the nodes (real numbers); and $j = 1, 2, \dots, N$.

Formula (4) is said to possess the Haar d -property if it is exact for any Haar polynomial $P(x_1, x_2)$ of degree at most d ; i.e., $Q_N[P] = I[P]$.

To establish the inequality (3) for cubature formulas (4) possessing the Haar d -property, we need to recall the following auxiliary statements.

It was shown in [2] that there exist Haar polynomials of degree m that satisfy the equality:

$$\kappa_{m,j}(x) = \begin{cases} 2^m, & \text{if } x \in l_{m+1,j}, \\ 2^{m-1}, & \text{if } x \in \bar{l}_{m+1,j} \setminus l_{m+1,j}, \\ 0, & \text{if } x \in [0, 1] \setminus \bar{l}_{m+1,j}, \end{cases} \quad (5)$$

where $m = 1, 2, \dots$ and $j = 1, 2, \dots, 2^m$.

Lemma 1 (see [2]). *The functions $\kappa_{m,1}(x)$, $\kappa_{m,2}(x)$, $\dots, \kappa_{m,2^m}(x)$ form a basis in the linear space of Haar polynomials of degrees not exceeding m .*

The κ -monomials of degree d are by definition the functions $\kappa_{d,k}(x_1)$, $\kappa_{d,k}(x_2)$, $\kappa_{l,i}(x_1)\kappa_{m,j}(x_2)$, where $k = 1, \dots, 2^d$, $l + m = d$, $i = 1, \dots, 2^l$ and $j = 1, \dots, 2^m$.

Lemma 1 implies

Lemma 2. *Cubature formula (4) possesses the Haar d -property if and only if it is exact for all κ -monomials of degree d .*

Identity (5) implies that each closed binary rectangle of area 2^{-d} is the support of some κ -monomial of degree d , namely, $\overline{l_{n+1,i} \times l_{m+1,j}} = \text{supp}\{\kappa_{n,i}(x_1)\kappa_{m,j}(x_2)\}$, $\overline{l_{n+1,i} \times [0, 1]} = \text{supp}\{\kappa_{n,i}(x_1)\}$, $[0, 1] \times \overline{l_{m+1,j}} = \text{supp}\{\kappa_{m,j}(x_2)\}$, where $n, m = 1, 2, \dots$, $i = 1, \dots, 2^n$, $j = 1, \dots, 2^m$.

Prove the following propositions.

Lemma 3. *If $K_d(x_1, x_2)$ is an arbitrary κ -monomial of degree d , then the following identity hold:*

$$I[K_d] = \int_0^1 \int_0^1 K_d(x_1, x_2) dx_1 dx_2 = 1. \quad (6)$$

Proof. It follows from (5) that $K_d(x_1, x_2) = 2^d$ for every interior point (x_1, x_2) of the set $\text{supp}\{K_d\}$. Since $\text{supp}\{K_d\}$ is a binary rectangle of area 2^{-d} , we obtain (6). \square

Lemma 4. *At the points of continuity of the Haar functions $\chi_{m,j}(x)$ ($m = 1, 2, \dots$, $j = 1, \dots, 2^{m-1}$) the identity*

$$\chi_{m,j}^2(x) = \kappa_{m-1,j}(x) \quad (7)$$

holds true. Everywhere except the points at which the functions $\chi_{k,i}(x)$ and $\chi_{m,j}(x)$ are simultaneously discontinuous (if such points exist), the product of these functions can be written as

$$\chi_{k,i}(x)\chi_{m,j}(x) = \begin{cases} 2^{\frac{k-1}{2}}\chi_{m,j}(x), & \text{if } l_{m,j} \subseteq l_{k,i}^-, \\ -2^{\frac{k-1}{2}}\chi_{m,j}(x), & \text{if } l_{m,j} \subseteq l_{k,i}^+, \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

where $m \geq k$, $i \neq j$ as $m = k$.

Equality (7) follows from (5), the equality (8) is proved directly.

Lemma 5. *If the cubature formula (4) possesses the Haar d -property, then every closed binary rectangle of area 2^{-d} contains at least one node of this formula.*

Proof. Suppose that some closed binary rectangle of area 2^{-d} does not contain any node of cubature formula (4). Let $K_d(x_1, x_2)$ be the κ -monomial of degree d , such that $\text{supp}\{K_d(x_1, x_2)\}$ is the mentioned rectangle.

Then $Q_N[K_d] = 0$, while by Lemma 3 $I[K_d] = 1$. This contradiction to the condition of the accuracy of the formula for $K_d(x_1, x_2)$ proves the lemma. \square

2. A lower estimate of the number of nodes of cubature formulas possessing the Haar d -property

In the present section we prove Theorems 1, 2. These theorems imply a lower estimate of the number of nodes of the cubature formula (4) possessing the Haar d -property (Theorem 3).

Theorem 1. *Let the cubature formula (4) possesses the Haar d -property ($d \geq 2$). If the coordinates of nodes of this formula not are the points of discontinuity of any Haar function of the $(d-1)$ -th group, then the number N of nodes of the formula satisfies the following inequality:*

$$N \geq 2^d.$$

Proof. We use the technique applied in [6, 7].

Let $f_1(x_1, x_2) \equiv 1$, $f_2(x_1, x_2) = \chi_{1,1}(x_1)$, $f_3(x_1, x_2) = \chi_{1,1}(x_2)$, $f_4(x_1, x_2) = \chi_{1,1}(x_1)\chi_{1,1}(x_2)$, $f_5(x_1, x_2) = \chi_{2,1}(x_1)$, $f_6(x_1, x_2) = \chi_{2,2}(x_1)$, $f_7(x_1, x_2) = \chi_{2,1}(x_1)\chi_{1,1}(x_2)$, $f_8(x_1, x_2) = \chi_{2,2}(x_1)\chi_{1,1}(x_2), \dots$, $f_{2^{d-1}+1}(x_1, x_2) = \chi_{d-1,1}(x_1)$, $f_{2^{d-1}+2}(x_1, x_2) = \chi_{d-1,2}(x_1)$, \dots , $f_{3 \cdot 2^{d-2}}(x_1, x_2) = \chi_{d-1,2^{d-2}}(x_1)$, $f_{3 \cdot 2^{d-2}+1}(x_1, x_2) = \chi_{d-1,1}(x_1)\chi_{1,1}(x_2)$, $f_{3 \cdot 2^{d-2}+2}(x_1, x_2) = \chi_{d-1,2}(x_1)\chi_{1,1}(x_2), \dots$, $f_{2^d}(x_1, x_2) = \chi_{d-1,2^{d-2}}(x_1)\chi_{1,1}(x_2)$.

Consider the equality

$$\sum_{j=1}^N C_j f_l(x_1^{(j)}, x_2^{(j)}) f_{l'}(x_1^{(j)}, x_2^{(j)}) = Q_N[f_l f_{l'}], \quad l, l' = 1, \dots, 2^d.$$

There are no points of discontinuity of the functions $\chi_{m,j}$ ($m = 1, 2, \dots, d-1$, $j = 1, 2, \dots, 2^{m-1}$) among the $x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \dots, x_1^{(N)}, x_2^{(N)}$. Hence, by Lemma 4 the function $f_l f_{l'}$ is a Haar polynomial of the degree at most d . Then by the fact that the cubature formula (4) possesses the Haar d -property the following equalities holds:

$$Q_N[f_l f_{l'}] = I[f_l f_{l'}],$$

where $l, l' = 1, \dots, 2^d$. Since the system of Haar functions is orthonormal (see [5]) then

$$\int_0^1 \int_0^1 f_l(x_1, x_2) f_{l'}(x_1, x_2) dx_1 dx_2 = \begin{cases} 0, & \text{if } l \neq l', \\ 1, & \text{if } l = l'. \end{cases}$$

Thus, we have

$$\sum_{j=1}^N C_j f_l(x_1^{(j)}, x_2^{(j)}) f_{l'}(x_1^{(j)}, x_2^{(j)}) = \delta_{l,l'}, \quad (9)$$

where $\delta_{l,l'}$ is the Kronecker delta, and $l, l' = 1, \dots, 2^d$. Let

$$F = \begin{pmatrix} f_1(x_1^{(1)}, x_2^{(1)}) & f_1(x_1^{(2)}, x_2^{(2)}) & \dots & f_1(x_1^{(N)}, x_2^{(N)}) \\ f_2(x_1^{(1)}, x_2^{(1)}) & f_2(x_1^{(2)}, x_2^{(2)}) & \dots & f_2(x_1^{(N)}, x_2^{(N)}) \\ \dots & \dots & \dots & \dots \\ f_{2^d}(x_1^{(1)}, x_2^{(1)}) & f_{2^d}(x_1^{(2)}, x_2^{(2)}) & \dots & f_{2^d}(x_1^{(N)}, x_2^{(N)}) \end{pmatrix}, \text{ and } \tilde{C} = \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_N \end{pmatrix}.$$

Then the equalities (9) can be rewritten as

$$F \tilde{C} F^T = E,$$

where E is an identity matrix of order 2^d . Since the rank of a product of matrices does not exceed the rank of each factors, then

$$N \geq \text{rank } F \geq \text{rank}(F \tilde{C} F^T) = \text{rank } E = 2^d.$$

□

Theorem 2. *Let cubature formula (4) possesses the Haar d -property ($d \geq 2$). If one of the coordinates at least one of the nodes of this formula is a point of discontinuity of Haar function of the $(d-1)$ -th group, then the number N of nodes of the formula holds the following inequality:*

$$N \geq 2^{d-1} + 1. \quad (10)$$

Proof. For definiteness, assume that the first coordinate of a node $(x_1^{(p)}, x_2^{(p)})$ of the cubature formula (4) is a point of discontinuity of some Haar functions of the $(d-1)$ -th group, i.e., $x_1^{(p)} = \frac{j_p}{2^{d-1}}$, $1 \leq j_p \leq 2^{d-1} - 1$.

Consider the following closed binary rectangles of area 2^{-d} : $[0, \frac{1}{2^d}] \times [0, 1]$, $[\frac{2}{2^d}, \frac{3}{2^d}] \times [0, 1]$, \dots , $[\frac{2j_p-2}{2^d}, \frac{2j_p-1}{2^d}] \times [0, 1]$, $[\frac{2j_p+1}{2^d}, \frac{2j_p+2}{2^d}] \times [0, 1]$, $[\frac{2j_p+3}{2^d}, \frac{2j_p+4}{2^d}] \times [0, 1]$, \dots , $[1 - \frac{1}{2^d}, 1] \times [0, 1]$. By Lemma 5 each of them contains at least one node of the cubature formula (4). These rectangles are pairwise disjoint and number of their is equal to 2^{d-1} . Taking into account that no one of these rectangles does not contain the node $(x_1^{(p)}, x_2^{(p)})$ we obtain (10). \square

The Theorems 1 and 2 imply

Theorem 3. *If the cubature formula (4) possesses the Haar d -property, then the number N of its nodes satisfies the inequality (10).*

3. Examples of minimal cubature formulas possessing the Haar d -property

In the cases of $d = 2$ and $d = 3$ we give examples of cubature formulas (4) possessing the Haar d -property. It is easy to see that the formula given in Example 1 is exact for all κ -monomials of degree 2, and the formula given in Example 2 is exact for all κ -monomials of degree 3. Hence by Lemma 2 they possess the Haar d -property for $d = 2$ and $d = 3$, respectively. The number N of nodes of these formulas satisfies the equality $N = 2^{d-1} + 1$, therefore (10) implies their minimality. The formulas mentioned were obtained by experiment (see Fig 1).

Example 1. $d = 2$, $N = 3$: $(x_1^{(1)}, x_2^{(1)}) = (1/4, 1/2)$, $(x_1^{(2)}, x_2^{(2)}) = (5/8, 1/8)$, $(x_1^{(3)}, x_2^{(3)}) = (7/8, 7/8)$, $C_1 = 1/2$, $C_2 = 1/4$, $C_3 = 1/4$.

Example 2. $d = 3$, $N = 5$: $(x_1^{(1)}, x_2^{(1)}) = (1/8, 1/2)$, $(x_1^{(2)}, x_2^{(2)}) = (5/16, 1/16)$, $(x_1^{(3)}, x_2^{(3)}) = (1/2, 7/8)$, $(x_1^{(4)}, x_2^{(4)}) = (3/4, 1/4)$, $(x_1^{(5)}, x_2^{(5)}) = (15/16, 11/16)$, $C_1 = 1/4$, $C_2 = 1/8$, $C_3 = 1/4$, $C_4 = 1/4$, $C_5 = 1/8$.

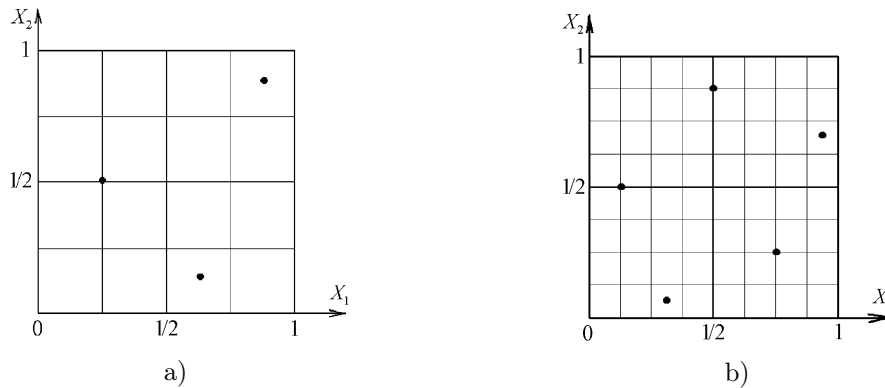


Fig. 1. Location scheme of nodes of minimal cubature formulas possessing the Haar d -property: a) — for the formula given in Example 1; b) — for the formula given in Example 2

Conclusions

In [1] one considered the following cubature formulas

$$\int_0^1 \dots \int_0^1 f(x_1, \dots, x_n) dx_1 \dots dx_n \approx \frac{1}{N} \sum_{j=1}^N f(x_1^{(j)}, \dots, x_n^{(j)}) \quad (11)$$

with nodes $(x_1^{(j)}, \dots, x_n^{(j)}) \in [0, 1]^n$, generating Π_τ -nets, i.e., nets that consist of $N = 2^\nu$ nodes such that each of the binary parallelepipeds of volume $2^{\tau-\nu}$ contains 2^τ nodes ($\nu > \tau$).

In [1] it is proved that these formulas possess the Haar $(\nu - \tau)$ -property, and in the cases of $n = 2, 3$ Π_τ -nets with an arbitrarily large number $N = 2^\nu$ of nodes exist for any $\tau = 0, 1, 2, \dots$. Therefore, in the two-dimensional case for a fixed d minimal cubature formula among the formulas (11) possessing the Haar d -property is a formula, such that consists of $N = 2^d$ nodes, generating Π_τ -nets. Note that the number of nodes of cubature formulas given in the Examples 1 and 2 is less than 2^d .

References

- [1] I.M.Sobol', Multidimensional Quadrature Formulas and Haar Functions, Nauka, Moscow, 1969 (in Russian).
- [2] K.A.Kirillov, M.V.Noskov, Minimal quadrature formulas accurate for Haar polynomials, *Comput. Math. Phys.*, **42**(2002), no. 6, 758–766.
- [3] K.A.Kirillov, Lower estimates for node number of cubature formulas exact for Haar polynomials in two-dimensional case, *Vychislitelnye tekhnologii*, special issue, **9**(2004), 62–71 (in Russian).
- [4] K.A.Kirillov, Construction of minimal cubature formulas exact for Haar polynomials of high degrees in two-dimensional case, *Vychislitelnye tekhnologii*, special issue, **10**(2005), 29–47 (in Russian).
- [5] A.Haar, Zur Theorie der Orthogonalen Funktionensysteme, *Math. Ann.*, **69**(1910), 331–371.
- [6] I.P.Mysovskikh, Interpolation Cubature Formulas, Nauka, Moscow, 1981 (in Russian).
- [7] I.P.Mysovskikh, Cubature formulas exact for trigonometric polynomials, *Metody Vychislenii*, Leningrad, Leningradsky Gosudarstvennyy Universitet, **15**(1988), 7–19 (in Russian).

О минимальных кубатурных формулах, точных для полиномов Хаара малых степеней в двумерном случае

Кирилл А. Кириллов

Получена нижняя оценка числа узлов кубатурных формул, точных для полиномов Хаара в двумерном случае. Для $d = 2$ и $d = 3$ приведены примеры минимальных формул, обладающих d -свойством Хаара.

Ключевые слова: минимальные кубатурные формулы, функции Хаара, полиномы Хаара, d -свойство Хаара.