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Determination of Source Functions in Composite Type System of Equations

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The problem of identification of the source function for semievolutionary system of two partial differential equations is considered in the paper. The investigated system of equations is obtained from the original system by adding the time derivative containing a small parameter $\varepsilon > 0$ to the elliptic equation. The Cauchy problem and the second boundary-value problem are considered.

Keywords: identification, inverse problem, parabolic equation, the method of weak approximation, small parameter, convergence.

We obtain a priori (uniform in $\varepsilon > 0$) estimates of solutions of approximate problems. We prove convergence of solutions approximating the inverse problems to solutions of original problems when $\varepsilon \to 0$ on the basis of the obtained a priori estimates. We obtain that the rate of convergence of solutions of approximate problems is $O(\varepsilon^{1/2})$ in class of continuous functions. The case of the first boundary-value problem has been studied by Yu.Ya.Belov.

An identification problem of source functions in the composite type system is considered in [1–3].

1. Formulation of the problem and reduction it to the direct problem

Consider in the strip $G_{[0,T]} = \{(t,x) \mid 0 \leqslant t \leqslant T,\ x \in E_1\}$ the problem of determining real-valued functions $\left(\overset{\varepsilon}{u}(t,x), \overset{\varepsilon}{v}(t,x), \overset{\varepsilon}{g}(t)\right)$, satisfying the system of equations

$$\begin{cases} \overset{\varepsilon}{u}_t(t,x) + a_{11}(t)\overset{\varepsilon}{u}(t,x) + a_{12}(t)\overset{\varepsilon}{v}(t,x) = \mu_1 \overset{\varepsilon}{u}_{xx}(t,x) + \overset{\varepsilon}{g}(t) f(t,x), \\ \varepsilon\overset{\varepsilon}{v}_t(t,x) + a_{21}(t)\overset{\varepsilon}{u}(t,x) + a_{22}(t)\overset{\varepsilon}{v}(t,x) = \mu_2 \overset{\varepsilon}{v}_{xx}(t,x) + F(t,x), \end{cases} \tag{1}$$

where constant $\varepsilon \in (0,1]$. Initial conditions are

$$\overset{\varepsilon}{u}(0,x) = u_0(x), \quad \overset{\varepsilon}{v}(0,x) = v_0(x), \tag{2}$$

and the overdetermination condition are

$$\overset{\varepsilon}{u}\big(t,x^0\big) = \varphi(t), \quad \varphi \in C^2[0,T], \tag{3}$$

where $\varphi(t)$ is a given function, $0 \leqslant t \leqslant T$ and x^0 is some fixed point.

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Let the functions $a_{ij}(t), i, j = 1, 2$, be defined on the interval $[0, T]$ and let the functions $f(t, x)$, $F(t, x)$ be defined on the strip $G_{[0,T]}$. Let μ_1, $\mu_2 > 0$ be given constants.

Let the relationship

$$\left|f(t, x^0)\right| \geqslant \delta > 0, \quad t \in [0, T], \quad \delta > 0 - \text{const.} \tag{4}$$

is fulfilled.

Assuming sufficient smoothness of the input data

– we prove the solvability of the problem (1)–(3) for each fixed $\varepsilon \in (0, 1]$;

– provided periodic in x and smooth input data f, F, u_0, v_0 we prove the existence of a sufficiently smooth solution $\overset{\varepsilon}{u}, \overset{\varepsilon}{v}, \overset{\varepsilon}{g}$ in $\overline{Q}_T = [0, T] \times [0, l]$ for the boundary conditions

$$\overset{\varepsilon}{u}_x(t, 0) = \overset{\varepsilon}{v}_x(t, 0) = \overset{\varepsilon}{u}_x(t, l) = \overset{\varepsilon}{v}_x(t, l) = 0, \quad t \in [0, T];$$

– we prove the existence of solution u, v, g to the second boundary problem (1^0) $(2^0), (3^0)$, where

$$u = \lim_{\varepsilon \to 0} \overset{\varepsilon}{u}, \quad v = \lim_{\varepsilon \to 0} \overset{\varepsilon}{v}, \quad g = \lim_{\varepsilon \to 0} \overset{\varepsilon}{g},$$

and $(1^0), (2^0), (3^0)$ denote $(1), (2), (3)$ with $\varepsilon = 0$ (as $\overset{\varepsilon}{u} = u, \overset{\varepsilon}{v} = v, \overset{\varepsilon}{g} = g$);

– we obtain an estimate of the rate of convergence of $\overset{\varepsilon}{u}, \overset{\varepsilon}{v}, \overset{\varepsilon}{g}$ to u, v, g, respectively , when $\varepsilon \to 0$.

Let us assume that the following consistency condition is fulfilled

$$u_0(x^0) = \varphi(0), \tag{5}$$

functions $a_{ij}(t),\ i, j = 1, 2$, are of class $C^2[0, T]$. Let us also assume that matrix

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$$

generates a symmetric and coercive bilinear form $a(t, \xi, \chi) = (A(t)\xi, \chi)$ and

$$\begin{gathered} a(t, \xi, \chi) = a(t, \chi, \xi), \quad \forall \xi, \chi \in E_2, \\ a(t, \xi, \xi) \geqslant \kappa \left|\xi\right|^2 \quad \forall \xi = (\xi_1, \xi_2) \in E_2, \quad t \in [0, T], \quad \kappa > 0 - \text{const.} \end{gathered} \tag{6}$$

Let us reduce inverse problem (1)–(2) to an subsidiary direct problem. In system (1) we set $x = x^0$:

$$\begin{cases} \varphi_t(t) + a_{11}(t)\varphi(t) + a_{12}(t)\overset{\varepsilon}{v}\big(t, x^0\big) = \mu_1 \overset{\varepsilon}{u}_{xx}\big(t, x^0\big) + \overset{\varepsilon}{g}(t) f\big(t, x^0\big), \\ \varepsilon \overset{\varepsilon}{v}_t\big(t, x^0\big) + a_{21}(t)\varphi(t) + a_{22}(t)\overset{\varepsilon}{v}\big(t, x^0\big) = \mu_2 \overset{\varepsilon}{v}_{xx}\big(t, x^0\big) + F\big(t, x^0\big). \end{cases} \tag{7}$$

From (7) we obtain

$$\overset{\varepsilon}{g}(t) = \frac{\psi(t) + a_{12}(t)\overset{\varepsilon}{v}\big(t, x^0\big) - \mu_1 \overset{\varepsilon}{u}_{xx}\big(t, x^0\big)}{f\big(t, x^0\big)}, \tag{8}$$

where

$$\psi(t) = \varphi_t(t) + a_{11}(t)\varphi(t)$$

is known.

Substituting expression for $\overset{\varepsilon}{g}(t)$ in (1) we obtain the following direct problem:

$$\begin{cases} \overset{\varepsilon}{u}_t(t,x) + a_{11}(t)\overset{\varepsilon}{u}(t,x) + a_{12}(t)\overset{\varepsilon}{v}(t,x) = \\ \qquad = \mu_1 \overset{\varepsilon}{u}_{xx}(t,x) + \dfrac{\psi(t) + a_{12}(t)\overset{\varepsilon}{v}(t,x^0) - \mu_1 \overset{\varepsilon}{u}_{xx}(t,x^0)}{f(t,x^0)} f(t,x), \\ \varepsilon \overset{\varepsilon}{v}_t(t,x) + a_{21}(t)\overset{\varepsilon}{u}(t,x) + a_{22}(t)\overset{\varepsilon}{v}(t,x) = \mu_2 \overset{\varepsilon}{v}_{xx}(t,x) + F(t,x), \end{cases} \tag{9}$$

$$\overset{\varepsilon}{u}(0,x) = u_0(x), \tag{10}$$

$$\overset{\varepsilon}{v}(0,x) = v_0(x). \tag{11}$$

2. Proof the solvability of problem (1)–(3) for $\varepsilon \in (0,1]$

We use the method of weak approximation [4,5] to prove the existence of a solution of direct problem (9)–(11). We reduce problem (9)–(11) to the problem

$$\begin{cases} \overset{\varepsilon\tau}{u}_t(t,x) = 3\mu_1 \overset{\varepsilon\tau}{u}_{xx}(t,x), \\ \varepsilon \overset{\varepsilon\tau}{v}_t(t,x) = 3\mu_2 \overset{\varepsilon\tau}{v}_{xx}(t,x), \quad j\tau < t \leqslant \left(j + \frac{1}{3}\right)\tau, \end{cases} \tag{12}$$

$$\begin{cases} \overset{\varepsilon\tau}{u}_t(t,x) + 3a_{11}(t)\overset{\varepsilon\tau}{u}(t,x) = 0, \\ \varepsilon \overset{\varepsilon\tau}{v}_t(t,x) + 3a_{22}(t)\overset{\varepsilon\tau}{v}(t,x) = 0, \quad \left(j + \frac{1}{3}\right)\tau < t \leqslant \left(j + \frac{2}{3}\right)\tau, \end{cases} \tag{13}$$

$$\begin{cases} \overset{\varepsilon\tau}{u}_t(t,x) = -3a_{12}(t)\overset{\varepsilon\tau}{v}(t - \frac{\tau}{3}, x) + 3\dfrac{\psi(t) + a_{12}(t)\overset{\varepsilon\tau}{v}(t - \frac{\tau}{3}, x^0) - \mu_1 \overset{\varepsilon\tau}{u}_{xx}(t - \frac{\tau}{3}, x^0)}{f(t,x^0)} f(t,x), \\ \varepsilon \overset{\varepsilon\tau}{v}_t(t,x) = -3a_{21}(t)\overset{\varepsilon\tau}{u}(t - \frac{\tau}{3}, x) + F(t,x), \end{cases}$$

$$\left(j + \frac{2}{3}\right)\tau < t \leqslant (j+1)\tau, \tag{14}$$

$$\overset{\varepsilon\tau}{u}(t,x)|_{t\leqslant 0} = u_0(x), \tag{15}$$

$$\overset{\varepsilon\tau}{v}(t,x)|_{t\leqslant 0} = v_0(x). \tag{16}$$

Here $j = 0, 1, \ldots, N-1$ and $\tau N = T$.

The input data are sufficiently smooth functions. They have all continuous derivatives occurring in given below relations (17)–(19) and satisfy these relations:

$$|a_{ij}(t)| \leqslant C, \quad i = 1,2, \quad j = 1,2, \tag{17}$$

$$\left|\frac{\partial^k}{\partial x^k} f(t,x)\right| + \left|\frac{\partial^k}{\partial x^k} F(t,x)\right| + \left|\frac{d^k}{dx^k} u_0(x)\right| + \left|\frac{d^k}{dx^k} v_0(x)\right| \leqslant C, \quad k = 0, \ldots, p+6, \tag{18}$$

$$|\varphi(t)| + \left|\varphi'(t)\right| + \left|\varphi''(t)\right| \leqslant C, \quad (t,x) \in G_{[0,T]}. \tag{19}$$

Next we consider some proofs assuming, for convenience, that the constant C is greater than unity and the constant $p \geqslant 5$ is an odd number.

We obtain uniform estimates with respect τ

$$\left|\frac{\partial^k}{\partial x^k} \overset{\varepsilon\tau}{u}(t,x)\right| + \left|\frac{\partial^k}{\partial x^k} \overset{\varepsilon\tau}{v}(t,x)\right| \leqslant C(\varepsilon), \quad k = 0, \ldots, p+6, \quad (t,x) \in G_{[0,T]}, \tag{20}$$

$$\left|\frac{\partial^j}{\partial x^j}\overset{\varepsilon}{u}{}^{\tau}_t(t,x)\right| + \left|\frac{\partial^j}{\partial x^j}\overset{\varepsilon}{v}{}^{\tau}_t(t,x)\right| \leqslant C(\varepsilon), \quad j=0,...,p+4, \quad (t,x)\in G_{[0,T]}. \tag{21}$$

Taking into account (20), (21), the Arzela's theorem [6] and the convergence theorem of the weak approximation method [4], we can prove the following theorem.

Theorem 1. *Let conditions* (4)–(6), (17)–(19) *are fulfilled. Then there exists a unique solution* $\overset{\varepsilon}{u}(t,x), \overset{\varepsilon}{v}(t,x), \overset{\varepsilon}{g}(t)$ *of problem* (1)–(3) *in the class*

$$Z(T) = \left\{\overset{\varepsilon}{u}(t,x), \overset{\varepsilon}{v}(t,x), \overset{\varepsilon}{g}(t) \middle| \overset{\varepsilon}{u}(t,x)\in C^{1,p+4}_{t,x}(G_{[0,T]}),\ \overset{\varepsilon}{v}(t,x)\in C^{1,p+4}_{t,x}(G_{[0,T]}),\ \overset{\varepsilon}{g}(t)\in C([0,T])\right\},$$

and the following relations hold

$$\sum_{k=0}^{p+6}\left(\left|\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{u}(t,x)\right| + \left|\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{v}(t,x)\right|\right) + \left\|\overset{\varepsilon}{g}\right\|_{C^1[0,T]} + \\ \left|\frac{\partial}{\partial t}\overset{\varepsilon}{u}(t,x)\right| + \left|\frac{\partial}{\partial t}\overset{\varepsilon}{v}(t,x)\right| \leqslant C(\varepsilon), \quad (t,x)\in G_{[0,T]}, \tag{22}$$

where

$$C^{1,p+4}_{t,x}\big(G_{[0,T]}\big) = \left\{f(t,x,z)|f_t\in C\big(G_{[0,T]}\big), \frac{\partial^k}{\partial x^k}f\in C\big(G_{[0,T]}\big),\ k=0,...,p+4\right\}.$$

In the general case, the constant $C(\varepsilon)$ in (22) depends on ε and the input data.

3. Periodicity

Assumption 1. Let us assume that the constant $x^0\in(0,l)$, the input data $u_0(x), v_0(x)$, $f(t,x), F(t,x)$ are periodic in x functions with period $2l>0$ and the series

$$\begin{aligned} u_0(x) &= \sum_{k=0}^{\infty}\alpha_k\cos\frac{k\pi}{l}x,\\ v_0(x) &= \sum_{k=0}^{\infty}\beta_k\cos\frac{k\pi}{l}x,\\ f(t,x) &= \sum_{k=0}^{\infty}f_k(t)\cos\frac{k\pi}{l}x,\\ F(t,x) &= \sum_{k=0}^{\infty}F_k(t)\cos\frac{k\pi}{l}x \end{aligned} \tag{23}$$

converge uniformly on $[0,l]$ and $\overline{Q}_T$, together with their derivatives with respect to x of order $p+4$.

Solution $\overset{\varepsilon}{u}{}^{\tau}(t,x)$, $\overset{\varepsilon}{v}{}^{\tau}(t,x)$ converges uniformly in $G^M_{[0,T]}$ for any fixed $M>0$ together with its derivatives with respect to x up to order $p+4$ to $\overset{\varepsilon}{u}(t,x), \overset{\varepsilon}{v}(t,x)$. Considering Assumptions 1, the components of solution $\overset{\varepsilon}{u}(t,x), \overset{\varepsilon}{v}(t,x)$ are periodic functions with respect to variable x with period $2l$. Then we have the following theorem.

Theorem 2. *Let Assumption 1 and the conditions of Theorem 1 hold. Then for any fixed* $\varepsilon>0$ *the components* $\overset{\varepsilon}{u}, \overset{\varepsilon}{v}$ *of the solution* $\left(\overset{\varepsilon}{u}, \overset{\varepsilon}{v}, \overset{\varepsilon}{g}\right)$ *to problem* (1)–(3) *are periodic functions with respect to variable* x *with period* $2l$ *and they satisfy the following relations*

$$\frac{\partial^{2m+1}\overset{\varepsilon}{u}(t,0)}{\partial x^{2m+1}} = \frac{\partial^{2m+1}\overset{\varepsilon}{u}(t,l)}{\partial x^{2m+1}} = \frac{\partial^{2m+1}\overset{\varepsilon}{v}(t,0)}{\partial x^{2m+1}} = \frac{\partial^{2m+1}\overset{\varepsilon}{v}(t,l)}{\partial x^{2m+1}} = 0, \quad m=0,1,\dots,\frac{p+3}{2}. \tag{24}$$

Remark 1. It follows from relation (22) and system (1) that $\frac{\partial^m}{\partial t^m}\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{u}(t,x)$, $\frac{\partial^m}{\partial t^m}\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{v}(t,x)$, $2m+k\leqslant p+6$ exist, they are continuous in $G_{[0,T]}$ and

$$\left|\frac{\partial^m}{\partial t^m}\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{u}(t,x)\right|+\left|\frac{\partial^m}{\partial t^m}\frac{\partial^k}{\partial x^k}\overset{\varepsilon}{v}(t,x)\right|\leqslant C(\varepsilon),\quad (t,x)\in G_{[0,T]}. \tag{25}$$

Let us assume that

$$\begin{gathered}M_{1,j}=\max_{\overline{Q}_T}\left|\frac{\partial^j}{\partial x^j}f(t,x)\right|,\quad M_{1,j}^1=\max_{\overline{Q}_T}\left|\frac{\partial}{\partial t}\frac{\partial^j}{\partial x^j}f(t,x)\right|,\\ M_{2,j}=\max_{\overline{Q}_T}\left|\frac{\partial^j}{\partial x^j}F(t,x)\right|,\quad M_{2,j}^1=\max_{\overline{Q}_T}\left|\frac{\partial}{\partial t}\frac{\partial^j}{\partial x^j}F(t,x)\right|,\\ M_1=\max_{0\leqslant j\leqslant p+2}\left\{M_{1,j},M_{1,j}^1\right\},\quad M_2=\max_{0\leqslant j\leqslant p+2}\left\{M_{2,j},M_{2,j}^1\right\}.\end{gathered} \tag{26}$$

In what follows we assume that the conditions of Theorems 1 and 2 are fulfilled.

4. The solution existence of the second boundary problem

Let us consider problem (1)–(3) in $G_{[0,T]}$ with boundary condition

$$\overset{\varepsilon}{u}_x(t,0)=\overset{\varepsilon}{v}_x(t,0)=\overset{\varepsilon}{u}_x(t,l)=\overset{\varepsilon}{v}_x(t,l)=0,\quad t\in[0,T]. \tag{27}$$

Let us prove that the uniform with respect to ε family of estimates $\left\{\overset{\varepsilon}{w}\right\}=\left\{\overset{\varepsilon}{u},\overset{\varepsilon}{v}\right\}$ solutions of (1)–(3), (27) exists under conditions (5)–(13).

Let us introduce

$$\overset{\varepsilon}{w}_j=\left(\overset{\varepsilon}{u}_j,\overset{\varepsilon}{v}_j\right)=\left(\frac{\partial^j}{\partial x^j}\overset{\varepsilon}{u},\frac{\partial^j}{\partial x^j}\overset{\varepsilon}{v}\right)=\frac{\partial^j}{\partial x^j}\left(\overset{\varepsilon}{u},\overset{\varepsilon}{v}\right).$$

Let us differentiate j times ($j\leqslant p$) problem (1), (2) with respect to x. Then we multiply the result of differentiating by $e^{-\theta t}\frac{\partial}{\partial t}\overset{\varepsilon}{w}_{j+2}=e^{-\theta t}\left(\frac{\partial}{\partial t}\overset{\varepsilon}{u}_{j+2},\frac{\partial}{\partial t}\overset{\varepsilon}{v}_{j+2}\right)$, where constant $\theta>0$, and integrate over $Q_t=(0,t)\times(0,l)$, $t\in(0,T)$. This can be done by virtue of Remark 1.

We have the following relations

$$\begin{gathered}\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu+\varepsilon\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu+\\ +\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_j,\frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+2}\right)dxd\nu-\mu_1\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{u}_{j+2}\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu-\\ -\mu_2\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{v}_{j+2}\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu=\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{g}f_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu+\int_{Q_t}e^{-\theta\nu}F_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu,\end{gathered} \tag{28}$$

$$I_1=-\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu=\int_{Q_t}e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+1}\right)^2dxd\nu, \tag{29}$$

$$I_2=-\varepsilon\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu=\varepsilon\int_{Q_t}e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+1}\right)^2dxd\nu, \tag{30}$$

$$I_3=-\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_j,\frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+2}\right)dxd\nu=\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_{j+1},\frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+1}\right)dxd\nu=$$

$$= \frac{1}{2}\int_{Q_t}\frac{\partial}{\partial\nu}\left[e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)\right]dxd\nu+\frac{\theta}{2}\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu-$$

$$-\frac{1}{2}\int_{Q_t}e^{-\theta\nu}a^{'}\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu=\frac{1}{2}e^{-\theta t}\int_0^l a\left(t,\overset{\varepsilon}{w}_{j+1}(t,x),\overset{\varepsilon}{w}_{j+1}(t,x)\right)dx-$$

$$-\frac{1}{2}\int_0^l a\left(\nu,\overset{0}{w}_{j+1}(x),\overset{0}{w}_{j+1}(x)\right)dx+\frac{\theta}{2}\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu-$$

$$-\frac{1}{2}\int_{Q_t}e^{-\theta\nu}a^{'}\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu, \quad (31)$$

$$I_4=\frac{\mu_1}{2}\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\left(\overset{\varepsilon}{u}_{j+2}\right)^2dxd\nu=\frac{\mu_1}{2}e^{-\theta t}\int_0^l\left(\overset{\varepsilon}{u}_{j+2}(t,x)\right)^2dx+$$

$$+\frac{\theta\mu_1}{2}\int_{Q_t}e^{-\theta\nu}(\overset{\varepsilon}{u}_{j+2})^2dxd\nu-\frac{\mu_1}{2}\int_0^l\left(\overset{0}{u}_{j+2}(x)\right)^2dx, \quad (32)$$

$$I_5=\frac{\mu_2}{2}\int_{Q_t}e^{-\theta\nu}\frac{\partial}{\partial\nu}\left(\overset{\varepsilon}{v}_{j+2}\right)^2dxd\nu=\frac{\mu_2}{2}e^{-\theta t}\int_0^l\left(\overset{\varepsilon}{v}_{j+2}(t,x)\right)^2dx+$$

$$+\frac{\theta\mu_2}{2}\int_{Q_t}e^{-\theta\nu}\left(\overset{\varepsilon}{v}_{j+2}\right)^2dxd\nu-\frac{\mu_2}{2}\int_0^l\left(\overset{0}{v}_{j+2}(x)\right)^2dx. \quad (33)$$

Substitution of (29)–(33) into (28) gives

$$\int_{Q_t}e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+1}\right)^2dxd\nu+\varepsilon\int_{Q_t}e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+1}\right)^2dxd\nu+$$

$$\frac{1}{2}e^{-\theta t}\int_0^l a\left(\nu,\overset{\varepsilon}{w}_{j+1}(t,x),\overset{\varepsilon}{w}_{j+1}(t,x)\right)dx++\frac{\theta}{2}\int_{Q_t}e^{-\theta\nu}a\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu-$$

$$\frac{1}{2}\int_{Q_t}e^{-\theta\nu}a^{'}\left(\nu,\overset{\varepsilon}{w}_{j+1},\overset{\varepsilon}{w}_{j+1}\right)dxd\nu+\frac{\mu_1}{2}e^{-\theta t}\int_0^l\overset{\varepsilon}{u}{}^2_{j+2}(t,x)dx+\frac{\mu_1\theta}{2}\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{u}{}^2_{j+2}dxd\nu+$$

$$+\frac{\mu_2}{2}e^{-\theta t}\int_0^l\overset{\varepsilon}{v}{}^2_{j+2}(t,x)dx+\frac{\mu_2\theta}{2}\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{v}{}^2_{j+2}dxd\nu=$$

$$=\int_{Q_t}e^{-\theta\nu}\overset{\varepsilon}{g}f_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu+\int_{Q_t}e^{-\theta\nu}F_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu+$$

$$\frac{1}{2}\int_0^l a\left(0,\overset{0}{w}_{j+1}(x),\overset{0}{w}_{j+1}(x)\right)dx++\frac{\mu_1}{2}\int_0^l\overset{0}{u}{}^2_{j+2}(x)dx+\frac{\mu_2}{2}\int_0^l\overset{0}{v}{}^2_{j+2}(x)dx. \quad (34)$$

In system (1) we set $x=x^0\in(0,l)$ and evaluate function $\overset{\varepsilon}{g}(t)$. We obtain

$$\overset{\varepsilon}{g}(t)=\frac{\varphi^{'}(t)+a_{11}\varphi(t)+a_{12}\overset{\varepsilon}{v}(t,x^0)-\mu_1\overset{\varepsilon}{u}_{xx}(t,x^0)}{f(t,x^0)}. \quad (35)$$

By virtue of (4) and conditions imposed on $\varphi, \varphi_t, a_{11}, a_{12}, \mu_1$ function $\overset{\varepsilon}{g}$ satisfies inequality

$$\left|\overset{\varepsilon}{g}(t)\right| \leqslant C_1 \left(1 + \left|\overset{\varepsilon}{v}(t, x^0)\right| + \left|\overset{\varepsilon}{u}_{xx}(t, x^0)\right|\right), \tag{36}$$

where

$$C_1 = \max\left\{\frac{k}{\delta}, \frac{\mu_1}{\delta}, \frac{|a_{12}|}{\delta}\right\}, \quad k = \max_{[0,T]}\left(\left|\varphi^{'}(t)\right| + |a_{11}\varphi(t)|\right).$$

There is the following Embedding theorem [7].

Theorem 3. *Let Ω be bounded sidereal relative to some sphere domain and $\Omega \subseteq R^n$. If $\varphi \in H^m(\Omega)$ and $n < 2m$ then φ is continuous function everywhere in the region Ω including the boundary of Ω. Herewith we have*

$$||\varphi||_{C(\overline{\Omega})} \leqslant K||\varphi||_{H^m(\Omega)}, \tag{37}$$

where $||\varphi||_{C(\overline{\Omega})} = \max\limits_{\overline{\Omega}}|\varphi|$ and K is a constant independent of the choice of function φ.

By virtue of Theorem 3 we obtain from (36)

$$\left|\overset{\varepsilon}{g}(t)\right| \leqslant C_1 \left(1 + \left|\overset{\varepsilon}{v}(t, x^0)\right| + \left|\overset{\varepsilon}{u}_{xx}(t, x^0)\right|\right) \leqslant C_1 \left(1 + \sum_{j=0}^{2}\left\|\overset{\varepsilon}{v}_j(t)\right\|_{C(0,l)} + \sum_{j=0}^{2}\left\|\overset{\varepsilon}{u}_j(t)\right\|_{C(0,l)}\right) \leqslant$$
$$\leqslant C_1\left(1 + K\left(\int_0^l \sum_{j=0}^{3} \overset{\varepsilon}{u}{}_j^2(t,x)dx\right)^{1/2} + K\left(\int_0^l \sum_{j=0}^{3} \overset{\varepsilon}{v}{}_j^2(t,x)dx\right)^{1/2}\right) =$$
$$= C_1\left(1 + K\left(\left\|\overset{\varepsilon}{u}(t)\right\|_{H^3(0,l)} + \left\|\overset{\varepsilon}{v}(t)\right\|_{H^3(0,l)}\right)\right) \leqslant C_1\left(1 + K\left(\left\|\overset{\varepsilon}{u}(t)\right\|_{H^{m+1}(0,l)} + \left\|\overset{\varepsilon}{v}(t)\right\|_{H^{m+1}(0,l)}\right)\right),$$
$$m \geqslant 2. \tag{38}$$

where K is a constant that depends only on l.

Upon substituting (38) into (34) and summing over j from 0 to m, we obtain

$$\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+1}\right)^2 dxd\nu + \frac{1}{2}e^{-\theta t}\sum_{j=0}^{m}\int_0^l a\left(\nu, \overset{\varepsilon}{w}_{j+1}(t,x), \overset{\varepsilon}{w}_{j+1}(t,x)\right)dx+$$
$$+\frac{\theta}{2}\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu} a\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}_{j+1}\right)dxd\nu - \frac{1}{2}\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}_{j+1}\right)dxd\nu+$$
$$+\frac{\mu_1}{2}e^{-\theta t}\sum_{j=0}^{m}\int_0^l \overset{\varepsilon}{u}{}_{j+2}^2(t,x)dx + \frac{\mu_1\theta}{2}\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{u}{}_{j+2}^2 dxd\nu+$$
$$+\frac{\mu_2}{2}e^{-\theta t}\sum_{j=0}^{m}\int_0^l \overset{\varepsilon}{v}{}_{j+2}^2(t,x)dx + \frac{\mu_2\theta}{2}\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{v}{}_{j+2}^2 dxd\nu \leqslant$$
$$\leqslant \sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu} C_1\left(1 + K\left\|\overset{\varepsilon}{u}(\nu)\right\|_{H^{m+1}(0,l)} + K\left\|\overset{\varepsilon}{v}(\nu)\right\|_{H^{m+1}(0,l)}\right) f_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}dxd\nu+$$
$$+\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}F_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}dxd\nu + C_2. \tag{39}$$

where C_2 is a constant that depends only on the input data and constants T and l.

We have the following relations

$$\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{u}{}_j^2(\nu,x)dxd\nu = \int_0^t e^{-\theta\nu}d\nu\sum_{j=0}^{m}\int_0^l \overset{\varepsilon}{u}{}_j^2(\nu,x)dx = \int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^m(0,l)}d\nu, \tag{40}$$

$$\sum_{j=0}^{m}\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{v}{}_j^2(\nu,x)dxd\nu = \int_0^t e^{-\theta\nu}d\nu\sum_{j=0}^{m}\int_0^l \overset{\varepsilon}{v}{}_j^2(\nu,x)dx = \int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^m(0,l)}d\nu. \tag{41}$$

By virtue of (6) we obtain from (39)

$$\begin{gathered}\frac{1}{2}\int_0^t e^{-\theta\nu}\left\|\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+1}(0,l)}d\nu + \left(\frac{\theta\kappa}{2} - mC_1M_1K\right)\int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+1}(0,l)}d\nu + \\ + \frac{\kappa}{2}e^{-\theta t}\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+1}(0,l)} + \left(\frac{\kappa}{2}-\alpha\right)e^{-\theta t}\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+1}(0,l)} + \\ + \left(\theta(\frac{\kappa}{2}-\alpha) - mM_1C_1K - \alpha\right)\int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+1}(0,l)}d\nu + \\ + \frac{\mu_1}{2}e^{-\theta t}\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+2}(0,l)} + \frac{\mu_2}{2}e^{-\theta t}\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+2}(0,l)} + \frac{\theta\mu_1}{2}\int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+2}(0,l)}d\nu + \\ + \frac{\theta\mu_2}{2}\int_0^t e^{-\theta\nu}\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+2}(0,l)}d\nu \leqslant C_3(l,T,\alpha,M_1,M_2,C_1),\end{gathered} \tag{42}$$

where C_3 is a constant depending on $l,T,\delta,\alpha,\mu_1,\varphi,M_{1,j},M_{2,j}$ and constant $\alpha>0$.

Let us assume that $\alpha = \dfrac{\kappa}{4}$. Then we choose $\theta=\widetilde{\theta}$ such that all the coefficients before integrals in the left-hand side of previous inequality are the coefficients more than

$$\gamma = \min\left\{\frac{1}{2},\frac{\kappa}{4},\frac{\mu_1}{2},\frac{\mu_2}{2}\right\}.$$

Then we obtain the following inequality

$$\gamma e^{-\widetilde{\theta}T}\left\{\int_0^t\left\|\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+1}(0,l)}d\nu + \left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+2}(0,l)} + \left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+2}(0,l)}\right\} \leqslant$$
$$\leqslant C_3(l,T,\alpha,M_{1,j},M_{2,j},C_1), \quad t\in[0,T].$$

Hence the following estimates is implied

$$\int_0^t\left\|\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+1}(0,l)}d\nu + \int_0^t\left\|\overset{\varepsilon}{u}(\nu)\right\|^2_{H^{m+2}(0,l)}d\nu + \int_0^t\left\|\overset{\varepsilon}{v}(\nu)\right\|^2_{H^{m+2}(0,l)}d\nu \leqslant M,\ \forall t\in[0,T], \tag{43}$$

$$\left\|\overset{\varepsilon}{u}(t)\right\|^2_{H^{m+2}(0,l)} + \left\|\overset{\varepsilon}{v}(t)\right\|^2_{H^{m+2}(0,l)} \leqslant M, \quad \forall t\in[0,T], \quad m=\overline{0,p}, \tag{44}$$

where $M = C_3(l,T,\alpha,M_{1,j},M_{2,j},C_1)e^{\widetilde{\theta}T}/\gamma$.

From the last estimate we obtain for $t\in[0,T]$ the following uniform with respect to ε inequality

$$\left\|\overset{\varepsilon}{u}_j(t)\right\|_{C([0,l])} + \left\|\overset{\varepsilon}{v}_j(t)\right\|_{C([0,l])} \leqslant C, \quad j=0,...,p+1, \quad t\in[0,T]. \tag{45}$$

By virtue of (44) we obtain

$$\left\|\overset{\varepsilon}{g}(t)\right\|_{C[0,T]} \leqslant C. \tag{46}$$

This inequality is uniform with respect to ε.

We assume the following compatibility conditions for the input data

$$\mu_2 \overset{0}{v}_{xx}(x) + F(0,x) - a_{21}(0)\overset{0}{u}(x) - a_{22}(0)\overset{0}{v}(x) = 0. \tag{47}$$

Upon differentiating system (1) with respect to t, we obtain the system

$$\begin{cases} \dfrac{\partial}{\partial t}\overset{\varepsilon}{u}' + a_{11}\overset{\varepsilon}{u}' + a_{12}\overset{\varepsilon}{v}' + a_{11}'\overset{\varepsilon}{u} + a_{12}'\overset{\varepsilon}{v} = \mu_1 \overset{\varepsilon}{u}_{xx}' + \overset{\varepsilon}{g}'(t)f + \overset{\varepsilon}{g}(t)f', \\ \varepsilon\dfrac{\partial}{\partial t}\overset{\varepsilon}{v}' + a_{21}\overset{\varepsilon}{u}' + a_{22}\overset{\varepsilon}{v}' + a_{21}'\overset{\varepsilon}{u} + a_{22}'\overset{\varepsilon}{v} = \mu_2 \overset{\varepsilon}{v}_{xx}' + F', \end{cases} \tag{48}$$

with initial data

$$\overset{\varepsilon}{u}'(0,x) = -a_{11}(0)\overset{0}{u}(x) - a_{12}(0)\overset{0}{v}(x) + \mu_1 \overset{0}{u}_{xx} + \overset{\varepsilon}{g}(0)f(0,x), \tag{49}$$

$$\overset{\varepsilon}{v}'(0,x) = 0. \tag{50}$$

Let us differentiate problem (48)–(50) j times ($j \leqslant p$) with respect to x , multiply the result of differentiating by $-e^{-\theta t}\Big(\frac{\partial}{\partial t}\Big(\overset{\varepsilon}{u}_{j+2}'\Big), \frac{\partial}{\partial t}\Big(\overset{\varepsilon}{v}_{j+2}'\Big)\Big)$ and integrate $Q_t, t \in (0,T)$. This can be done by virtue of Remark 1. Then we have the following relations

$$\begin{aligned} -\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}'dxd\nu - \varepsilon\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}'dxd\nu - \\ -\int_{Q_t} e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_j', \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+2}'\right)dxd\nu - \int_{Q_t} e^{-\theta\nu}a'\left(\nu, \overset{\varepsilon}{w}_j, \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+2}'\right)dxd\nu + \\ +\mu_1\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{u}_{j+2}'\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}'dxd\nu + \mu_2\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{v}_{j+2}'\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}'dxd\nu = \\ = -\int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{g}'f_j\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}'dxd\nu - \int_{Q_t} e^{-\theta\nu}\overset{\varepsilon}{g}f_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}'dxd\nu - \int_{Q_t} e^{-\theta\nu}F_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}'dxd\nu, \end{aligned} \tag{51}$$

$$I_6 = -\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+2}'dxd\nu = \int_{Q_t} e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{u}_{j+1}'\right)^2 dxd\nu, \tag{52}$$

$$I_7 = -\varepsilon\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_j'\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+2}'dxd\nu = \varepsilon\int_{Q_t} e^{-\theta\nu}\left(\frac{\partial}{\partial\nu}\overset{\varepsilon}{v}_{j+1}'\right)^2 dxd\nu, \tag{53}$$

$$\begin{aligned} I_8 = -\int_{Q_t} e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_j', \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+2}'\right)dxd\nu = \int_{Q_t} e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_{j+1}', \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}_{j+1}'\right)dxd\nu = \\ = \frac{1}{2}\int_{Q_t}\frac{\partial}{\partial t}\left[e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_{j+1}', \overset{\varepsilon}{w}_{j+1}'\right)\right]dxd\nu + \frac{\theta}{2}\int_{Q_t} e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_{j+1}', \overset{\varepsilon}{w}_{j+1}'\right)dxd\nu - \\ -\frac{1}{2}\int_{Q_t} e^{-\theta\nu}a'\left(\nu, \overset{\varepsilon}{w}_{j+1}', \overset{\varepsilon}{w}_{j+1}'\right)dxd\nu = \frac{1}{2}e^{-\theta t}\int_0^l a\left(t, \overset{\varepsilon}{w}_{j+1}'(t,x), \overset{\varepsilon}{w}_{j+1}'(t,x)\right)dx - \\ -\frac{1}{2}\int_0^l a\left(\nu, \overset{0}{w}_{j+1}'(x), \overset{0}{w}_{j+1}'(x)\right)dx + \frac{\theta}{2}\int_{Q_t} e^{-\theta\nu}a\left(\nu, \overset{\varepsilon}{w}_{j+1}', \overset{\varepsilon}{w}_{j+1}'\right)dxd\nu - \\ -\frac{1}{2}\int_{Q_t} e^{-\theta\nu}a'\left(\nu, \overset{\varepsilon}{w}_{j+1}', \overset{\varepsilon}{w}_{j+1}'\right)dxd\nu, \end{aligned} \tag{54}$$

$$I_9 = -\int_{Q_t} e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_j, \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}{}^{'}_{j+2}\right) dxd\nu = \int_{Q_t} e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \frac{\partial}{\partial\nu}\overset{\varepsilon}{w}{}^{'}_{j+1}\right) dxd\nu =$$
$$= \frac{1}{2}\int_{Q_t} \frac{\partial}{\partial t}\left[e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}{}^{'}_{j+1}\right)\right] dxd\nu + \frac{\theta}{2}\int_{Q_t} e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}{}^{'}_{j+1}\right) dxd\nu -$$
$$-\frac{1}{2}\int_{Q_t} e^{-\theta\nu} a^{''}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}{}^{'}_{j+1}\right) dxd\nu = \frac{1}{2}e^{-\theta t}\int_0^l a^{'}\left(t, \overset{\varepsilon}{w}_{j+1}(t,x), \overset{\varepsilon}{w}{}^{'}_{j+1}(t,x)\right) dx -$$
$$-\frac{1}{2}\int_0^l a^{'}\left(\nu, \overset{0}{w}_{j+1}(x), \overset{0}{w}{}^{'}_{j+1}(x)\right) dx + \frac{\theta}{2}\int_{Q_t} e^{-\theta\nu} a^{'}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}{}^{'}_{j+1}\right) dxd\nu -$$
$$-\frac{1}{2}\int_{Q_t} e^{-\theta\nu} a^{''}\left(\nu, \overset{\varepsilon}{w}_{j+1}, \overset{\varepsilon}{w}{}^{'}_{j+1}\right) dxd\nu, \quad (55)$$

$$I_{10} = \frac{\mu_1}{2}\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\left(\overset{\varepsilon}{u}{}^{'}_{j+2}\right)^2 dxd\nu = \frac{\mu_1}{2}e^{-\theta t}\int_0^l \left(\overset{\varepsilon}{u}{}^{'}_{j+2}(t,x)\right)^2 dx +$$
$$+\frac{\theta\mu_1}{2}\int_{Q_t} e^{-\theta\nu}\left(\overset{\varepsilon}{u}{}^{'}_{j+2}\right)^2 dxd\nu - \frac{\mu_1}{2}\int_0^l \left(\overset{0}{u}{}^{'}_{j+2}(x)\right)^2 dx, \quad (56)$$

$$I_{11} = \frac{\mu_2}{2}\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial\nu}\left(\overset{\varepsilon}{v}{}^{'}_{j+2}\right)^2 dxd\nu = \frac{\mu_2}{2}e^{-\theta t}\int_0^l \left(\overset{\varepsilon}{v}{}^{'}_{j+2}(t,x)\right)^2 dx +$$
$$+\frac{\theta\mu_2}{2}\int_{Q_t} e^{-\theta\nu}\left(\overset{\varepsilon}{v}{}^{'}_{j+2}\right)^2 dxd\nu - \frac{\mu_2}{2}\int_0^l \left(\overset{0}{v}{}^{'}_{j+2}(x)\right)^2 dx. \quad (57)$$

On the basis of (17)–(19), (49), (50), (51)–(57) and repeating arguments used to obtain inequality (44), we have

$$\left\|\overset{\varepsilon}{u}{}^{'}(t)\right\|^2_{H^{m+2}(0,l)} + \left\|\overset{\varepsilon}{v}{}^{'}(t)\right\|^2_{H^{m+2}(0,l)} \leqslant C, \quad \forall t \in [0,T], \quad m = \overline{0,p}. \quad (58)$$

From (58) we obtain uniform with respect to ε inequality

$$\left\|\overset{\varepsilon}{u}{}^{'}_j(t)\right\|_{C([0,l])} + \left\|\overset{\varepsilon}{v}{}^{'}_j(t)\right\|_{C([0,l])} \leqslant C, \quad j = 0,...,p+1, \quad t \in [0,T]. \quad (59)$$

By virtue of (45), (59) and taking into account Arzela's theorem [6], we can choose the subsequence $\left(\overset{\mu}{u}, \overset{\mu}{v}\right)$ such that it converges to the vector function (u,v) as $\mu \to 0$:

$$\overset{\mu}{u}_j \to u_j, \quad \overset{\mu}{v}_j \to v_j \quad \text{uniformly in } C(\overline{Q}_T), \quad j = 0,...,p-1. \quad (60)$$

Taking the limit $\mu \to 0$ in system (1) (with $\varepsilon = \mu$) and taking into account estimate (59) (with $j = 0$), by virtue of (60) we find that the vector (u,v) satisfies in $\overline{Q}_T$ the following system of equations

$$\begin{cases} u_t(t) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + g(t)f(t,x), \\ \qquad a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + F(t,x), \quad (t,x) \in \overline{Q}_T \end{cases} \quad (61)$$

with initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in [0,l], \quad (62)$$

boundary conditions

$$u_x(t,0) = v_x(t,0) = u_x(t,l) = v_x(t,l) = 0, \tag{63}$$

and the overdetermination condition

$$u(t,x^0) = \varphi(t). \tag{64}$$

Based on overdetermination condition (64), we have

$$g(t) = \frac{\varphi^{'} + a_{11}\varphi + a_{12}v(t,x^0) - \mu_1 u_{xx}(t,x^0)}{f(t,x^0)}$$

in system (61).

Hence, by virtue of (60) we have

$$\overset{\varepsilon}{g}(t) \to g(t), \text{ uniformly in } \quad C[0,T]. \tag{65}$$

Let us prove the uniqueness of the solution to problem (61)–(64).

Let us assume that $(\overset{1}{u}, \overset{1}{v}, \overset{1}{g})$, $(\overset{2}{u}, \overset{2}{v}, \overset{2}{g})$ are two solutions of problem (61)–(64) and $\widetilde{u} = \overset{1}{u} - \overset{2}{u}$, $\widetilde{v} = \overset{1}{v} - \overset{2}{v}$, $\widetilde{g} = \overset{1}{g} - \overset{2}{g}$.

The vector $(\widetilde{u}, \widetilde{v}, \widetilde{g})$ satisfies the system of equations

$$\begin{cases} \widetilde{u}_t(t) + a_{11}\widetilde{u} + a_{12}\widetilde{v} = \mu_1\widetilde{u}_{xx} + \widetilde{g}f, \\ \qquad a_{21}\widetilde{u} + a_{22}\widetilde{v} = \mu_2\widetilde{v}_{xx}. \end{cases} \tag{66}$$

with initial conditions

$$\widetilde{u}(0,x) = \widetilde{v}(0,x) = 0, \tag{67}$$

boundary conditions

$$\widetilde{u}_x(t,0) = \widetilde{v}_x(t,0) = \widetilde{u}_x(t,l) = \widetilde{v}_x(t,l) = 0 \tag{68}$$

and the overdetermination condition

$$\widetilde{u}(t,x^0) = 0. \tag{69}$$

By virtue of conditions (67)–(69) we obtain

$$\widetilde{g} = \frac{a_{12}\widetilde{v}(t,x^0) - \mu_1\widetilde{u}_{xx}(t,x^0)}{f(t,x^0)}. \tag{70}$$

Let us differentiate problem (66)–(69) j times ($j \leqslant p$) with respect to x, multiply the result of differentiating by $e^{-\theta t}\widetilde{w}_j = e^{-\theta t}\left(\widetilde{u}_j, \widetilde{v}_j\right)$ and integrate over $Q_t = (0,t) \times (0,l), t \in (0,T)$. This can be done by virtue of Remark 1. Then we have the following relations:

$$\int_{Q_t} e^{-\theta\nu}\widetilde{u}_j\frac{\partial}{\partial t}\widetilde{u}_j dxd\nu + \int_{Q_t} e^{-\theta\nu} a\left(\nu, \widetilde{w}_j, \widetilde{w}_j\right) dxd\nu - \\ - \mu_1 \int_{Q_t} e^{-\theta\nu}\widetilde{u}_{j+2}\widetilde{u}_j dxd\nu - \mu_2\int_{Q_t} e^{-\theta\nu}\widetilde{v}_{j+2}\widetilde{v}_j dxd\nu = \int_{Q_t} e^{-\theta\nu}\widetilde{g}f_j\widetilde{u}_j dxd\nu, \tag{71}$$

$$I_{12} = \int_{Q_t} e^{-\theta\nu}\widetilde{u}_j\frac{\partial}{\partial t}\widetilde{u}_j dxd\nu = \frac{1}{2}\int_{Q_t} e^{-\theta\nu}\frac{\partial}{\partial t}\widetilde{u}_j^2 dxd\nu = \frac{1}{2}e^{-\theta t}\int_0^l \widetilde{u}_j^2(t,x)dx + \\ + \frac{\theta}{2}\int_{Q_t} e^{-\theta\nu}\widetilde{u}_j^2 dxd\nu - \frac{1}{2}\int_0^l \overset{0}{u}{}_j^2(x)dx, \tag{72}$$

$$I_{13} = -\mu_1 \int_{Q_t} e^{-\theta\nu} \widetilde{u}_{j+2} \widetilde{u}_j dx d\nu = \mu_1 \int_{Q_t} e^{-\theta\nu} \widetilde{u}^2_{j+1} dx d\nu, \tag{73}$$

$$I_{14} = -\mu_2 \int_{Q_t} e^{-\theta\nu} \widetilde{v}_{j+2} \widetilde{v}_j dx d\nu = \mu_2 \int_{Q_t} e^{-\theta\nu} \widetilde{v}^2_{j+1} dx d\nu. \tag{74}$$

Taking into account (70), conditions (4) and the condition imposed on μ_1, function $\widetilde{g}$ satisfies the following inequality

$$|\widetilde{g}(t)| \leqslant C_4(\left|\widetilde{v}\left(t, x^0\right)\right| + \left|\widetilde{u}_{xx}\left(t, x^0\right)\right|),$$

where $C_4 = \max\left\{\dfrac{\mu_1}{\delta}, \dfrac{|a_{12}|}{\delta}\right\}$.

Hence, by virtue of Theorem 3 we obtain

$$|\widetilde{g}(t)| \leqslant C_4 K \left(||\widetilde{u}(t)||_{H^m(0,l)} + ||\widetilde{v}(t)||_{H^m(0,l)}\right).$$

Considering relationships (40), (41), we obtain

$$\left(\frac{\theta}{2} + \kappa - \frac{mM_1C_4K\alpha}{2} - \frac{1}{2\alpha}\right) \int_0^t e^{-\theta\nu} \left\|\widetilde{u}\right\|^2_{H^m(0,l)} d\nu + \frac{1}{2} e^{-\theta t} \left\|\widetilde{u}\right\|^2_{H^m(0,l)} +$$
$$+ \left(\kappa - \frac{mM_1C_4K\alpha}{2}\right) \int_0^t e^{-\theta\nu} ||\widetilde{v}||^2_{H^m(0,l)} d\nu + \mu_1 \int_0^t e^{-\theta\nu} ||\widetilde{u}||^2_{H^{m+1}(0,l)} d\nu +$$
$$+ \mu_2 \int_0^t e^{-\theta\nu} ||\widetilde{v}||^2_{H^{m+1}(0,l)} d\nu \leqslant 0. \tag{75}$$

We choose the constant $\alpha > 0$ to be sufficiently small such that

$$\kappa - \frac{mM_1C_4K\alpha}{2} > 0. \tag{76}$$

Then we choose the constant θ to be sufficiently large such that

$$\frac{\theta}{2} + \kappa - \frac{mM_1C_4K\alpha}{2} - \frac{1}{2\alpha} > 0. \tag{77}$$

From relation (75) we obtain

$$||\widetilde{u}||_{L_2(Q_T)} + ||\widetilde{v}||_{L_2(Q_T)} \leqslant 0. \tag{78}$$

It follows that $\widetilde{u} = \widetilde{v} = 0$ in $\overline{Q}_T$.

Thus we have proved the following result.

Theorem 4. *Let us assume that conditions* (3)–(6), (24), (25), (47) *and Assumption 1 are satisfied. Then there exists a unique solution* (u, v, g) *of problem* (61)–(64) *in the class*

$$X(T) = \left\{u(t,x), v(t,x), g(t) | u(t,x) \in C^{1,p-1}_{t,x}\left(\overline{Q}_T\right), v(t,x) \in C^{0,p-1}_{t,x}\left(\overline{Q}_T\right), g(t) \in C\left([0,T]\right)\right\}.$$

The solution (u, v, g) to problem (61)–(64) is unique and belongs to the class of $X(T)$. The sequence $\left(\overset{\varepsilon}{u}, \overset{\varepsilon}{v}, \overset{\varepsilon}{g}\right)$ converges to (u, v, g) as well as the subsequence $\left(\overset{\mu}{u}, \overset{\mu}{v}, \overset{\mu}{g}\right)$ given above. There is the following theorem.

Theorem 5. *Let us assume that the conditions of Theorem 4 are satisfied. When* $\varepsilon \to 0$

$$\overset{\varepsilon}{u}_j \to u_j, \quad \overset{\varepsilon}{v}_j \to v_j \quad \text{in } C\left(\overline{Q}_T\right), \quad j = 0, ..., p-1, \tag{79}$$

$$\overset{\varepsilon}{g} \to g \quad \text{in } C([0,T]). \tag{80}$$

5. Degree of convergence as $\varepsilon \to 0$

Let us subtract system (61) from system (1) and denote $\left(\overset{\varepsilon}{u} - u, \overset{\varepsilon}{v} - v\right) = \left(\overset{\varepsilon}{r}^{(1)}, \overset{\varepsilon}{r}^{(2)}\right) = \overset{\varepsilon}{r}$. Then we obtain the following system of equations

$$\begin{cases} \dfrac{\partial}{\partial t}\overset{\varepsilon}{r}^{(1)} + a_{11}\overset{\varepsilon}{r}^{(1)} + a_{12}\overset{\varepsilon}{r}^{(2)} = \mu_1 \dfrac{\partial^2 \overset{\varepsilon}{r}^{(1)}}{\partial x^2} + \overset{\varepsilon}{G}(t) f(t,x), \\ \varepsilon \dfrac{\partial \overset{\varepsilon}{v}}{\partial t} + a_{21}\overset{\varepsilon}{r}^{(1)} + a_{22}\overset{\varepsilon}{r}^{(2)} = \mu_2 \dfrac{\partial^2 \overset{\varepsilon}{r}^{(2)}}{\partial x^2}, \end{cases} \tag{81}$$

for the vector $\overset{\varepsilon}{r} = \left(\overset{\varepsilon}{r}^{(1)}, \overset{\varepsilon}{r}^{(2)}\right)$ that satisfies conditions

$$\overset{\varepsilon}{r}(0,x) = 0, \quad x \in [0,l], \tag{82}$$

$$\frac{\partial^{2m+1}}{\partial x^{2m+1}}\overset{\varepsilon}{r}(t,0) = \frac{\partial^{2m+1}}{\partial x^{2m+1}}\overset{\varepsilon}{r}(t,l) = 0, \quad m = 0,1,2. \tag{83}$$

In (81) the function $\overset{\varepsilon}{G}$ is of the following form:

$$\overset{\varepsilon}{G}(t) = \overset{\varepsilon}{g}(t) - g(t) = \frac{a_{12}\overset{\varepsilon}{r}^{(2)}(t,x^0) - \mu_1 \overset{\varepsilon}{r}^{(1)}_{xx}(t,x^0)}{f(t,x^0)}, \tag{84}$$

Let us differentiate problem (81)–(83) three times with respect to x, multiply the differentiated system by $e^{-\theta t}\overset{\varepsilon}{r}_3$ and integrate over $Q_t, t \in (0,T)$. Repeating arguments used in obtaining relation (44), we have the inequality

$$\left\| \overset{\varepsilon}{r}^{(1)}(t) \right\|^2_{H^3(0,l)} + \int_0^t e^{-\theta\nu} \left\| \overset{\varepsilon}{r}^{(1)}(t) \right\|^2_{H^4(0,l)} d\nu + \int_0^t e^{-\theta\nu} \left\| \overset{\varepsilon}{r}^{(2)}(t) \right\|^2_{H^4(0,l)} d\nu \leqslant \varepsilon C. \tag{85}$$

By virtue of (85) it follows that

$$\left\| \overset{\varepsilon}{u}_j - u_j \right\|_{C(\overline{Q}_T)} \leqslant \varepsilon^{1/2} C, \quad j = \overline{0,2}, \tag{86}$$

$$\left\| \overset{\varepsilon}{v}_m - v_m \right\|_{L_2(Q_T)} \leqslant \varepsilon^{1/2} C, \quad m = \overline{0,4}. \tag{87}$$

Taking into account (86) and (87), we have from (84)

$$\left\| \overset{\varepsilon}{g}(t) - g(t) \right\|_{L_2([0,T])} \leqslant \varepsilon^{1/2} C. \tag{88}$$

Thus we have proved the following result.

Theorem 6. *Let us assume that the conditions of Theorem 4 are satisfied. Then relations* (86), (87) *hold.*

Let us consider problem (1)–(3) and assume that the conditions of Theorem 2 are satisfied. By virtue of the periodicity of the input data there is the following theorem.

Theorem 7. *Let us assume that the conditions of Theorem 4 are satisfied. Then the solution* (u,v,g) *of the problem*

$$\begin{cases} u_t(t) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + g(t)f(t,x), \\ a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + F(t,x), \end{cases}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in [0,l], \quad u(t,x^0) = \varphi(t)$$

exists and it is unique in the class

$$X(T) = \left\{ u(t,x), v(t,x), g(t) | u(t,x) \in C^{1,p-1}_{t,x}(G_{[0,T]}), v(t,x) \in C^{0,p-1}_{t,x}(G_{[0,T]}), \quad g(t) \in C([0,T]) \right\}.$$

Relations (86), (87) *are satisfied and when* $\varepsilon \to 0$ $\overset{\varepsilon}{u}_j \to u_j$, $\overset{\varepsilon}{v}_j \to v_j$ *uniformly in* $G_{[0,T]}$, $j = 0, ..., p-1$, $\overset{\varepsilon}{g} \to g$, *uniformly in* $C[0,T]$, $\left| \overset{\varepsilon}{u}(t,x) - u(t,x) \right| \leqslant \varepsilon^{1/2} C$, $(t,x) \in G_{[0,T]}$.

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Определение функций источника систем уравнений составного типа

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Рассмотрена задача идентификации функций источника одномерной полуэволюционной системы уравнений для двух уравнений в частных производных. Исследована система уравнений, полученная из исходной системы, в которой в эллиптическое уравнение добавлена производная по времени, содержащая малый параметр $\varepsilon > 0$. *Рассмотрены задача Коши и вторая краевая задача.*

Ключевые слова: идентификация, обратная задача, уравнение параболического типа, метод слабой аппроксимации, малый параметр, сходимость.